PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 2, Pages 539–542
S 0002-9939(00)05592-1
Article electronically published on September 20, 2000

THE RIESZ DECOMPOSITION PROPERTY FOR THE SPACE OF REGULAR OPERATORS

NICOLAE DĂNEŢ

(Communicated by David R. Larson)

Dedicated to Prof. Romulus Cristescu on his 70th birthday

ABSTRACT. If E and F are Banach lattices such that E is separable and F has the countable interpolation property, then the space of all continuous regular operators $\mathcal{L}^r(E,F)$ has the Riesz decomposition property. This result is a positive answer to a conjecture posed by A. W. Wickstead.

1. Introduction

It is well known that the space of all regular operators $L^r(E,F)$ between two Riesz spaces E and F is a *Riesz space* if the range space F is Dedekind complete. (Actually, the space $L^r(E,F)$ is a Dedekind complete Riesz space ([4], Th.1.3.2).) If F is not a Dedekind complete Riesz space, there exists little knowledge about the order structure of the space $L^r(E,F)$. In [1] Y. A. Abramovich and A. W. Wickstead showed that the space $L^r(l_0^\infty, l_0^\infty)$ fails to have the Riesz decomposition property. (l_0^∞ is the space of all real sequences which are constant except for a finite number of terms and it is the smallest possible non-Dedekind complete Riesz space.) In [6] A. W. Wickstead began the study of the Riesz decomposition property in the space of continuous regular operators between Banach lattices. He proved that the space $\mathcal{L}^r(c,F)$ has the Riesz decomposition property if and only if the Banach lattice F has the countable interpolation property. He showed also that it is possible for such a space to have the Riesz decomposition property without being a lattice, and that not all spaces of regular operators have the Riesz decomposition property.

In this paper we give a positive answer to the following conjecture posed by A. W. Wickstead in [6]: the space of all continuous regular operators $\mathcal{L}^r(E,F)$ between the Banach lattices E and F has the Riesz decomposition property whenever E is separable and F has the countable interpolation property.

For the unexplained terms in the theory of Banach lattices and positive operators we refer to [4].

2. Some definitions and a preliminary result

We say that the ordered vector space E has the *Riesz decomposition property* if there exist $x_1, x_2 \in E$ such that $0 \le x_1 \le z_1, 0 \le x_2 \le z_2$ and $x = x_1 + x_2$ for

Received by the editors December 17, 1998 and, in revised form, May 4, 1999.
1991 *Mathematics Subject Classification.* Primary 47B60; Secondary 47B65, 46B42, 47L05.
Key words and phrases. Banach lattices, regular operators, Riesz decomposition property.

$x, z_1, z_2 \in E_+$ and $0 \leq x \leq z_1+z_2$. Every Riesz space E has the Riesz decomposition property.

We recall that an Archimedean Riesz space F has the *countable interpolation property* (or Cantor property) if for all sequences (x_n) and (z_n) in F such that $x_n \leq z_m$ for all $n, m \in \mathbb{N}$ there exists $y \in F$ such that $x_n \leq y \leq z_m$ for all $n, m \in \mathbb{N}$ (equivalently, if, whenever $x_n \uparrow, z_n \downarrow$ and $x_n \leq z_m$ for all $n, m \in \mathbb{N}$, there exists $y \in F$ such that $x_n \leq y \leq z_n$ for all $n \in \mathbb{N}$). C. B. Huijsmans and B. de Pagter have shown that an Archimedean Riesz space F has the countable interpolation property if and only if F is uniformly complete and $F = \{x^+\}^\perp + \{x^-\}^\perp$ for all $x \in F$ ([3], Th. 9.15). If X is a completely regular Hausdorff space, then the space $C(X)$ has countable interpolation property if and only if X is an F-space, i.e., any pair of disjoint open F_σ subsets of X have disjoint closures ([5]). For other equivalent conditions with the countable interpolation property for $C(X)$ see [3], Th. 10.5.

Using the countable interpolation property, Y. A. Abramovich and A. W. Wickstead proved in [2] the following new version of the Hahn-Banach-Kantorovich extension theorem.

Theorem 2.1 ([2], Th. 3.5)**.** *Let E and F be Banach lattices, such that E is separable and F has the countable interpolation property, and let $P : E \to F$ be a continuous sublinear operator. If G is a vector subspace of E and $T : G \to F$ is a continuous linear operator satisfying $T(x) \leq P(x)$ for all $x \in G$, then there exists an extension $\hat{T}$ of T to all of E also satisfying $\hat{T}(x) \leq P(x)$ for all $x \in E$.*

We mention that by a *sublinear* operator we understand an operator $P : E \to F$ that is subadditive and positively homogeneous, i.e., $P(x_1 + x_2) \leq P(x_1) + P(x_2)$ and $P(\alpha x) = \alpha P(x)$ for all $x_1, x_2, x \in E$ and $\alpha \geq 0$. In [2] the authors use the name of sublinear operator for an operator that is subadditive and $P(\alpha x) =\mid \alpha \mid P(x)$ for all $x \in E$ and $\alpha \in \mathbb{R}$. But their proof works for a sublinear operator too.

Let E and F be two Archimedean Riesz spaces. A linear operator $T : E \to F$ is called positive if $T(x) \geq 0$ for all $x \in E_+$. We will denote this by $T \geq 0$. T is called *regular* if T is the difference of two positive linear operators. We denote by $L^r(E, F)$ the space of all regular operators. If E is a Banach lattice and F a normed Riesz space, then every positive linear operator is continuous ([4], Proposition 1.3.5). In this situation we have $L^r(E, F) = \mathcal{L}^r(E, F)$, where $\mathcal{L}^r(E, F)$ denotes the space of all continuous regular operators.

3. The main theorem

The goal of this section is to demonstrate the following theorem.

Theorem 3.1. *Let E and F be two Banach lattices such that E is separable and F has the countable interpolation property. Then the space of all continuous regular operators $\mathcal{L}^r(E, F)$ has the Riesz decomposition property.*

Proof. The proof follows the idea of demonstration of Corollary 1.5.6 from [4] and uses the extension Theorem 2.1. It is quite different from Wickstead's proof for the particular case $E = c$.

Let T, S_1, S_2 be three positive linear operators in $\mathcal{L}^r(E, F)$ such that

$$T \leq S_1 + S_2.$$

We have to prove that there exist two linear operators $T_1, T_2 \in \mathcal{L}^r(E,F)$ with the properties

$$\begin{aligned} 0 &\le T_i \le S_i, \quad i = 1,2, \\ T &= T_1 + T_2. \end{aligned}$$

Consider the Banach lattice $E \times E$ with the canonical order and norm. Define $P : E \times E \to F$ by

$$P(x_1, x_2) = S_1(x_1^+) + S_2(x_2^+), \qquad (x_1, x_2) \in E \times E.$$

P is a sublinear operator. Since the lattice operations are continuous and S_1, S_2 are continuous operators, it follows that P is continuous too.

Consider now the subspace $G = \{(x,x) \mid x \in E\}$ of $E \times E$ and define on G the operator $T_0 : G \to F$ by

$$T_0(x,x) = T(x), \qquad x \in E.$$

T_0 is a continuous linear operator and

$$\begin{aligned} T_0(x,x) &= T(x) = T(x^+) - T(x^-) \le T(x^+) \\ &\le S_1(x^+) + S_2(x^+) = P(x,x) \end{aligned}$$

on G. By Theorem 2.1 there exists an extension $\hat{T}$ of T_0 to all $E \times E$ which satisfies the inequality

$$\hat{T}(x_1, x_2) \le P(x_1, x_2), \qquad (x_1, x_2) \in E \times E.$$

Define

$$\begin{aligned} T_1(x) &= \hat{T}(x, 0), \qquad T_1 : E \to F, \\ T_2(x) &= \hat{T}(0, x), \qquad T_2 : E \to F. \end{aligned}$$

T_1, T_2 are the desired operators. Indeed, for every $x \in E$ we have

$$\begin{aligned} T_1(x) &= \hat{T}(x,0) \le P(x,0) = S_1(x^+), \\ -T_1(x) &= T_1(-x) = \hat{T}(-x,0) \le P(-x,0) = S_1((-x)^+) = S_1(x^-), \end{aligned}$$

from which it follows that

$$-S_1(x^-) \le T_1(x) \le S_1(x^+), \qquad x \in E.$$

If $x \ge 0$, then $x^+ = x$ and $x^- = 0$. Therefore, we obtain

$$0 \le T_1(x) \le S_1(x), \qquad x \in E_+.$$

Similarly $0 \le S_2 \le T_2$. Finally, we have

$$T(x) = T_0(x,x) = \hat{T}(x,x) = \hat{T}(x,0) + \hat{T}(0,x) = T_1(x) + T_2(x)$$

for all $x \in E$. The proof is complete. □

Finally, we can give the following characterization of a Banach lattice F which has the countable interpolation property. The implication (iii) $\Rightarrow$ (i) is proved in [6], Theorem 3.1.

Corollary 3.2. *The following conditions on a Banach lattice F are equivalent:*

(i) *F has the countable interpolation property;*

(ii) *For every separable Banach lattice E the space $\mathcal{L}^r(E,F)$ has the Riesz decomposition property;*

(iii) *$\mathcal{L}^r(c,F)$ has the Riesz decomposition property.*

References

[1] Y. A. Abramovich and A. W. Wickstead, *Regular operators from and into a small Riesz space*, Indag. Mathem., N.S. **2** (1991), no.3, 257-274. MR **93i:**47052

[2] Y. A. Abramovich and A. W. Wickstead, *The regularity of order bounded operators into C(K)*, II, Quart. J. Math. Oxford (2), **44** (1993), 257-270. MR **94h:**47067

[3] C. B. Huijsmans and B. de Pagter, *On z-ideals and d-ideals in Riesz spaces*, II, Indag. Mathem. **42** (1980), no.4, 391-408. MR **83c:**46004a

[4] Peter Meyer-Nieberg, *Banach Lattices,* Springer-Verlag, Berlin Heidelberg New York, 1991. MR **93f:**46025

[5] G. L. Seever, *Measures on F-spaces*, Trans. Amer. Math. Soc.**133** (1968), 267-280. MR **37:**1976

[6] A. W. Wickstead, *Spaces of operators with the Riesz separation property*, Indag. Mathem., N.S. **6** (1995), no.2, 235-245. MR **96g:**47032

Technical University of Civil Engineering of Bucharest, 122-124, Lacul Tei Blvd., 72302 Bucharest, Romania

E-mail address: ndanet@fx.ro