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ON THE DIMENSION OF A HOMEOMORPHISM GROUP

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Abstract. We prove that the homeomorphism group of each one of a collection of continua constructed in a paper by the first author (Trans. Amer. Math. Soc. **121** (1966), 516–548) is one dimensional. This answers a question posed in that paper.

1. Introduction

For a compact metric space X, $G(X)$ denotes the space of all autohomeomorphisms on X with the compact-open topology. It is a separable completely metrizable topological group. When X is a topological manifold, the space $G(X)$ is infinite dimensional, while there exists a compact metrizable space Y with $G(Y) = \{id_Y\}$. Also it is known that the homeomorphism groups of the Sierpinski curve, as well as the universal Menger compactum μ^n ([1], [2]), are totally disconnected and exactly one-dimensional. (The proof of total disconnectivity for μ^n is similar to the proof for μ^1 in [3]. The proof of nonzero dimensionality is the same as the proof for μ^1 in [3], using Theorem 3.2.2 of Bestvina [2] which says that each μ^n, $n \geq 1$, is strongly locally homogeneous.)

In an attempt to construct a compact connected metric space which has a homeomorphism group of finite, but at least two dimensions, the first author constructed in [3] compact connected metric spaces M_n, for all $n \geq 1$, with the following property: there exist compact metric spaces $L_1, L_2, \ldots, L_n, \ldots$ such that

1. $G(M_n)$ is topologically isomorphic to the product $\prod_{i=1}^{n} G(L_i)$.
2. Each $G(L_i)$ is totally disconnected and one-dimensional. Hence $G(M_n)$ is at most n-dimensional.

She asked whether the dimension of $G(M_n)$ is indeed n or not. The purpose of this note is to prove that $\dim G(M_n) = 1$. Throughout, a *continuum* means a compact connected metric space.

2. Background

2.1. **The continua M_n.** Each compact metric space L_i above is obtained by a single type of construction which is briefly described below. (A *dendroid* is a nondegenerate continuum X such that each pair of distinct points $x, y \in X$ is

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contained in a unique continuum $[x, y]$ which is minimal with respect to containing x, y. $[x, y]$ must be an arc, when X is metric.)

There exists an infinite collection $\{D_j\}_{j=1}^{\infty}$ of locally connected dendroids such that $G(D_j) = \{id\}$. Take an increasing sequence $\{p_j\}_{j=1}^{\infty}$ of primes greater than two. For a fixed j, dendroid D_j, and prime p_j, the construction below produces the compactum, $K = L_j$.

Fix j. Take two distinct end points a_j and b_j of D_j and take a continuous surjection $f_j : D_j \to [0, 1]$ such that $f_j^{-1}(0) = a_j$ and $f_j^{-1}(1) = b_j$. Let E_j be the continuum obtained from D_j by identifying a_j and b_j, let ϕ_j be this identification map, and let $g_j : E_j \to S_j$ be the induced map from E_j onto a simple closed curve S_j. Let $v_j = g_j(\phi_j(a_j)) = g_j(\phi_j(b_j))$.

Let $\pi_{j_i} : S_{j_i} \to S_j$ be the $(p_j)^i$-fold covering of the simple closed curve to itself. Let K_{j_i} be the fibre product of the maps π_{j_i} and g_j and let $h_{j_i} : K_{j_i} \to S_{j_i}$ be the projection of K_{j_i} onto the $(p_j)^i$-fold cover of S_j. The points $h_{j_i}^{-1}\pi_{j_i}^{-1}(v_j)$ are called the *vertices* of K_{j_i}. The sequence $\{S_{j_i}\}_{i=1}^{\infty}$ can be identified with a sequence of concentric circles centered at the origin o of the plane converging to the unit circle T centered at o.

It is easy to see that $G(K_{j_i}) = \mathbf{Z}_{(p_j)^i}$, the cyclic group of order p_j^i, generated by a rotation r_{j_i}. The topological sum $\bigoplus K_{j_i}$ is mapped onto the union $\bigcup_{i=1}^{\infty} S_{j_i}$ by $\bigoplus h_{j_i}$ and hence compactified by adding the unit circle T in the obvious way. The resulting compactum is denoted by K. The compactum K is realized as a subset of the plane and what is important to us is the following property:

(*) For each homeomorphism $h \in G(K)$ and for each i, the restriction $h|K_{j_i}$ is a plane rotation $r_{j_i}^k$ (= k-fold iteration of r_{j_i}) for some k.

Roughly speaking, for a fixed prime p, one may think of a sequence of concentric, locally connected, "circular dendroids", $\{D_i\}_{i=1}^{\infty}$, the ith one D_i being the union of p^i dendroids, and admitting exactly p^i homeomorphisms, and constructed so that this sequence converges to the unit circle $T = T_1$. Then for $n = 1$, the compactum is

$$K = K_1 = T_1 \cup \bigcup_{i=1}^{\infty} D_i.$$

For $n \geq 2$, one may obtain compacta K_n by generalizing this construction, using limit tori T_n, instead of the circle $T = T_1$, and using primes $p_1, \ldots, p_n$ in the n product directions, respectively. For $n = 2$, it looks like a sequence of barbed wire grids converging to a limit torus. The continua M_n are obtained from the K_n's by taking a cone over the vertices and limit tori, but replacing the free arcs of the cone by locally connected dendroids, each admitting exactly one homeomorphism. Then $G(M_n) \cong G(K_n)$, both topologically and algebraically.

For more detail, see [3], p. 540. In the following, we let d denote the restriction of the standard metric of the plane to K.

2.2. **Almost zero dimensional spaces.** The key notion for our argument is the almost zero dimensionality [6]. A separable metrizable space X is said to be *almost zero dimensional* if it admits a countable base $\mathcal{B}$ with the following property:

(**) For each pair $B_1, B_2 \in \mathcal{B}$ with $cl(B_1) \cap cl(B_2) = \emptyset$, there exists a clopen set G such that $B_1 \subset G$ and $B_2 \subset X \setminus G$.

Theorem 2.1 ([6], [5])**.** *Each almost zero-dimensional separable metrizable space is at most one-dimensional.*

3. Main theorem

The following lemma is immediate from the definition of almost zero dimensional.

Lemma 3.1. *Let $\{X_i \mid i = 1, 2, 3, \dots\}$ be a countable collection of almost zero dimensional separable metrizable spaces. Then the product $\prod_{i=1}^{\infty} X_i$ is almost zero dimensional.*

It follows from the above that, for the proof of main theorem, it suffices to prove the following.

Theorem 3.2. *Let K be the compact metrizable space constructed in Section 2. Then the space $G(K)$ is almost zero dimensional.*

Proof. The sup metric on $G(K)$ induced by the metric d is also denoted by d for simplicity. The closed ε-ball at $f \in G(K)$ is denoted by $\hat{B}(f, \varepsilon)$. By Theorem 4 of [6], it suffices to prove the following claim:

for each $f \notin \hat{B}(id, \varepsilon)$, there exists a clopen set $V \subset G(K) \setminus \hat{B}(id, \varepsilon)$ such that $f \in V$.

Suppose that $d(f, id) = \varepsilon + \delta$ for $\delta > 0$. There exists a positive integer i such that $d(f|K_{j_i}, id) > \varepsilon$. Since $f|K_{j_i}$ is a planar rotation (see the property (*)), there exists a vertex $x \in K_{j_i}$ such that $d(f(x), x) > \varepsilon$. Also by the property (*), $f|K_{j_i} = r_{j_i}^k$ for some k.

Let $i_0 = min\{|k| \,|\, d(r_{j_i}^k, id) > \varepsilon\}$ and define $U = \{h \in G(K) \,|\, h|K_{j_i} = r_{j_i}^k$ for some k, with $|k| \leq i_0 - 1\}$ and $V = \{h \in G(K) \,|\, h|K_{j_i} = r_{j_i}^k$ for some k, with $|k| \geq i_0\}$. It is easy to see that U and V are disjoint clopen sets such that $G(K) = U \cup V$. Further $\hat{B}(id, \varepsilon) \subset U$ and $f \in V$. This completes the proof.

Our main theorem now can be stated as follows:

Theorem 3.3. *The space $G(M_n)$ is almost zero dimensional and hence* $\dim G(M_n) = 1$.

The proof follows immediately from Theorem 3.2 above.

4. Open problem

The problem raised in [3], namely whether there exists a compact metric space, or metric continuum, K, such that $1 < \dim G(K) < \infty$, remains open.

We note that in [4], Keesling and Wilson construct examples, X_n, of $(n-1)$ dimensional, connected, locally connected subgroups of R^n, whose homeomorphism groups are also of dimension $(n-1)$. Thus, they establish the existence of spaces whose homeomorphism groups are of dimension n, for all $n \geq 1$. However, the spaces are neither compact nor even locally compact.

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