

## FINITE RANK BUTLER GROUPS AND TORSION-FREE MODULES OVER A DISCRETE VALUATION RING

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ABSTRACT. Fully faithful functors from isomorphism at  $p$  categories of finite rank Butler groups to torsion-free modules of finite rank over the integers localized at a prime  $p$  are constructed via categories of representations of antichains over discrete valuation rings. Descriptions and properties of modules in the images of these functors are given, including a characterization of finite representation type and a complete list of indecomposables in that case.

The two subjects of torsion-free abelian groups and  $p$ -local torsion-free abelian groups, more generally torsion-free modules over a discrete valuation ring, have traditionally had little in common. For instance, the set of types, isomorphism classes of rank-1 torsion-free abelian groups, is an uncountable lattice [6] while there are only two types in the  $p$ -local case, namely the isomorphism classes of  $Z_p$ , the integers localized at a prime  $p$ , and  $Q$ , the additive group of rationals. Pure subgroups of finite direct sums of rank-1 groups, called Butler groups, are known to be complicated [4] but  $p$ -local Butler groups are direct sums of rank-1  $p$ -local groups [5]. On the other hand,  $p$ -local torsion-free abelian groups of finite rank can be viewed, up to quasi-isomorphism, as finite rank Butler modules over a Dedekind domain [13].

In this paper, we demonstrate the existence of a variety of fully faithful embeddings from isomorphism at  $p$  categories of finite rank Butler groups to the category  $TF(Z_p)$  of  $p$ -local torsion-free abelian groups of finite rank. Under these embeddings, pure subgroups of the  $p$ -adic completion of the integers correspond to rank-1 Butler groups.

Let  $T$  be a finite lattice of types with least element  $\tau_0$ ,  $j$  a non-negative integer, and  $JI(T)$  the partially ordered set of join irreducible elements of  $T$ . Define  $B(T, j)_p$  to be the isomorphism at  $p$  category of finite rank Butler groups  $G$  with typeset contained in  $T$  and  $p^j G \leq \sum \{G(\tau) : \tau \in JI(T)\} \leq G$ . Morphism sets for this category are  $\text{Hom}(G, H)_p$ , the localization of  $\text{Hom}_Z(G, H)$  at a prime  $p$ . Isomorphism at  $p$  is the local version of near isomorphism and is strictly weaker than quasi-isomorphism [8] or [2].

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A finite lattice  $T$  of types is  $p$ -locally free if  $X_p$  is isomorphic to  $Z_p$  for each rank-1 group  $X$  with type  $X \in T$ . Our results require  $T$  to be a finite Boolean algebra, in this case  $JI(T)$  is an antichain consisting of the atoms of  $T$ . The  $p$ -rank of a torsion-free abelian group  $M$  is the  $Z/pZ$  dimension of  $M/pM$ .

**Theorem I.** *Suppose  $T$  is a  $p$ -locally free finite Boolean algebra of types and  $j$  is a non-negative integer. There are uncountably many fully faithful functors  $F_A$  from  $B(T, j)_p$  into  $TF(Z_p)$  indexed by finite subsets  $A$  of the  $p$ -adic integers. Moreover,*

$$\text{rank } F_A(G) \geq 2(p\text{-rank } F_A(G))$$

for each  $G$  in  $B(T, j)$ .

The definition of  $F_A$ , including an independence condition on  $A$ , is given in Theorem 1.3. Rank 1 groups in  $B(T, j)_p$  correspond under  $F_A$  to purely indecomposable groups, those indecomposable  $p$ -local groups of finite rank with  $p$ -rank 1. On the other hand, if  $M \in TF(Z_p)$  is co-purely indecomposable, indecomposable with rank  $-p\text{-rank} = 1$  [1], and if  $3 \leq \text{rank } M$ , then  $M$  is not in the image of any  $F_A$ . However, co-purely indecomposable groups can be realized as the image of Butler groups via another class of functors defined on  $B(T, 0)_p$ .

**Theorem II.** *Assume that  $T$  is a  $p$ -locally free finite Boolean algebra of types. There are uncountably many faithful functors  $G_A$  from  $B(T, 0)_p$  into  $TF(Z_p)$ , indexed by finite subsets  $A$  of the  $p$ -adic integers. Furthermore,*

$$\text{rank } G_A(H) \leq 2(p\text{-rank } G_A(H))$$

for each  $H$  in  $B(T, 0)$ .

Suppose  $H = (X_1 \oplus \cdots \oplus X_k)/K \in B(T, 0)$  is an indecomposable group, where  $K$  is a pure rank-1 subgroup and each  $X_i$  is a rank-1 group with type  $X_i$  an atom of  $T$ . Then  $G_A(H)$  is a co-purely indecomposable  $p$ -local group. Butler groups  $H$  of this kind have been studied extensively [4]. Theorems I and II and the duality between purely indecomposable and co-purely indecomposable  $p$ -local groups given in [1] reflect the “dual” nature of rank-1 groups  $X$  that are images of a finite direct sum of rank-1 groups  $X_1, \dots, X_n$  and groups of the form  $H = (X_1 \oplus \cdots \oplus X_k)/K$ ,  $K$  a pure rank-1 subgroup.

The functors of Theorems I and II are derived from fully faithful embeddings, also denoted by  $F_A$  and  $G_A$ , from  $\text{rep}(S_n, R, j)$ , a category of representations of an antichain  $S_n$  over a discrete valuation ring  $R$ , into the category  $TF(R)$  of finite rank torsion-free  $R$ -modules. As for images of these functors, we have (Corollary 2.6):

- (i) Each  $M \in \text{image } F_A \cup \text{image } G_A$  has a fully invariant basic submodule.
- (ii) Each  $M \in \text{image } F_A \cap \text{image } G_A$  is a free  $R$ -module.
- (iii) If  $M \in \text{image } F_A$ , then  $M$  is a homomorphic image of a finite direct sum of purely indecomposable modules with rank  $\leq 2$  and endomorphism ring  $R$ .
- (iv) If  $M \in \text{image } G_A$ , then  $M$  is a pure submodule of a finite direct sum of purely indecomposable modules with rank  $\leq 2$  and endomorphism ring  $R$ .

Theorems I and II allow the translation of known results for isomorphism at  $p$  categories of Butler groups directly into the image of  $F_A$  and the image of  $G_A$ . A full subcategory  $\mathcal{C}$  of  $TF(Z_p)$  has *finite representation type* if there are only finitely many isomorphism classes of indecomposables in  $\mathcal{C}$  and *wild mod  $p$  representation type* if for each  $Z_p$ -algebra  $\Lambda$  that is finitely generated and free as a  $Z_p$ -module,

there is a ring homomorphism from  $\text{End } M$  onto  $\Lambda/p\Lambda$ . Representation types of categories of  $Z_p$ -representations are computed in [3]. This yields:

**Corollary III.** *The full subcategory of  $TF(Z_p)$  consisting of  $p$ -local groups in the image of  $F_A$ , respectively the image of  $G_A$ , has*

- (a) *finite representation type if and only if  $|A| \leq 2$ ;*
- (b) *wild mod  $p$  representation type if and only if  $|A| \geq 3$ .*

In Section 2, a complete list of isomorphism classes of indecomposable groups in the image of  $F_A$  and the image of  $G_A$  is given for the case that  $|A| = 2$  (Corollary 2.4). Indecomposable groups in the image of  $F_A$  have either  $p$ -rank 1 and rank  $\leq 3$  or else  $p$ -rank 2 and rank 4 while indecomposables in the image of  $G_A$  are either isomorphic to  $Z_p$  or have  $p$ -rank 2 and rank 3.

Computations of representation type and classification of indecomposables for quasi-homomorphism categories in  $TF(R)$  are given in terms of dimensions of splitting fields in [9, Theorem 5.11]. These results are obtained by embedding the quasi-homomorphism subcategory  $QTF_K$  of modules with a fixed splitting field  $K$  as a full subcategory of finite dimensional modules over a finite dimensional  $Q$ -algebra. By way of comparison with the results in this paper, let  $n = [K : Q]$ . Then  $QTF_K$  has finite representation type if and only if  $n \leq 3$ . Moreover, if  $n = 3$ , then a strongly indecomposable reduced  $M$  in  $QTF_K$  has  $p$ -rank 1 and rank  $\leq 2$  or  $p$ -rank 2 and rank 3.

The results of Section 1 carry over to torsion-free modules of finite rank over almost maximal valuation domains with appropriate modifications of definitions and proofs. For relevant results, see [7, Chapter 14].

## 1. TORSION-FREE MODULES OVER DISCRETE VALUATION RINGS AND REPRESENTATIONS

Let  $R$  be a discrete valuation ring (commutative domain), with prime  $p$  and quotient field  $Q$ , and denote the  $p$ -adic completion of  $R$  by  $R^*$ . The following fundamental properties and definitions of torsion-free  $R$ -modules can be found, for example, in [1] or [9]. If  $M$  is a torsion-free reduced  $R$ -module of finite rank, then  $M$  contains a *basic submodule*  $B$ , i.e.  $B$  is a finitely generated free  $R$ -module that is pure in  $M$  with  $M/B$  divisible. The  *$p$ -rank of  $M$*  is the  $R/pR$ -dimension of  $M/pM$ , in fact  $p\text{-rank } M = \text{rank } B$  for a basic submodule  $B$  of  $M$ . Moreover,  $M$  is a pure submodule of  $M^*$ , the  $p$ -adic completion of  $M$ ,  $M^*$  is a free  $R^*$ -module with rank the rank of  $B$ , and  $M^*/M$  is divisible. Thus, we may assume that  $M$  is a pure submodule of  $M^* = B^* = B \otimes_R R^*$ . Every  $R$ -endomorphism of  $M$  extends uniquely to an  $R^*$ -endomorphism of  $M^*$ .

Suppose  $M$  is a finite rank torsion-free  $R$ -module with  $p\text{-rank } M = n$  and  $\text{rank } M = n + k$ . Then  $M$  is free if and only if  $k = 0$  and divisible if and only if  $n = 0$ . If  $n = 1$ , then  $M = N \oplus D$ , where  $D$  is divisible and  $N$  is *purely indecomposable*, reduced and indecomposable with  $p$ -rank 1. On the other hand, if  $k = 1$ , then  $M = L \oplus F$ , where  $F$  is a free  $R$ -module and  $L$  is *co-purely indecomposable*, indecomposable with  $\text{rank } L = p\text{-rank } L + 1$ . Basic properties of purely and co-purely indecomposable groups are given in [1] or [9].

For an example of a purely indecomposable module, let  $t \in R^* \setminus R$  and let  $R_t$  denote the pure submodule of  $R^*$  generated by  $\{1, t\}$ . Then  $Z_p$  is a basic submodule of  $R_t$ ,  $R_t^* = R^*$ ,  $p\text{-rank } R_t = 1$ , and  $\text{rank } R_t = 2$ .

The following proposition is well known.

**Proposition 1.1.** *Suppose  $M$  and  $N$  are two purely indecomposable  $R$ -modules with  $\text{rank } M = \text{rank } N$ .*

(a) [1] *The following are equivalent:*

- (i)  *$M$  is isomorphic to  $N$ ;*
- (ii)  *$M$  is quasi-isomorphic to  $N$ ;*
- (iii)  *$\text{Hom}_R(M, N) \neq 0$ .*

(b) [14] *Let  $M = R_s$  and  $N = R_t$  for  $s, t \in R^* \setminus R$ . Then  $M$  and  $N$  are isomorphic if and only if there are elements  $a, b, c, d$  in  $Q$  with  $t = (c + ds)/(a + bs)$ .*

The properties of purely indecomposables given in Proposition 1.1(a) are analogous to those of rank-1 torsion-free abelian groups. Proposition 1.2 gives analogs of the pure fully invariant subgroups  $G(\tau)$ , for  $G$  a torsion-free abelian group and  $\tau$  a type, where an element  $t$  of  $R^*$  plays the role of a type  $\tau$ . Let  $M$  be a torsion-free module of finite rank with a basic submodule  $B$  and assume  $B \leq M \leq M^* = R^* \otimes_R B$ . For  $t \in R^*$ , define  $M(t) = \{m \in M : tm \in M\}$ . Notice that if  $t \in R$ , then  $M(t) = M$ .

**Proposition 1.2.** *Assume  $M$  and  $N$  are finite rank reduced torsion-free  $R$ -modules and  $t \in R^* \setminus R$ .*

(a)  *$M(t)$  is a pure submodule of  $M$  and if  $f: M \rightarrow N$  is an  $R$ -homomorphism, then  $f(M(t)) \leq N(t)$ .*

(b)  *$R_t M(t) = \text{Hom}(R_t, M)R_t$ , the image of the evaluation map  $\text{Hom}(R_t, M) \otimes_R R_t \rightarrow M$ .*

(c)  *$M(t) \cap M(s) \leq M(t + s)$ .*

(d)  *$M(t)(s) = M(ts) \cap M(t)$ .*

*Proof.* (a) Suppose  $m \in M$  and  $p^i m \in M(t)$ . Then  $p^i t m \in p^i M^* \cap M = p^i M$  so that  $t m \in M$ . Thus,  $m \in M(t)$  and so  $M(t)$  is pure in  $M$ . Recall that  $f: M \rightarrow N$  extends to an  $R^*$ -homomorphism  $f^*$  from  $M^*$  to  $N^*$ . Hence, if  $m \in M$  with  $tm \in M$ , then  $f(tm) = f^*(tm) = t f^*(m) = t f(m) \in N$ . This shows that  $f(M(t)) \leq N(t)$ .

(b) Let  $m \in M(t)$ . Then  $R_t m \leq M$  since  $R_t = \langle R \oplus Rt \rangle_*$  and  $M$  is pure in  $M^*$ . Define  $f: R_t \rightarrow M$  by  $f(y) = ym$ , an  $R$ -homomorphism with  $m \in \text{image } f$ . Consequently,  $M(t) \leq \text{Hom}(R_t, M)R_t$  and  $R_t M(t) \leq \text{Hom}(R_t, M)R_t$ .

Conversely, let  $0 \neq f: R_t \rightarrow M$ . Then  $f$  is a monomorphism, otherwise  $\ker f$  is a pure cyclic submodule of  $R_t$  and image  $f$  is divisible, a contradiction to the assumption that  $M$  is reduced. Hence, image  $f = R_t m \leq M$  for  $m = f(1) \in M$ . In particular,  $m \in M(t)$  and image  $f \leq R_t M(t)$  as desired.

(c) If  $m \in M$  and  $tm$  and  $sm$  are in  $M$ , then  $(t + s)m$  is in  $M$ .

(d) Let  $m \in M(t)(s)$ . Then  $m \in M(t)$  with  $sm \in M(t)$  and so  $tsm \in M$ . This shows that  $M(t)(s) \leq M(st) \cap M(t)$ . Conversely, if  $m \in M(st) \cap M(t)$ , then  $stm \in M$  and  $tm \in M$ . Hence,  $m \in M(t)(s)$ .  $\square$

Let  $S$  be a finite partially ordered set and  $\text{rep}(S, R, j)$  the category with objects  $(U_0, U_i : i \in S)$ , where  $U_0$  is a finitely generated free  $R$ -module, each  $U_i$  is an  $R$ -summand of  $U_0$ ,  $U_i$  is contained in  $U_s$  if  $i \leq s$  in  $S$ , and  $p^j U_0 \leq \sum \{U_i : i \in S\} \leq U_0$ . A morphism  $f: U \rightarrow V$  is an  $R$ -homomorphism  $f: U_0 \rightarrow V_0$  such that  $f(U_i) \leq V_i$  for each  $i$ .

The category of torsion-free  $R$ -modules of finite rank is denoted by  $TF(R)$ . An additive functor  $F: C \rightarrow D$  is *fully faithful* if  $F: \text{Hom}(U, V) \rightarrow \text{Hom}(F(U), F(V))$  is an isomorphism for each  $U, V$  in  $C$ .

**Theorem 1.3.** *Let  $S_k = \{1, \dots, k\}$  be an antichain and  $A = \{t_1, \dots, t_k\}$  a subset of  $R^* \setminus R$  such that  $\{1, t_j, t_i t_j : 1 \leq j \leq k\}$  is a  $Q$ -independent set for each  $1 \leq i \leq k$ . For each non-negative integer  $j$ , there is a fully faithful embedding  $F_A: \text{rep}(S_k, R, j) \rightarrow TF(R)$  with image consisting of those  $M$  with fully invariant basic submodule  $B$  such that  $M/(B + R_{t_1}M(t_1) + \dots + R_{t_k}M(t_k))$  is torsion, each  $M(t_i)$  is contained in  $B$ , and  $p^j B \leq M(t_1) + \dots + M(t_k) \leq B$ .*

*Proof.* Let  $U = (U_0, U_i : 1 \leq i \leq k)$  and define

$$M = F_A(U) = \langle U_0 + U_1 t_1 + \dots + U_k t_k \rangle_* \leq U_0^* = R^* \otimes_R U_0.$$

Given a representation morphism  $f: U \rightarrow V$ , let  $F_A(f)$  be the restriction of the  $R^*$ -homomorphism  $1 \otimes f: U_0^* \rightarrow V_0^*$  to  $F_A(U)$ . It follows that  $F_A(f): F_A(U) \rightarrow F_A(V)$  and that  $F_A$  is a faithful functor.

The free module  $U_0$  is a basic submodule of  $M$ . This is because  $U_0$  is a pure submodule of  $U_0^*$ , hence of  $M$ , and  $M/U_0$  is divisible since  $U_0^*/U_0$  is divisible and  $M$  is a pure submodule of  $U_0^*$ . Moreover, for each  $i$ ,  $U_i$  is contained in  $M(t_i)$ . To see that  $M(t_i) = U_i$ , assume  $m \in M(t_i)$ . Then  $m = q(u_0 + u_1 t_1 + \dots + u_k t_k)$  with  $t_i m = q'(u'_0 + u'_1 t_1 + \dots + u'_k t_k) = q(u_0 t_i + u_1 t_1 t_i + \dots + u_k t_k t_i)$  for some  $q, q' \in Q$  and  $u_j, u'_j \in U_j$ . Choose an  $R$ -basis for  $U_0$ , necessarily an  $R^*$ -basis for  $U_0^*$ , and write each  $u_i$  in terms of this basis, recalling that  $U_i \leq U_0$ . Now equate coefficients and use the assumption that the set  $\{1, t_j, t_i t_j : 1 \leq j \leq k\}$  is  $Q$ -independent to see that  $m \in U_i$ .

We now have that  $B = U_0$  is a basic submodule of  $M$  and  $U_i = M(t_i) \leq B$ . Then  $M/(B + R_{t_1}M(t_1) + \dots + R_{t_k}M(t_k))$  is torsion since  $U_i = M(t_i) \leq R_{t_i}M(t_i)$  by Proposition 1.2(b) and  $p^j U_0 = p^j B \leq M(t_1) + \dots + M(t_k) = U_1 + \dots + U_k \leq U_0 = B$ . By Proposition 1.2(a), each  $M(t_i)$  is fully invariant in  $M$  so that  $B$  is also fully invariant in  $M$ .

To see that  $F$  is a full functor, let  $g: F_A(U) \rightarrow F_A(V)$  and let  $f$  be the restriction of  $g$  to  $U_0$ , a basic submodule of  $F_A(U)$ . Now  $f: U_0 \rightarrow V_0$  with  $f(U_i) = f(F_A(U)(t_i)) \leq F_A(V)(t_i) = V_i$  for each  $i$ , by Proposition 1.2(a). Thus,  $f: U = (U_0, U_i) \rightarrow V = (V_0, V_i)$  is a representation morphism with  $F_A(f) = g$ , as desired.

Finally, let  $M$  be in  $TF(R)$  satisfying the above conditions. Then  $U = (B, M(t_i)) \in \text{rep}(S_k, R, j)$  since  $M(t_i)$  is pure in, hence a summand of, the finitely generated free  $R$ -module  $B$  and  $p^j B \leq M(t_1) + \dots + M(t_k) \leq B$ . Moreover,  $F_A(U) = \langle B + M(t_1)t_1 + \dots + M(t_k)t_k \rangle_* = \langle B + R_{t_1}M(t_1) + \dots + R_{t_k}M(t_k) \rangle_* = M$ , as desired.  $\square$

**Corollary 1.4.** *Let  $F_A: \text{rep}(S_k, R, j) \rightarrow TF(R)$  be the functor given in Theorem 1.3.*

(a) *If  $F_A(U) = M$ , with  $U = (U_0, U_i)$ , then  $p$ -rank  $M = \text{rank } U_0$  and  $\text{rank } M = \text{rank } U_0 + \text{rank } U_1 + \dots + \text{rank } U_k \geq 2(p\text{-rank } M)$ .*

(b) *If  $U = (U_0, U_i)$  with  $\text{rank } U_0 = 1$ , then  $M = F_A(U)$  is purely indecomposable with  $\text{End } M = R$ .*

(c) *If  $F_A(U) = M$  is co-purely indecomposable, then  $\text{rank } M = 2$  and  $M$  is isomorphic to  $R_t$  for some  $t \in A$ .*

*Proof.* (a) By Theorem 1.3,  $U_0$  is a basic submodule of  $M$  so that  $\text{rank } U_0 = p\text{-rank } M$ . Moreover,  $U_1t_1 + \dots + U_k t_k = U_1t_1 \oplus \dots \oplus U_k t_k$  by the independence condition on  $A$  and  $\text{rank } U_0 \leq \text{rank } U_1 + \dots + \text{rank } U_k$ . Hence,  $\text{rank } M = \text{rank } U_0 + \text{rank } U_1 + \dots + \text{rank } U_k \geq 2(p\text{-rank } M)$ .

(b) is a consequence of (a) and the fact that  $R = \text{End } U \cong \text{End } F_A(U)$ , since  $F_A$  is a fully faithful functor.

(c) Since  $M$  is co-purely indecomposable,  $\text{rank } M = p\text{-rank } M + 1$ . Apply (a) to see that  $\text{rank } M = 2$ , whence  $p\text{-rank } M = 1$  and  $M$  is isomorphic to  $\langle 1, t \rangle_* = R_t$  for some  $t \in A$ . □

Define  $\Sigma(S_{k+1}, R)$  to be the full subcategory of  $\text{rep}(S_{k+1}, R, 0)$  with objects  $V = (V_0, V_1, \dots, V_k, V_\#)$  such that  $V_0 = V_1 \oplus \dots \oplus V_k$ ,  $V_\#$  is a pure submodule of  $V_0$ , and  $V_\# \cap V_i = 0$  for each  $1 \leq i \leq k$ .

**Lemma 1.5** ([3]). *There is a category equivalence from  $\text{rep}(S_k, R, 0)$  to  $\Sigma(S_{k+1}, R)$ . The functor is given by  $(U_0, U_i) \rightarrow (U_1 \oplus \dots \oplus U_k, U_1, \dots, U_k, U_\#)$ , where  $U_\#$  is the kernel of the epimorphism  $U_1 \oplus \dots \oplus U_k \rightarrow U_0$  induced by the inclusion of the  $U_i$ 's in  $U_0$ .*

Let  $V = (V_0, V_1, \dots, V_k, V_\#) \in \Sigma(S_{k+1}, R)$ . Given a subset  $A = \{t_1, \dots, t_k\}$  of  $R^*$ , define  $V_\#(A) = \{(t_1v_1, \dots, t_kv_k) \in V_0^* = V_1^* \oplus \dots \oplus V_k^* : (v_1, \dots, v_k) \in V_\# \leq V_0 = V_1 \oplus \dots \oplus V_k\}$ . Let  $M \in TF(R)$  with a basic submodule  $B = B_1 \oplus \dots \oplus B_k$ . Then  $M \leq M^* = B^* = B_1^* \oplus \dots \oplus B_k^*$ . Define  $M_A = \{(b_1^*, \dots, b_k^*) \in M : (t_1b_1^*, \dots, t_kb_k^*) \in M\}$  and  $M_A(A) = \{(t_1b_1^*, \dots, t_kb_k^*) : (b_1^*, \dots, b_k^*) \in M_A\}$ , a submodule of  $M$  with  $\text{rank } M_A = \text{rank } M_A(A)$ . Clearly,  $M_A$  and  $M_A(A)$  depend on a choice of basic submodule  $B$  and a decomposition  $B = B_1 \oplus \dots \oplus B_k$ .

**Theorem 1.6.** *Let  $A = \{t_1, \dots, t_k\}$  be a subset of  $R^* \setminus R$  such that  $\{1, t_j, t_it_j : 1 \leq j \leq k\}$  is a  $Q$ -independent set for each  $1 \leq i \leq k$ . There is a faithful embedding*

$$G_A: \text{rep}(S_k, R, 0) \rightarrow TF(R)$$

*that is full for all those  $U \in \text{rep}(S_k, R, 0)$  with no rank-1 summands. The image of  $G_A$  consists of all those  $M$  such that  $M/(B + M_A(A))$  is torsion, where  $M_A$  is contained in a basic submodule  $B = B_1 \oplus \dots \oplus B_k$  of  $M$  with each  $B_i$  fully invariant in  $M$ .*

*Proof.* In view of Lemma 1.5, it suffices to define a functor  $G_A: \Sigma(S_{k+1}, R) \rightarrow TF(R)$ . Let  $V = (V_0, V_i, V_\#) \in \Sigma(S_{k+1}, R)$  and define

$$M = G_A(V) = \langle V_0 + V_\#(A) \rangle_* \leq V_0^* = R^* \otimes_R V_0.$$

Given a representation morphism  $f: V \rightarrow V'$ , extend  $f: V_0 \rightarrow V'_0$  to an  $R^*$ -homomorphism  $f^*: V_0^* \rightarrow V_0'^*$  and define  $G_A(f)$  to be the restriction of  $f^*$  to  $G_A(V)$ . It follows that  $G_A(f): G_A(V) \rightarrow G_A(V')$  and  $G_A$  is a faithful functor.

Observe that  $B = V_0 = V_1 \oplus \dots \oplus V_k = B_1 \oplus \dots \oplus B_k$  is a basic submodule of  $M$ . Moreover,  $V_\# \leq M_A$  since  $V_\#(A) \leq M$ . Conversely,  $M_A \leq V_\#$ . This can be seen by choosing an  $R$ -basis of  $V_0$  consisting of a basis for each  $V_i$ . This is then an  $R^*$ -basis of  $V_0^* = V_1^* \oplus \dots \oplus V_k^*$ . Write  $m \in M_A$  in terms of that basis and, just as in the proof of Theorem 1.3, equate coefficients and use the independence conditions on  $A$  to conclude that  $m \in V_\#$ . Thus,  $V_\# = M_A$  is contained in  $B = V_0$ , and  $M/(B + M_A(A)) = \langle V_0 + V_\#(A) \rangle_*/(V_0 + V_\#(A))$  is torsion.

To see that  $G_A$  is full, assume that  $V = (V_0, V_i, V_\#)$  and  $V' = (V'_0, V'_i, V'_\#) \in \Sigma(S_{k+1}, R)$  have no rank-1 summands and let  $g: M = G_A(V) \rightarrow M' = G_A(V')$

be an  $R$ -homomorphism. Extend  $g$  to an  $R^*$ -homomorphism  $g^*: M^* = V_0^* \rightarrow (M')^* = (V'_0)^*$ . Then  $g: M_A = V_\# \rightarrow M'_A = V'_\#$ . Let  $\pi_i: V_0 \rightarrow V_i$  be a projection of  $V_0$  onto the summand  $V_i$ . Since  $V$  has no rank-1 summands, each  $V_i/\pi_i(V_\#)$  must be torsion, otherwise there is a cyclic summand of  $V_i$  that gives rise to a rank-1 representation summand of  $V$ . Consequently,  $g: V_i \rightarrow V'_i$  for each  $i$ . But  $V_0 = V_1 \oplus \cdots \oplus V_k$ , so  $g$  induces a representation morphism  $f: V \rightarrow V'$  with  $G_A(f) = g$ , as desired.

Finally, let  $M$  be as described in the statement of the theorem. Then  $V = (B, B_i, M_A) \in \Sigma(S_{k+1}, R)$ , noting that  $M_A$  is pure in  $B$  and  $M_A \cap B_i = 0$  for each  $i$  since  $A$  is a subset of  $R^* \setminus R$ . Hence,  $G_A(V) = \langle B + M_A(A) \rangle_* = M$ .  $\square$

**Corollary 1.7.** *Let  $G_A: \text{rep}(S_k, R, 0) \rightarrow TF(R)$  be the functor given in Theorem 1.6.*

(a) *If  $M = G_A(U)$  with  $U = (U_0, U_i)$ , then  $p\text{-rank } M = \text{rank } U_1 + \cdots + \text{rank } U_k$  and  $\text{rank } M \leq 2(p\text{-rank } M)$ . If  $U$  has no rank-1 summands, then  $\text{rank } M = 2(p\text{-rank } M) - \text{rank } U_0$ .*

(b) *Let  $U = (U_0, U_i) \in \text{rep}(S_k, Z_p, 0)$  with  $k \geq 2$  with  $U_0 = R^{k-1} = R \oplus \cdots \oplus R$ ,  $U_1 = R \oplus 0 \oplus \cdots \oplus 0, \dots, U_{k-1} = 0 \oplus 0 \oplus \cdots \oplus R$ , and  $U_k = R(1, 1, \dots, 1)$ .*

*Then  $M = G_A(U)$  is co-purely indecomposable with  $\text{End } M = R$ .*

(c) *If  $M = G_A(U)$  is purely indecomposable, then  $M$  is isomorphic to  $R$ .*

*Proof.* (a) is a consequence of Lemma 1.5 and Theorem 1.6. Specifically,  $\text{rank } V_0 = p\text{-rank } M$ , where  $V = (U_1 \oplus \cdots \oplus U_k, U_1, \dots, U_k, U_\#) = (V_0, V_1, \dots, V_k, V_\#)$ ,  $\text{rank } M = \text{rank } V_0 + \text{rank } V_\#(A)$ , and  $\text{rank } V_\#(A) \leq \text{rank } V_0$ . Finally, if  $U$ , hence  $V$ , has no rank-1 summands, then  $\text{rank } M = \text{rank } V_0 + \text{rank } V_\#(A) = \text{rank } V_0 + \text{rank } V_\# = \text{rank } V_0 + \text{rank } V_0 - \text{rank } U_0 = 2(p\text{-rank } M) - \text{rank } U_0$ .

(b) Apply the functor of Lemma 1.5 to see that  $U \rightarrow V = (V_0, V_i)$ , where

$$V = (R^k = R \oplus \cdots \oplus R, R \oplus 0 \cdots \oplus 0, \dots, 0 \oplus \cdots \oplus R, R(1, \dots, 1)) \in \Sigma(S_{k+1}, R).$$

By (a),  $M = G_A(V) = \langle R^k, R(t_1, \dots, t_k) \rangle_*$  has  $p\text{-rank } k$ ,  $\text{rank } k + 1$ , and endomorphism ring isomorphic to  $\text{End } V = R$ . In particular,  $M$  is co-purely indecomposable.

(c) As a consequence of (a),  $\text{rank } M = p\text{-rank } M = 1$ , noting that  $U_0$  must be non-zero. Thus,  $M$  is isomorphic to  $R$ , as desired.  $\square$

The next proposition demonstrates that co-purely indecomposables (purely indecomposables) are generators (co-generators), for modules in  $TF(R)$ .

**Proposition 1.8.** *Suppose  $M \in TF(R)$  with  $p\text{-rank } n$  and  $\text{rank } n + k$  for  $n, k \geq 1$ .*

(a) *There is an exact sequence*

$$0 \rightarrow F \rightarrow C_1 \oplus \cdots \oplus C_k \rightarrow M \rightarrow 0$$

*where  $F$  is a free  $R$ -module and each  $C_i$  has  $p\text{-rank } n$  and  $\text{rank } n + 1$ .*

(b) *The module  $M$  is a pure submodule of some  $N = N_1 \oplus \cdots \oplus N_m$  with  $M/N$  divisible and  $p\text{-rank } N_i = 1$  for each  $i$ .*

*Proof.* (a) Let  $B$  be a basic submodule of  $M$  of rank  $n$  and extend  $B$  to a maximal free submodule  $K = B \oplus Rm_1 \oplus \cdots \oplus Rm_k$  of  $M$ . Then  $M$  is the purification of  $K$  in  $M^*$  and  $M/B$  is isomorphic to  $Q^k$ . Define  $C_i$  to be the purification of  $B \oplus Rm_i$  in  $M$ . Then  $B$  is a basic submodule of  $C_i$  with  $C_i/B = Q$  so that  $C_i$  has rank  $n + 1$  and  $p\text{-rank } n$ . Now  $N = C_1 + \cdots + C_k$  is a submodule of  $M$  containing  $B$ . In fact,  $N = M$ , as  $N/B$  is a  $k$ -dimensional  $Q$ -space contained in  $M/B$ . Inclusion induces an exact sequence  $0 \rightarrow F \rightarrow C = C_1 \oplus \cdots \oplus C_k \rightarrow M \rightarrow 0$ .

Now  $p$ -rank  $C = kn$  and rank  $C = k(n+1)$ . Thus,  $p$ -rank  $F = kn - n = (k-1)n$  and rank  $F = k(n+1) - (n+k) = (k-1)n$ , whence  $F$  is a free  $R$ -module.

(b) [1, Proposition 4.4.a].  $\square$

## 2. BUTLER GROUPS AND $p$ -LOCAL GROUPS

A *type* is an isomorphism class of a torsion-free abelian group of rank 1. The set of all types form a lattice [6]. A *Butler group*  $G$  is a pure subgroup of a finite direct sum  $C_1 \oplus \cdots \oplus C_m$  of rank 1 torsion-free abelian groups  $C_i$  [5]. If  $G$  is a Butler group, then *typeset*  $G$ , the set of types of pure rank-1 subgroups of  $G$ , is contained in the lattice of types  $T$  generated by the types of  $C_i$ 's. In particular, typeset  $G$  is finite if  $G$  is a Butler group.

Let  $T$  be a finite sublattice of the lattice of all types with least element  $\tau_0$  and  $B(T)$  the category of finite rank Butler groups with typeset contained in  $T$ . Let  $JI(T)$  be the partially ordered set of join irreducible elements of  $T$  and fix a prime  $p$ . The lattice  $T$  is  *$p$ -locally free* if  $X_p \cong Z_p$  for each rank-1 group  $X$  with type  $X \in T$ . For  $j \geq 0$ , let  $B(T, j)$  be the full subcategory of  $B(T)$  consisting of those  $G$  with  $p^j G$  contained in  $\Sigma\{G(\tau) : \tau \in JI(T)\}$ . The isomorphism at  $p$  category of  $B(T, j)$  is denoted by  $B(T, j)_p$ ; the objects are groups in  $B(T, j)$  and morphism sets are  $\text{Hom}(G, H)_p = \text{Hom}(G, H) \otimes Z_p$ ; see [3].

Up to isomorphism at  $p$ , there is essentially no loss of generality in considering  $B(T, j)$ , since if  $G$  is in  $B(T)$  and has no rank-1 summand of type  $\tau_0$ , then  $G$  is isomorphic at  $p$  to a group  $H \in B(T, j)$  for some  $j \geq 0$  [3].

**Lemma 2.1** ([15]). *If  $T$  is a  $p$ -locally free finite lattice of types, then there is a category equivalence  $H : B(T, j)_p \rightarrow \text{rep}(J(T)^{op}, Z_p, j)$  given by  $H(G) = (G_p, G(\tau)_p : \tau \in JI(T))$ .*

Theorems I and II of the introduction follow from the next theorem.

**Theorem 2.2.** *Let  $T$  be a  $p$ -locally free finite Boolean algebra of types with atoms  $\tau_1, \dots, \tau_k$  and  $A = \{t_1, \dots, t_k\}$  a subset of  $Z_p^* \setminus Z_p$  such that  $\{1, t_j, t_i t_j : 1 \leq j \leq k\}$  is a  $Q$ -independent set for each  $i$ .*

(a) *For each non-negative integer  $j$ , there is a fully faithful embedding*

$$F_A : B(T, j)_p \rightarrow TF(Z_p)$$

given by  $F_A(G) = \langle G_p + G(\tau_1)_p t_1 + \cdots + G(\tau_k)_p t_k \rangle_* \leq G_p^*$ .

(b) *There is a faithful embedding*

$$G_A : B(T, 0)_p \rightarrow TF(Z_p)$$

that is full for groups  $H$  with no rank-1 summands given by

$$G_A(H) = \langle H(\tau_1)_p \oplus \cdots \oplus H(\tau_k)_p + H_{\#p}(A) \rangle_* \leq H(\tau_1)_p^* \oplus \cdots \oplus H(\tau_k)_p^*$$

where  $H_{\#}$  is the kernel of  $H(\tau_1) \oplus \cdots \oplus H(\tau_k) \rightarrow H \rightarrow 0$ .

*Proof.* The join irreducibles of a finite Boolean algebra  $T$  is an antichain  $S_k$  consisting of the atoms of  $T$ . The functor  $F_A$  is the composite of  $H : B(T, j)_p \rightarrow \text{rep}(S_k, Z_p, j)$  given in Lemma 2.1 and the functor  $F_A : \text{rep}(S_k, Z_p, j) \rightarrow TF(Z_p)$  given in Theorem 1.3. The functor  $G_A$  is the composite of  $H : B(T, 0)_p \rightarrow \text{rep}(S_k, Z_p, 0)$  given in Lemma 2.1 and the functor  $G_A : \text{rep}(S_k, Z_p, 0) \rightarrow TF$  given in Theorem 1.6.  $\square$

Corollary III of the introduction follows immediately from Theorem 2.2 and the following:

**Theorem 2.3** ([3]). *If  $T$  is a  $p$ -locally free finite Boolean algebra of types with  $k$  atoms, then  $B(T, j)_p$  has:*

- (a) *wild modulo  $p$  representation type if and only if  $k \geq 3, j \geq 0$ ,*
- (b) *finite representation type if and only if  $k \leq 2, j \geq 0$ .*

The following is a complete list of indecomposables in the image of  $F_A$  and the image of  $G_A$  for the case that  $|A| = 2$ , obtained from a complete list of indecomposables in  $\text{rep}(S_2, Z_p, j)$  given in [3].

**Corollary 2.4.** *Let  $T$  be a finite  $p$ -locally free Boolean algebra of types with atoms  $\tau_1, \tau_2$ , and let  $A = \{t_1, t_2\}$  be a subset of  $Z_p^* \setminus Z_p$  such that  $\{1, t_1, t_2, t_2 t_1, t_2^2, t_1^2\}$  is a  $Q$ -independent set.*

- (a) *If  $G \in B(T, j)$  and  $M = F_A(G)$  is indecomposable, then  $M$  is isomorphic to:*
  - (i)  $Z_p$  *if  $M$  has  $p$ -rank 1 and rank 1,*
  - (ii)  $\langle Z_p, Z_p t_1 \rangle_*$  *or*  $\langle Z_p, Z_p t_2 \rangle_*$  *if  $M$  has  $p$ -rank 1 and rank 2,*
  - (iii)  $\langle Z_p, Z_p t_1, Z_p t_2 \rangle_*$  *if  $M$  has  $p$ -rank 1 and rank 3.*
  - (iv)  $\langle B, Z_p t_1 \oplus 0, 0 \oplus Z_p t_2 \rangle_* \leq B^*$  *if  $M$  has  $p$ -rank 2 and rank 4, where*

$$B = Z_p \oplus Z_p + Z_p(1, 1)/p^j \leq Q \oplus Q.$$

- (b) *If  $H \in B(T, 0)$  and  $M = G_A(H)$  is indecomposable, then  $M$  is isomorphic to*

- (i)  $Z_p$  *if  $M$  has  $p$ -rank 1 and rank 1,*
- (ii)  $\langle Z_p \oplus Z_p + Z_p(t_1, t_2) \rangle_* \leq Z_p^* \oplus Z_p^*$  *if  $M$  has  $p$ -rank 2 and rank 3.*

**Corollary 2.5.** *Assume the notation of Theorem 2.2.*

- (a) *If  $M = F_A(G)$  for  $G$  in  $B(T, j)$ , then  $\text{rank } G = p\text{-rank } M$  and  $\text{rank } M = p\text{-rank } M + \text{rank } G(\tau_1) + \dots + \text{rank } G(\tau_k)$ .*
- (b) *If  $G \in B(T, j)$  has rank 1, then  $F_A(G)$  is purely indecomposable.*
- (c) *If  $H \in B(T, 0)$  and  $M = G_A(H)$ , then  $p\text{-rank } M = \text{rank } H(\tau_1) + \dots + \text{rank } H(\tau_k)$  and  $\text{rank } M \leq 2(p\text{-rank } M)$ . If  $H$  has no rank-1 summands, then  $\text{rank } M = 2(p\text{-rank } M) - \text{rank } H$ .*
- (d) *If  $H = (X_1 \oplus \dots \oplus X_k)/(1, 1, \dots, 1)_*$ , where  $X_i$  is a subgroup of  $Q$  containing 1,  $1/p \notin X_i$ , and type  $X_i = \tau_i$  is an atom of  $T$  for each  $i$ , then  $G_A(H)$  is co-purely indecomposable.*

*Proof.* (a)–(d) follow from Theorem 2.2 and Corollaries 1.4 and 1.7. □

As a consequence of Proposition 1.8 and the following corollary, each  $M$  in the image of  $F_A$  is both a pure submodule and a homomorphic image of a finite direct sum of modules with  $p$ -rank 1 and endomorphism ring  $R$  and each  $M$  in the image of  $G_A$  is both a pure submodule and a torsion-free homomorphic image of a finite direct sum of modules with  $\text{rank} - p\text{-rank} = 1$  and endomorphism ring  $R$ .

**Corollary 2.6.** *Assume the notation of Theorems 1.3 and 1.6.*

- (a) *If  $M = F_A(G)$ , then there is an epimorphism  $N_1 \oplus \dots \oplus N_m \rightarrow M \rightarrow 0$  of  $R$ -modules such that each  $N_i$  is a purely indecomposable module with  $\text{rank} \leq 2$  and  $\text{End } N_i = R$ .*

(b) If  $M = G_A(H)$ , then  $M$  is a pure submodule of  $N_1 \oplus \cdots \oplus N_m$  such that each  $N_i$  is a purely indecomposable module with  $\text{rank} \leq 2$  and  $\text{End } N_i = R$ .

(c) Suppose that either  $M = F_A(G)$  or else  $M = G_A(H)$  and let  $S$  be the pure submodule of  $R^*$  generated by  $\{1, t_1, \dots, t_k\}$ . Then  $S \otimes_R M \cong S^n$  for  $n = p\text{-rank } M$ .

(d) Suppose that  $M = F_A(G) = G_A(H)$ . Then  $M$  is a free  $R$ -module.

*Proof.* (a) Write  $M = F_A(U) = \langle U_0 + U_1 t_1 + \cdots + U_k t_k \rangle_*$  as in Theorem 1.3. Then  $M = N$  for  $N = U_0 + R_{t_1} U_1 + \cdots + R_{t_k} U_k$ . This is because  $N/U_0$  is torsion-free divisible with  $\text{rank} = \text{rank } M/U_0$ , recalling that  $U_0$  is a basic submodule of  $M$  and  $R_{t_i} U_i / U_i \cong Q \otimes_R U_i$ .

(b) Write  $M = G_A(V) = \langle V_0 + V_{\#}(A) \rangle_* \leq V_0^*$  as in Theorem 1.6. Then  $M$  embeds as a pure submodule of  $N = R_{t_1} V_1 \oplus \cdots \oplus R_{t_k} V_k$ , recalling that  $V_0 = V_1 \oplus \cdots \oplus V_k$  and  $N$  is pure in  $N^* = V_0^*$ .

(c) Let  $M = F_A(U) = \langle U_0 + U_1 t_1 + \cdots + U_k t_k \rangle_* \leq U_0^* = R^* \otimes_R U_0$  as in Theorem 1.3. Then  $S \otimes_R U_0$  is pure in  $U_0^*$ , since  $R$  is a discrete valuation ring,  $U_0$  is pure in  $U_0^*$ , and  $S$  is torsion free. Clearly,  $M \leq S \otimes_R U_0$  so that  $S \otimes_R M = S \otimes_R U_0 \cong S^n$  for  $n = \text{rank } U_0 = p\text{-rank } M$ , recalling that  $U_0$  is a basic submodule of  $M$ . A similar argument shows that if  $M = G_A(V) = \langle V_0 + V_{\#}(A) \rangle_* \leq V_0^*$  as in Theorem 1.6, then  $S \otimes_R M = S \otimes_R V_0$  with  $V_0$  a basic submodule of  $M$ .

(d) Note that if  $M = F_A(U)$ , then  $M(t_i) \neq 0$  for each  $t_i$  in  $A$ . On the other hand, if  $M = G_A(H)$ , then  $M(t_i) = 0$  unless  $A = \{t_1\}$ . In this case,  $M = F_A(U)$  is a finite direct sum of copies of  $R$  and modules of  $p$ -rank 1 and rank 2, since  $U \in \text{rep}(S_1, R, j)$  is a direct sum of rank-1 representations  $(R, 0)$  and  $(R, R)$ . On the other hand,  $R_t$  is not in the image of  $G_A$  by Corollary 1.7. This shows that  $M$  is a free  $R$ -module.  $\square$

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