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THE SIZE OF THE DINI SUBDIFFERENTIAL

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ABSTRACT. Given a lower semicontinuous function $f : \mathbb{R}^h \to \mathbb{R} \cup \{+\infty\}$, we prove that the points of $\mathbb{R}^h$, where the lower Dini subdifferential contains more than one element, lie in a countable union of sets which are isomorphic to graphs of some Lipschitzian functions defined on $\mathbb{R}^{h-1}$. Consequently, the set of all these points has a null Lebesgue measure.

1. INTRODUCTION

Let $f : \mathbb{R}^h \to \mathbb{R} \cup \{+\infty\}$ be any lower semicontinuous function. An important and often-featured notion of generalized (directional) derivative is the lower Dini derivative, which, for x in the domain of f, is defined by

$$d^- f(x; d) = \liminf_{u \to d, t \to 0^+} \frac{f(x + tu) - f(x)}{t}.$$

We associate to it the lower Dini subdifferential of f at a point x, in the domain of f, defined by

$$\partial^- f(x) = \{\xi \in \mathbb{R}^h \ : \ \langle \xi, d \rangle \leq d^- f(x, d), \ \forall \ d \in \mathbb{R}^h\}.$$

Let us put $\partial^- f(x) = \emptyset$ if $f(x) = +\infty$. The lower Dini subdifferential is a closed convex set, which can be described as follows:

$$\partial^- f(x) = \{\xi \in \mathbb{R}^h \ : \ \liminf_{y \to x} \frac{f(y) - f(x) - \langle \xi, y - x \rangle}{||y - x||} \geq 0\}.$$

We refer to the book [1] for the above characterization and other classical properties of this subdifferential. This paper deals with the "size" of the two following subsets which are included in the domain of the lower Dini subdifferential of f at x:

$$D(f) = \{x \in \mathbb{R}^h : \text{the interior of } \partial^- f(x) \text{ is nonempty}\};$$

$$\tilde{D}(f) = \{x \in \mathbb{R}^h : \partial^- f(x) \text{ contains at least two elements}\}.$$

Let us notice that the inclusion $D(f) \subset \tilde{D}(f)$ is always true; moreover if $h = 1$, we have $D(f) = \tilde{D}(f)$.

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2. Countability of the set $D(f)$

In this section, we focus on the cardinality of the set $D(f)$. As shown in the following example, the set $D(f)$ is in general not finite. Let us consider the function f whose epigraph is the convex hull of the set

$$\{(n, n^2) : n \in \mathbb{Z}\}.$$

In this case, $D(f) = \mathbb{N}$ and so $D(f)$ is infinite countable. The first result shows that this property is still true for any lower semicontinuous function.

Proposition 2.1. *For any lower semicontinuous function f, the set $D(f)$ is at most countable.*

Proof. Let us choose a dense sequence $(v_p)_{p\geq 1}$ in $\mathbb{R}^h$ dense in $\mathbb{R}^h$ and let us define, for $\omega = (p, n, \ell) \in (\mathbb{N}^\star)^3$, the subset D_ω constituted of elements x of $\mathbb{R}^h$ such that $f(x)$ is real and such that, for all $\xi \in \overline{B}\left(v_p, \frac{1}{n}\right)$ and for all $y \in \overline{B}\left(x, \frac{1}{\ell}\right)$,

$$f(y) \geq f(x) + \langle \xi, y - x\rangle - \frac{1}{3n}||y - x||.$$

Step 1. Let us prove the inclusion $D(f) \subset \bigcup_{\omega\in(\mathbb{N}^\star)^3} D_\omega$.

Let $x \in D(f)$; from the definition of the set $D(f)$, $f(x)$ is real. Since the interior of the subdifferential $\partial^- f(x)$ is nonempty and since the sequence $(v_p)_{p\geq 1}$ is dense in $\mathbb{R}^h$, there exist an integer $p \geq 1$ and $\epsilon > 0$ such that

$$\overline{B}\left(v_p, \epsilon\right) \subset \partial^- f(x). \tag{1}$$

Let $(e_1, \cdots, e_h)$ be the canonical basis of $\mathbb{R}^h$; for $i \in \{1, \cdots, h\}$, we put $e_{h+i} = -e_i$. According to relation (1), for every $i \in \{1, \cdots, 2h\}$, we have $v_p + \epsilon e_i \in \partial^- f(x)$, whence there exists a constant $r_i > 0$ such that

$$f(y) \geq f(x) + \langle v_p + \epsilon e_i, y - x\rangle - \frac{1}{3n}||y - x||, \tag{2}$$

for all $y \in \overline{B}(x, r_i)$. Let us choose an integer $\ell \geq 1$ such that

$$\frac{1}{\ell} \leq \min\{r_i : i \in \{1, \cdots, 2h\}\}.$$

By convex combinations of inequalities (2) ($i = 1, \cdots, 2h$), we obtain:

$$f(y) \geq f(x) + \langle v_p + u, y - x\rangle - \frac{1}{3n}||y - x||, \tag{3}$$

for all $y \in \overline{B}\left(x, \frac{1}{\ell}\right)$ and for all $u \in \mathrm{co}\{\epsilon e_1, \cdots, \epsilon e_{2h}\}$. It is well-known that the null vector of $\mathbb{R}^h$ belongs to the interior of $\mathrm{co}\{\epsilon e_1, \cdots, \epsilon e_{2h}\}$ and consequently, we can find an integer $n \geq 1$ such that

$$\overline{B}\left(0, \frac{1}{n}\right) \subset \mathrm{co}\{\epsilon e_1, \cdots, \epsilon e_{2h}\}.$$

Hence inequality (3) is in particular valid, for all $y \in \overline{B}\left(x, \frac{1}{\ell}\right)$ and for all $u \in \overline{B}\left(0, \frac{1}{n}\right)$. Then we conclude that $x \in D_{(p,n,\ell)}$ and the result is proved.

Step 2. Let us show that, for all $\omega = (p, n, \ell) \in (\mathbb{N}^\star)^3$ and for all elements x_1, x_2 of D_ω, the condition $x_1 \neq x_2$ implies $||x_2 - x_1|| > \frac{1}{\ell}$.

Suppose to the contrary that there exist $\omega = (p, n, \ell)$ in $(\mathbb{N}^\star)^3$ and two distinct elements x_1, x_2 in D_ω such that:

$$||x_2 - x_1|| \leq \frac{1}{\ell}.$$

On one hand, since $x_1 \in D_\omega$, by choosing $\xi = v_p + \frac{1}{n}\frac{x_2 - x_1}{||x_2 - x_1||}$, we obtain from step 1:

$$f(x_2) \geq f(x_1) + \langle v_p, x_2 - x_1 \rangle + \frac{2}{3n}||x_2 - x_1||.$$

On the other hand, since $x_2 \in D_\omega$, by choosing $\xi = v_p$, we obtain from step 1:

$$f(x_1) \geq f(x_2) + \langle v_p, x_1 - x_2 \rangle - \frac{1}{3n}||x_1 - x_2||.$$

Adding these two last inequalities, we conclude

$$0 \geq \frac{1}{3n}||x_1 - x_2||,$$

or equivalently $x_1 = x_2$, which is impossible.

Step 3. Let us now conclude.

From step 2, for all $\omega \in (\mathbb{N}^\star)^3$, the set D_ω is discrete in $\mathbb{R}^h$ and, in particular, it is countable. Then, the inclusion proved in the step 1 allows us to confirm that the set $D(f)$ is also countable. □

As we shall see in the next section, the structure of the set $\tilde{D}(f)$ is more complicated.

3. The structure of the set $\tilde{D}(f)$

Henceforward we assume that $h \geq 2$. Let us remark that, for $h = 2$ and $f(x_1, x_2) = |x_1|$, we have $\tilde{D}(f) = \{0\} \times \mathbb{R}$. In this example $\tilde{D}(f)$ is not countable, but it has a null Lebesgue measure. More generally, if f is a Lipschitzian[1] function on $\mathbb{R}^h$, Rademacher's Theorem states that f is differentiable at almost every point of $\mathbb{R}^h$ and in particular the set $\tilde{D}(f)$ has a null Lebesgue measure. Theorem 3.3 shows that this result is valid even if the Lipschitz property fails.

First, we introduce the following definition:

Definition 3.1. We say that a subset X of $\mathbb{R}^h$ is Lipschitzian if there exist a linear isomorphism $\phi : \mathbb{R}^h \to \mathbb{R}^h$ and a Lipschitzian function $g : \mathbb{R}^{h-1} \to \mathbb{R}$ satisfying the equality

$$X = \phi(\mathrm{Gr}(g)),$$

where $\mathrm{Gr}(g)$ denotes the graph of the function g.

According to Fubini's Theorem, the graph of a Lipschitzian function has a null Lebesgue measure; hence we have the following property.

Proposition 3.2. *Any Lipschitzian set in $\mathbb{R}^h$ has a null Lebesgue measure.*

We can now establish the main result of this paper.

Theorem 3.3. *If f is a lower semicontinuous function, the set $\tilde{D}(f)$ is included in a countable union of Lipschitzian sets of $\mathbb{R}^h$. In particular, $\tilde{D}(f)$ has a null Lebesgue measure.*

[1] I.e., there exists $K > 0$ such that $| f(x) - f(y) | \leq K||x - y||$ for all x, y in $\mathbb{R}^h$.

Proof. Let us choose a sequence $(v_p)_{p\geq 1}$ in $\mathbb{R}^h \setminus \{0\}$ which is dense in $\mathbb{R}^h$ and let us define, for $\omega = (n,p,q,\ell) \in (\mathbb{N}^\star)^4$, the compact $\tilde{D}_\omega$ constituted of elements x belonging to the closed ball $\overline{B}(0,n)$ such that $f(x) \leq n$ and such that there exist $d \in \overline{B}\left(v_p, \frac{1}{n}\right) \cap \left[\mathbb{R}^h \setminus B\left(0, \frac{9}{n}\right)\right]$ and $\xi \in \overline{B}\left(v_q, \frac{1}{2n}\right)$ satisfying:

$$\forall y \in B\left(x, \frac{1}{\ell}\right),\ f(y) \geq f(x) + \langle \xi, y-x\rangle + |\langle d, y-x\rangle| - \frac{1}{2n}||y-x||.$$

Step 1. Let us prove the inclusion $\tilde{D}(f) \subset \bigcup_{\omega\in\Omega} \tilde{D}_\omega$, where $\Omega = \{\omega \in (\mathbb{N}^\star)^4 : \tilde{D}_\omega \neq \emptyset\}$.

Let $x \in \tilde{D}(f)$; there exist two different elements ξ_1, ξ_2 belonging to $\partial^- f(x)$. We can write $\xi_1 = \xi + d$ and $\xi_2 = \xi - d$, with $d \neq 0$; let us choose an integer $n \geq 1$ such that $f(x) \leq n$, $||x|| \leq n$ and $||d|| \geq \frac{9}{n}$. On one hand, since the sequence $(v_p)_{p\geq 1}$ is dense, we can find two integers $p \geq 1$ and $q \geq 1$ such that $v_p \in \overline{B}\left(d, \frac{1}{n}\right)$ and $v_q \in B\left(\xi, \frac{1}{2n}\right)$. On the other hand, from the characterization of lower Dini subdifferential, there exists an integer $\ell \geq 1$ such that, for $\epsilon \in \{1, -1\}$,

$$\forall\ y \in B\left(0, \frac{1}{\ell}\right),\ f(y) \geq f(x) + \langle \xi, y-x\rangle + \epsilon\langle d, y-x\rangle - \frac{1}{2n}\ ||y-x||.$$

And so, the asserted inclusion holds.

Step 2. Let us fix $\omega = (n,p,q,\ell) \in \Omega$ and let us prove that, for all $x \in \tilde{D}_\omega$,

$$\left(\{x\} + [B(0, \frac{1}{\ell}) \cap K(v_p)]\right) \setminus \{x\} \subset \mathbb{R}^h \setminus \tilde{D}_\omega,$$

where $K(v_p) := \{k \in \mathbb{R}^h : \langle k, v_p\rangle \geq \frac{1}{2}\ ||k||\ ||v_p||\}$ denotes the closed convex cone of vectors k such that the (radian) measure of the angle between k and v_p is less than $\arccos\frac{1}{2}$ (remark that $v_p \neq 0$).

Let y be an element of $\tilde{D}_\omega$ such that $||y-x|| < \frac{1}{\ell}$ and

$$\langle y-x, v_p\rangle \geq \frac{1}{2}\ ||y-x||\ ||v_p||. \tag{4}$$

We must verify that $y = x$. On one hand, since $x \in \tilde{D}_\omega$, there exist $d_x \in \overline{B}(v_p, \frac{1}{n}) \cap \left[\mathbb{R}^h \setminus B\left(0, \frac{9}{n}\right)\right]$ and $\xi_x \in \overline{B}\left(v_q, \frac{1}{2n}\right)$ such that

$$f(y) \geq f(x) + \langle \xi_x, y-x\rangle + |\langle d_x, y-x\rangle| - \frac{1}{2n}||y-x||. \tag{5}$$

But, from relation (4) and the Cauchy-Schwarz inequality, we deduce the following chain of inequalities:

$$\begin{array}{rcl} |\langle d_x, y-x\rangle| & \geq & |\langle v_p, y-x\rangle| - |\langle d_x - v_p, y-x\rangle| \\ & \geq & \frac{1}{2}\ ||y-x||\ ||v_p|| - ||d_x - v_p||\ ||y-x||. \end{array}$$

Hence, remarking that $||d_x - v_p|| \leq \frac{1}{n}$ and $||v_p|| \geq ||d_x|| - ||v_p - d_x|| \geq \frac{8}{n}$, we deduce

$$|\langle d_x, y-x\rangle| \geq \frac{3}{n}\ ||y-x||. \tag{6}$$

On the other hand, since $y \in \tilde{D}_\omega$, there exists $\xi_y \in \overline{B}\left(v_q, \frac{1}{2n}\right)$ such that

$$f(x) \geq f(y) + \langle \xi_y, x-y\rangle - \frac{1}{2n}\,||x-y||. \tag{7}$$

Summing the inequalities (5), (6) and (7), we obtain

$$\begin{array}{rcl} 0 & \geq & \langle \xi_x - \xi_y, y-x\rangle + \frac{2}{n}||y-x|| \\ & \geq & \left(\frac{2}{n} - ||\xi_x - \xi_y||\right)||y-x|| \\ & \geq & \frac{1}{n}\,||y-x||, \end{array}$$

and then $y = x$.

Step 3. Let us now conclude. By a compactness argument, if $\omega \in \Omega$, there exists a sequence $(x^i_\omega)_{i\in\mathbb{N}} \subset \tilde{D}_\omega$ taking a finite number of values such that $\tilde{D}_\omega \subset \bigcup_{i\in\mathbb{N}} \overline{B}\left(x^i_\omega, \frac{1}{2\ell}\right)$. Then, let us consider, for $i \in \mathbb{N}$, the compact sets $\tilde{D}^i_\omega = \tilde{D}_\omega \cap \overline{B}\left(x^i_\omega, \frac{1}{2\ell}\right)$ which satisfy the equality $\tilde{D}_\omega = \bigcup_{i\in\mathbb{N}} \tilde{D}^i_\omega$.

From step 1, it follows that $\tilde{D}(f) \subset \bigcup_{(\omega,i)\in\Omega\times\mathbb{N}} \tilde{D}^i_\omega$ and so it is sufficient to show that, for all $(\omega, i) \in \Omega \times \mathbb{N}$, $\tilde{D}^i_\omega$ is included in a Lipschitzian set. Setting $Y = \tilde{D}^i_\omega - K(v_p)$, Y is a proper closed subset of $\mathbb{R}^\ell$ and $Y - K(v_p) \subset Y$. The fact that the boundary of Y, denoted ∂Y, is Lipschitzian (see Definition 2.2) is a straighforward consequence of the following lemma which gives a standard criterion for a set to be Lipschitzian; see for example [3].

Lemma 3.4. *Let Y be a nonempty closed subset of $\mathbb{R}^h$ such that $Y \neq \mathbb{R}^h$ and $Y - K \subset Y$, where K is a closed convex cone of $\mathbb{R}^h$ with a nonempty interior. Let e_h be a unit vector belonging to the interior of K and $(e_1, \cdots, e_{h-1})$ an orthonormal set of the vector subspace $e_h^\perp = \{x \in \mathbb{R}^h : \langle x, e_h\rangle = 0\}$. Then $g : \mathbb{R}^{h-1} \to \mathbb{R}$ defined by $(\lambda_1, \cdots, \lambda_{h-1}) \mapsto \sup\{t \in \mathbb{R} \ : \ \sum_{i=1}^{h-1} \lambda_i e_i + te_h \in Y\}$ is a Lipschitzian function and $\partial Y = \phi(Gr(g))$, where ϕ denotes the isomorphism from $\mathbb{R}^h$ to $\mathbb{R}^h$ defined by $(\lambda_1, \cdots, \lambda_n) \mapsto \sum_{i=1}^h \lambda_i e_i$.*

Our last task is to prove the inclusion $\tilde{D}^i_\omega \subset \partial Y$. If it is not true, there exist $b \in \tilde{D}^i_\omega$ and $\eta > 0$ such that $\overline{B}(b, \eta) \subset Y$. In particular $b + k$ (with $k = \eta v_p/||v_p||$) can be written as $b' - k'$, where $b' \in \tilde{D}^i_\omega$ and $k' \in K(v_p)$. Since $K(v_p)$ is a convex cone, $k + k' \in K(v_p)$. From equality $k + k' = b' - b$, we also have $k + k' \in \overline{B}(0, \frac{1}{\ell})$. Hence $b' = b + (k + k') \in \{b\} + [\overline{B}(0, \frac{1}{\ell}) \cap K(v_p)]$ and, from step 2, $b' = b$. We conclude that $0 \neq k = -k'$, which contradicts the fact that $K(v_p)$ is a pointed cone. □

Remark 3.5. In the same manner, the definition of the sets $D(f)$ and $\tilde{D}(f)$ can be adapted to any subdifferential included in Dini's one. In particular we can apply these results to the proximal subdifferential (see [2], [4], [5], and [6]).

References

[1] J.-P. Aubin and H. Frankowska, "Set-Valued Analysis", Birkhäuser, Basel, 1990. MR **91d:**49001

[2] J. Benoist, Intégration du sous-différentiel proximal: un contre exemple, Can. J. Math., 50 (2), 1988, p. 242-265. CMP 98:13

[3] J.M. Bonnisseau and B. Cornet, Existence of equilibria when firms follow bounded losses pricing rule, *Journal of Mathematical Economics*, 17, 1988, p. 119-147. MR **91b:**90051

[4] F.H. Clarke and R.M. Redheffer, The proximal subgradient and constancy, *Canad. Math. Bull.*, 36, 1993, p. 30-32. MR **93m:**28004

[5] F.H. Clarke, R.J. Stern and P.R. Wolenski, Subgradient criteria for monotonicity, the Lipschitz condition, and convexity, *Can. J. Math.*, 45, n^0 6, 1993, p. 1167-1183. MR **94j:**49018

[6] R.T. Rockafellar, Proximal subgradients, marginal values, and augmented lagrangians in nonconvex optimization, *Math. Op. Res.*, 6, 1981, p. 424-436. MR **83m:**90088

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