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ABSOLUTELY CONTINUOUS MEASURES ON NON QUASI-ANALYTIC CURVES WITH INDEPENDENT POWERS

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Abstract. We prove that every non quasi-analytic Carleman class contains functions whose graph supports measures that are absolutely continuous with respect to arc length measure and yet they have independent convolution powers in the measure algebra $M(\mathbb{R}^2)$. The proof relies on conditions which ensure that the canonical map between two Cantor sets can be extended to a function in an arbitrary prescribed non quasi-analytic Carleman class.

1. Introduction

Our main technical object with this note is to prove certain sufficient conditions on pairs of Cantor sets K_0 and K to ensure that the so-called canonical map $\psi : \mathcal{B}_{K_0} \to \mathcal{B}_K$ (see Definition 1 in Section 3) can be extended to a function in $C\{M_j\}$, where $C\{M_j\}$ is an arbitrary prescribed non quasi-analytic Carleman class of C^∞-functions (see below). In Section 3 our extension result for Cantor set maps is given in three different forms in Theorem 3, Corollary 1 and Theorem 4. The content of those results is that extension is possible if the image Cantor set K is sufficiently small relative to the domain Cantor set K_0. In some sense these extension results are rather sharp (see Remark 3 in Section 3). Our proofs of those results rely on the methodology used to to prove Whitney's extension theorem (see Stein [10], Chapter VI).

The motivation for this work comes from Björk [1], which was inspired by [2] and [11]. Let A be a closed subalgebra (with unit) of the measure algebra $M(\mathbb{R}^n)$. The Fourier transform allows us to consider $\mathbb{R}^n$ as a subset of the maximal ideal space of A. A is called a Wiener algebra if $\mathbb{R}^n$ is dense in the maximal ideal space of A; this is known to imply that every measure in A whose Fourier transform is bounded away from zero is invertible in A. It is proved in [1] that if A is the closed subalgebra of $M(\mathbb{R}^n)$ generated by all Dirac measures and all measures that are absolutely continuous with respect to Lebesgue surface area measure on some real analytic submanifold of $\mathbb{R}^n$, then A is a Wiener algebra. See also Remark 8

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in Section 4. In contrast to this we shall prove the following Wiener-Pitt type theorem:

Theorem 1. *Let $C\{M_j\}$ be non quasi-analytic. Then there exists a curve*

$$\Gamma = \left\{(x,y) \in \mathbb{R}^2 : \ y = f(x)\right\},$$

where $f \in C\{M_j\}$, and a compactly supported measure $\nu \in M(\mathbb{R}^2)$ absolutely continuous with respect to arc length measure on Γ such that ν has independent powers (Remark 6: Section 4) and is not an element in some Wiener subalgebra A of $M(\mathbb{R}^2)$.

This result sharpens the analogous result in [1] stating the same as Theorem 1 except that the curve Γ was only required to be of the class C^∞ there. The proof of Theorem 1 is given in Section 4. In the construction used to prove Theorem 1 our extension result Theorem 4 is used in an essential way.

Quasi-analytic classes. Let $\{M_j\}_{j=0}^\infty$ be a sequence of positive numbers. We denote by $C\{M_j\}$ the Carleman class consisting of all C^∞-functions f on $\mathbb{R}$ satisfying the estimates

$$|f^{(j)}(x)| \leq \alpha A^j M_j, \quad x \in \mathbb{R}, \ j \geq 0,$$

for some constants $0 < \alpha, A < \infty$. It is known that every such class $C\{M_j\}$ can be defined by means of a logarithmically convex sequence. In fact, $C\{M_j\} = C\{\underline{M}_j\}$, where $\{\underline{M}_j\}_{j=0}^\infty$ is the largest logarithmically convex minorant sequence of $\{M_j\}_{j=0}^\infty$ (see [6], Chapter V, Section E). Using logarithmic convexity of $\{M_j\}_{j=0}^\infty$ one proves that $C\{M_j\}$ is an algebra of functions (see [6], Chapter IV, Section E).

The class $C\{M_j\}$ is said to be quasi-analytic if the zero function $f \equiv 0$ is the only function f in $C\{M_j\}$ satisfying $f^{(j)}(0) = 0$ for all $j \geq 0$. The celebrated theorem of Denjoy-Carleman (see [6], Chapter IV, Section B) states that $C\{M_j\}$ is quasi-analytic if and only if

$$\sum_{j=1}^\infty \frac{\underline{M}_{j-1}}{\underline{M}_j} = \infty.$$

A closely related result which will also be referred to as the theorem of Denjoy-Carleman is that

$$\sum_{j=1}^\infty \frac{M_{j-1}}{M_j} < \infty$$

implies that $C\{M_j\}$ is non quasi-analytic. In fact, this implication follows from the well-known construction of test functions by means of repeated convolutions of normalized characteristic functions (see [4], Theorem 1.3.2).

2. Whitney extension of a flat map

The aim with this section is to prove a variant of Whitney's extension theorem (see [10], Chapter VI, Section 2 or [4], Theorem 2.3.6), Theorem 2 below. The novelty of Theorem 2 is that it is formulated in terms of flatness (see assumption (3) below) and Carleman classes. See also Remark 1 below.

Lemma 1. *Let $F \subset \mathbb{R}$, $F \neq \mathbb{R}$, be a non-empty closed set and $C\{M_j\}$ a non quasi-analytic Carleman class. Then there exists a locally finite partition of unity $\{\varphi_k\}_{k=1}^\infty$ on $\mathbb{R} \setminus F$ such that:*

1) *Every point in* $\mathbb{R} \setminus F$ *has a neighborhood that intersects at most two of the* $\operatorname{supp}(\varphi_k)$*'s.*
2) *There exists a positive real constant* C *not depending on* k *such that:*
$$\operatorname{diam}(\operatorname{supp}(\varphi_k)) \le C \operatorname{dist}(\operatorname{supp}(\varphi_k), F).$$
3) *There exist constants* $0 < \alpha, A < \infty$ *such that*
$$|\varphi_k^{(j)}(x)| \le \alpha A^j M_j \operatorname{dist}(x, F)^{-j}, \quad x \in \mathbb{R}, \quad j = 0, 1, 2, \dots, \tag{1}$$
holds for all k*.*

We mention that partitions of unity similar to that in Lemma 1 are constructed in much more generality (arbitrary finite dimensional vector space) by Hörmander in [4] (see Theorem 1.4.10). For the sake of completeness we include a sketch of the proof of Lemma 1.

Sketch of the proof of Lemma 1. Write $\mathbb{R} \setminus F = \bigcup_{k=1}^{\infty} I_k$, where the I_k's are closed intervals whose interiors are pairwise disjoint and such that
$$c_1 \operatorname{diam}(I_k) \le \operatorname{dist}(I_k, F) \le c_2 \operatorname{diam}(I_k), \quad k = 1, 2, \dots,$$
for some constants c_1 and c_2. Let I_k^* be the interval with the same center as I_k but expanded by a factor $1 + \varepsilon$ for some $\varepsilon > 0$. We now choose $\varepsilon > 0$ sufficiently small to ensure that I_k^* and I_j^* intersect only if the intervals I_k and I_j have a common endpoint.

Let $I = [-1, 1]$ and $I^* = [-(1 + \varepsilon), 1 + \varepsilon]$. Let $\psi \in C\{M_j\}$ be such that $\psi = 1$ on I, $\operatorname{supp}(\psi) \subset I^*$ and $0 \le \psi \le 1$. (Such a test function ψ can be constructed by convolving a non-negative compactly supported function in $C\{M_j\}$ with integral $= 1$ and small support with the characteristic function for an interval.) We now transfer ψ to I_k^* by defining $\psi_k(x) = \psi((x - a_k)/s_k)$, where $I_k = [a_k - s_k, a_k + s_k]$. We now define the φ_k's by
$$\varphi_1(x) = \psi_1(x), \quad \varphi_k(x) = \psi_k(x) \prod_{j=1}^{k-1} (1 - \psi_j(x)), \quad k \ge 2.$$

One checks that $\{\varphi_k\}$ is a partition of unity on $\mathbb{R} \setminus F$. 1) and 2) hold by the construction of the I_k and I_k^*'s. By the construction there are constants β, B such that
$$|\psi_k^{(j)}(x)| \le \beta B^j M_j \operatorname{dist}(x, F)^{-j}, \quad x \in \mathbb{R}, \quad j = 0, 1, \dots, \tag{2}$$
holds for all $k \ge 1$. Since locally φ_k is a product of at most two functions satisfying (2) it follows from the Leibniz formula for the derivative of a product that (1) holds. □

Lemma 2. *Let* F *and* $\{\varphi_k\}$ *be as in Lemma 1. For every* k*, let* p_k *be a point in* F *closest to* $\operatorname{supp}(\varphi_k)$*. Then*
$$|p_k - y| \le (2 + C)|x - y|$$
for all $x \in \operatorname{supp}(\varphi_k)$ *and* $y \in F$*, where* C *is as in Lemma 1.*

Proof. Choose $x^* \in \operatorname{supp}(\varphi_k)$ such that $|x^* - p_k| = \operatorname{dist}(\operatorname{supp}(\varphi_k), F)$. By part 2) of Lemma 1 we have
$$|p_k - y| \le |p_k - x^*| + |x^* - x| + |x - y| \le (2 + C)|x - y|.$$
□

Theorem 2. *Let $C\{M_j\}$ be non quasi-analytic. Let $F \subset \mathbb{R}$, $F \neq \mathbb{R}$, be a non-empty closed set and ψ a continuous function on F such that*

$$|\psi(x) - \psi(y)| \leq \Lambda_j |x-y|^j, \quad x, y \in F, \quad j = 1, 2, \ldots, \tag{3}$$

for some constants $0 < \Lambda_j < \infty$.

Then ψ has an extension to a C^∞-function Ψ on $\mathbb{R}$ satisfying the estimates

$$|\Psi^{(j)}(x)| \leq 2\alpha A^j (2+C)^j \Lambda_j M_j, \quad x \in \mathbb{R}, \quad j = 1, 2, \ldots, \tag{4}$$

where α, A and C are as in Lemma 1. If ψ is bounded, then $\Psi \in C\{\Lambda_j M_j\}$.

Proof. Choose $\{\varphi_k\}_{k=1}^\infty$ and $\{p_k\}_{k=1}^\infty$ as in the preceding lemmas. The extension Ψ of ψ is now defined by

$$\Psi(x) = \begin{cases} \sum_{k=1}^\infty \psi(p_k)\varphi_k(x), & x \notin F, \\ \psi(x), & x \in F. \end{cases}$$

It is clear that Ψ is C^∞ in $\mathbb{R} \setminus F$ and that $\Psi^{(j)}$ exists and is $= 0$ in the interior of F for all $j \geq 1$. For proving that Ψ is C^∞ it suffices to check that every $\Psi^{(j)}$ extends continuously to $\mathbb{R}$.

The case $j = 0$. It suffices to prove that $F \not\ni x \to y \in F$ implies $\Psi(x) \to \psi(y)$. We have that

$$\Psi(x) - \psi(y) = \sum_k (\psi(p_k) - \psi(y))\, \varphi_k(x).$$

By (3) and Lemma 2,

$$|\Psi(x) - \psi(y)| \leq \sum_k \Lambda_1 |p_k - y| \varphi_k(x) \leq (2+C)\Lambda_1 |x-y| \to 0$$

as $x \to y$.

The case $j > 0$. Assume $x \notin F$. Choose $y \in F$ as close to x as possible. Since $\sum_k \varphi_k^{(j)}(x) = 0$ we have that

$$\Psi^{(j)}(x) = \sum_k \psi(p_k)\varphi_k^{(j)}(x) = \sum_k (\psi(p_k) - \psi(y))\, \varphi_k^{(j)}(x). \tag{5}$$

By (3) and Lemma 2,

$$|\Psi^{(j)}(x)| \leq \sum_k \Lambda_{j+1} |p_k - y|^{j+1} |\varphi_k^{(j)}(x)| \leq \Lambda_{j+1}(2+C)^{j+1} |x-y|^{j+1} \sum_k |\varphi_k^{(j)}(x)|.$$

By (1) and part 1) of Lemma 1 we can estimate $\sum_k |\varphi_k^{(j)}(x)|$ by $2\alpha A^j M_j |x-y|^{-j}$. We conclude that

$$|\Psi^{(j)}(x)| \leq 2\alpha A^j (2+C)^{j+1} \Lambda_{j+1} M_j |x-y| \to 0$$

as $x \to F$.

We now prove (4). Assume $x \notin F$. Choose $y \in F$ as close to x as possible. By (5) we have

$$|\Psi^{(j)}(x)| \leq \sum_k \Lambda_j |p_k - y|^j |\varphi_k^{(j)}(x)| \leq \Lambda_j (2+C)^j |x-y|^j \sum_k |\varphi_k^{(j)}(x)|.$$

Estimating $\sum_k |\varphi_k^{(j)}(x)|$ by $2\alpha A^j M_j |x-y|^{-j}$ we get (4). $\square$

Remark 1. The case of interest in Theorem 2 is when the set F is highly irregular. For instance, if $F = [0,1]$, then the assumption (3) for $j = 2$ implies that ψ is constant. In the proof of Theorem 3 in Section 3 we apply Theorem 2 to the case when F is a Cantor set and ψ is a so-called canonical map (Definition 1: Section 3).

3. Extension of maps between Cantor sets

A class of Cantor sets on $[0,1]$**.** For every $n \geq 1$ let there be given 2^{n-1} open intervals

$$\Delta_j^{(n)} = (\alpha_j^{(n)}, \beta_j^{(n)}), \quad j = 1, \dots, 2^{n-1},$$

such that

$$0 < \alpha_1^{(n)} < \beta_1^{(n)} < \alpha_2^{(n)} < \beta_2^{(n)} < \cdots < \alpha_{2^{n-1}}^{(n)} < \beta_{2^{n-1}}^{(n)} < 1.$$

Write $S^{(0)} = [0,1] = S_1^{(0)}$ and set

$$S^{(n)} = S^{(n-1)} \setminus \bigcup_{j=1}^{2^{n-1}} \Delta_j^{(n)} = \bigcup_{j=1}^{2^n} S_j^{(n)}, \quad n \geq 1,$$

where the $S_j^{(n)}$, $1 \leq j \leq 2^n$, are pairwise disjoint closed intervals. We demand further in the construction that the intervals $\Delta_j^{(n)}$ are such that

$$\Delta_j^{(n+1)} \subset S_j^{(n)}, \quad 1 \leq j \leq 2^n, \ n \geq 1.$$

The corresponding Cantor set is now defined by

$$K = \bigcap_{n=0}^{\infty} S^{(n)} = [0,1] \setminus \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{2^{n-1}} \Delta_j^{(n)}.$$

We define

$$\lambda_K(n) = \max_{1 \leq j \leq 2^n} |S_j^{(n)}| \quad (n \geq 0),$$

$$\delta_K(n) = \min_{1 \leq j \leq 2^{n-1}} |\Delta_j^{(n)}| \quad (n \geq 1)$$

and

$$\mathcal{B}_K = \left\{ 0, 1, \alpha_j^{(n)}, \beta_j^{(n)} : \ 1 \leq j \leq 2^{n-1}, \ n \geq 1 \right\}.$$

We call K an admissible Cantor set if K is constructed as above and $\lambda_K(n) \to 0$ as $n \to \infty$.

Remark 2. Note that $\mathcal{B}_K$ is dense in K if K is an admissible Cantor set. Let us also notice that there exist admissible Cantor sets with positive Lebesgue measure. For example when every $\Delta_j^{(n)}$, $1 \leq j \leq 2^{n-1}$, has length $1/4^n$ and the $S_j^{(n)}$, $1 \leq j \leq 2^n$, are of equal length, then the above construction gives an admissible Cantor set of measure $1/2$.

For easy reference we record the following lemma whose proof is trivial.

Lemma 3. *Let K be an admissible Cantor set on $[0,1]$. Let $x, y \in K$, $x \neq y$, and denote by $n(x,y)$ the largest integer $n \geq 0$ such that both $x \in S_j^{(n)}$ and $y \in S_j^{(n)}$ for some j. Then,*

$$\delta_K(n(x,y)+1) \leq |x-y| \leq \lambda_K(n(x,y)).$$

Definition 1. Let K_0 and K be admissible Cantor sets on $[0,1]$, where K is constructed as above and K_0 is obtained at step $n \geq 1$ by removing the intervals

$$(a_j^{(n)}, b_j^{(n)}), \quad j = 1, \dots, 2^{n-1}.$$

The canonical map $\psi : \mathcal{B}_{K_0} \to \mathcal{B}_K$ is defined by $\psi(0) = 0$, $\psi(1) = 1$, $\psi(a_j^{(n)}) = \alpha_j^{(n)}$, $\psi(b_j^{(n)}) = \beta_j^{(n)}$.

Note that ψ is strictly increasing and thereby preserves the construction.

Proposition 1. *Let K_0 and K be admissible Cantor sets on $[0,1]$ and let $\psi : \mathcal{B}_{K_0} \to \mathcal{B}_K$ be the corresponding canonical map (Definition 1). Then the following estimate holds:*

$$|\psi(x) - \psi(y)| \leq \Lambda_j |x-y|^j, \quad x, y \in \mathcal{B}_{K_0}, \quad j = 0, 1, 2, \dots, \tag{6}$$

where

$$\Lambda_j = \sup_{n \geq 0} \frac{\lambda_K(n)}{\delta_{K_0}(n+1)^j}. \tag{7}$$

Proof. Since ψ is strictly increasing, $n(\psi(x), \psi(y)) = n(x,y)$, where $n(\cdot,\cdot)$ is as in Lemma 3. For $x, y \in \mathcal{B}_{K_0}$, $x \neq y$, we have

$$|\psi(x) - \psi(y)| \leq \lambda_K(n(\psi(x), \psi(y))) = \frac{\lambda_K(n(x,y))}{|x-y|^j} |x-y|^j$$

$$\leq \frac{\lambda_K(n(x,y))}{\delta_{K_0}(n(x,y)+1)^j} |x-y|^j \leq \Lambda_j |x-y|^j.$$

□

We are mainly interested in the case when $\Lambda_j < \infty$ for all j. Note however that ψ extends uniquely to a Lipschitz continuous function $\psi : K_0 \to K$ if $\Lambda_1 < \infty$.

Remark 3. Let the notation be as in Proposition 1 and assume that $\Lambda_j < \infty$ for every j. If Ψ is a C^∞-extension of ψ, then $\Psi^{(j)} \equiv 0$ on K_0 for every $j \geq 1$. In particular, such an extension Ψ (if it exists) cannot be in some quasi-analytic class. In Theorem 4 below we show that if $\lambda_K(n) \to 0$ sufficiently fast, then there exists an extension $\Psi \in C\{M_j\}$ of ψ, where $C\{M_j\}$ is an arbitrary prescribed non quasi-analytic Carleman class.

Also, since $\Psi^{(j)}(0) = 0$ for all $j \geq 0$, Ψ can be glued together with any C^∞-function, all of whose derivatives vanish at 0 to form a C^∞-extension of ψ.

Applying Theorem 2 to the canonical map between two Cantor sets we obtain:

Theorem 3. *Let $C\{M_j\}$ be non quasi-analytic and let $\{\Lambda_j\}_{j=0}^\infty$ be a sequence of positive real numbers with $\lim_{j\to\infty} \Lambda_j^{1/j} = \infty$. Let K_0 and K be admissible Cantor sets on $[0,1]$ and $\psi : \mathcal{B}_{K_0} \to \mathcal{B}_K$ be the corresponding canonical map (Definition 1).*

If the set K is constructed in such a way that

$$\lambda_K(n) \leq \min_{j \geq 0} \delta_{K_0}(n+1)^j \Lambda_j \tag{8}$$

for all n sufficiently large, then ψ has an extension to a function Ψ in $C\{\Lambda_j M_j\}$.

Proof. By (8) and Proposition 1 we can uniquely extend ψ to a continuous function $\psi : K_0 \to K$ such that for some $C > 0$,

$$|\psi(x) - \psi(y)| \leq C^{j+1}\Lambda_j |x-y|^j, \quad j \geq 0,$$

for all $x, y \in K_0$. By Theorem 2, ψ has an extension to a function Ψ in the class $C\{C^{j+1}\Lambda_j M_j\} = C\{\Lambda_j M_j\}$. □

Remark 4. The assumption on $\{\Lambda_j\}_{j=0}^{\infty}$ that $\lim_{j\to\infty} \Lambda_j^{1/j} = \infty$ is needed to guarantee that the minima in (8) are attained.

Corollary 1. *Let $C\{M_j\}$ be non quasi-analytic. Let K_0, K and ψ be as in Theorem 3. Let*

$$\Lambda_j = \sup_{n\geq 0} \frac{\lambda_K(n)}{\delta_{K_0}(n+1)^j}.$$

If $C\{M_j/\Lambda_j\}$ is non quasi-analytic, then ψ has an extension to a function Ψ in $C\{M_j\}$.

Proof. Let $L_j = M_j/\Lambda_j$. By our choice of Λ_j,

$$\lambda_K(n) \leq \inf_{j\geq 0} \delta_{K_0}(n+1)^j \Lambda_j.$$

By Theorem 3 we see that ψ extends to a function in $C\{\Lambda_j L_j\} = C\{M_j\}$. □

The assumption in Corollary 1 that $C\{M_j/\Lambda_j\}$ is non quasi-analytic is, by the Denjoy-Carleman theorem, satisfied if

$$\sum_{j=1}^{\infty} \frac{\Lambda_j}{\Lambda_{j-1}} \frac{M_{j-1}}{M_j} < \infty.$$

This condition is fulfilled if $\lambda_K(n) \to 0$ sufficiently fast. We state this explicitly in the next theorem.

Theorem 4. *Let $C\{M_j\}$ be non quasi-analytic and let K_0 be an admissible Cantor set on $[0,1]$.*

If K is an admissible Cantor set on $[0,1]$ such that $\lambda_K(n) \to 0$ sufficiently fast (the rate of convergence depends only on $\{\delta_{K_0}(n)\}_{n=1}^{\infty}$ and $\{M_j\}_{j=0}^{\infty}$), then the canonical map $\psi : \mathcal{B}_{K_0} \to \mathcal{B}_K$ can be extended to a function Ψ in $C\{M_j\}$.

Proof. We may assume that $\{M_j\}_{j=0}^{\infty}$ is logarithmically convex. Since, by the theorem of Denjoy-Carleman, $\sum M_{j-1}/M_j < \infty$, we can choose $0 < \omega_j \to \infty$ such that

$$\sum_{j=1}^{\infty} \omega_j \frac{M_{j-1}}{M_j} < \infty. \tag{9}$$

Set $\Lambda_0 = 1$ and

$$\Lambda_j = \prod_{l=1}^{j} \omega_l, \quad j \geq 1.$$

By the theorem of Denjoy-Carleman (the non log-convex case, see the introductory remark on quasi-analytic classes), (9) implies that the class $C\{M_j/\Lambda_j\}$ is non quasi-analytic.

If K is such that

$$\lambda_K(n) \leq \min_{j\geq 0} \delta_{K_0}(n+1)^j \Lambda_j, \quad n \text{ large},$$

then by Theorem 3, ψ extends to a function in $C\{\Lambda_j M_j/\Lambda_j\} = C\{M_j\}$. □

4. Application to Wiener algebras

In this section we give the proof of Theorem 1 in the introduction. First we state some preliminary results.

Lemma 4. *There exists a Cantor set K on $[0,1]$ such that $K \setminus \{0\}$ is linearly independent over $\mathbb{Q}$ and $\lambda_K(n) \to 0$ faster than any prescribed quantity.*

Proof. By translating and dilating, this lemma is an immediate consequence of the construction of a linearly independent Cantor set (see [3], pages 187-189 or [5], pages 20-21). □

Lemma 5 (Theorem 5.3.2 in [8])**.** *Let G be a non-discrete locally compact abelian group and E an independent (see Remark 5) compact subset of G. If $\mu \in M(G)$ is a continuous measure concentrated on $E \cup (-E)$, then the measures $\delta_0, \mu, \mu^{*2}, \mu^{*3}, \ldots$ are mutually singular.*

Remark 5. A set $E \subset G$ is said to be independent if for every choice of distinct points $x_1, \ldots, x_k$ in E and $n_1, \ldots, n_k \in \mathbb{Z}$, $n_1x_1 + \cdots + n_kx_k = 0$ implies $n_1x_1 = \cdots = n_kx_k = 0$.

Remark 6. Recall that $\mu \in M(G)$ is said to have independent powers if the measures $\delta_0, \mu, \mu^{*2}, \ldots$ are mutually singular.

Below we use the notation $\sigma_A(x)$ for the spectrum of the element x in the Banach algebra A.

Lemma 6. *Let A be a closed subalgebra of a Banach algebra B. Let $x \in A$. Then the boundary of $\sigma_A(x)$ is contained in $\sigma_B(x)$.*

Proof. This result is well known and can be found as part of Theorem 10.18 in [9]. □

Proof of Theorem 1. Let $C\{M_j\}$ be non quasi-analytic. Let K_0 be an admissible Cantor set on $[0,1]$ with positive Lebesgue measure (see Remark 2 in Section 3). By Lemma 4 we can find an admissible Cantor set K on $[0,1]$ such that $K \setminus \{0\}$ is linearly independent over $\mathbb{Q}$ and $\lambda_K(n) \to 0$ sufficiently fast in the sense of Theorem 4. By Theorem 4, the canonical map $\psi : \mathcal{B}_{K_0} \to \mathcal{B}_K$ (Definition 1: Section 3) has an extension to a function Ψ in the class $C\{M_j\}$. Let $P = K_0 \times K$ and define

$$f(x) = \begin{cases} \Psi(x) & \text{if} \quad x \geq 0, \\ -\Psi(-x) & \text{if} \quad x \leq 0. \end{cases}$$

Then, by Remark 3, f is a function in $C\{M_j\}$. Let

$$\Gamma = \left\{(x,y) \in \mathbb{R}^2 : \ y = f(x), \ x \in \mathbb{R}\right\}$$

be the graph of f. Let σ denote the arc length measure on Γ. Define the measure ν by

$$\nu(E) = \sigma\left(E \cap (P \cup (-P))\right)$$

for every Borel set E in $\mathbb{R}^2$. It is obvious that ν is absolutely continuous with respect to σ. The proof is now completed by the following theorem:

Theorem 5. *In the above situation, ν has independent powers (Remark 6: Section 4) and is not an element in some Wiener subalgebra A of $M(\mathbb{R}^2)$.*

Proof. By Lemma 5 with $G = \mathbb{R}^2$ and $E = P \cap \Gamma = \{(x, f(x)) : \ x \in K_0\}$, ν has independent powers. Using Remark 3 a computation shows that the action of ν on test functions is given by

$$\int \varphi d\nu = \int_{K_0 \cup (-K_0)} \varphi(x, f(x)) dx. \tag{10}$$

Since K_0 has positive Lebesgue measure, $\nu \neq 0$. Assume for reaching a contradiction that $\nu \in A$ for some Wiener algebra A in $M(\mathbb{R}^2)$. Since f is odd it follows from (10) that the Fourier transform of ν is real-valued. Since A is Wiener, $\sigma_A(\nu) \subset \mathbb{R}$.

Let A_0 be the closed subalgebra of $M(\mathbb{R}^2)$ generated by ν. Since ν has independent powers, A_0 consists of all measures of the form

$$\mu = \sum_{j=0}^{\infty} a_j \nu^{*j}, \tag{11}$$

with

$$\|\mu\| = \sum_{j=0}^{\infty} |a_j| \|\nu\|^j < \infty.$$

It is well known that the multiplicative linear functionals on such an algebra are of the form

$$\mu \mapsto \sum_{j=0}^{\infty} a_j \lambda^j,$$

where λ is a complex number such that $|\lambda| \leq \|\nu\|$ and μ is given by (11). In particular,

$$\sigma_{A_0}(\nu) = \{\lambda \in \mathbb{C} : \ |\lambda| \leq \|\nu\|\}.$$

By Lemma 6 this implies that

$$\{\lambda \in \mathbb{C} : \ |\lambda| = \|\nu\|\} \subset \sigma_A(\nu),$$

which contradicts $\sigma_A(\nu) \subset \mathbb{R}$. □

Remark 7. By (10), ν is the push-forward measure $F_*(dx|K_0)$, where $F : x \mapsto (x, f(x))$.

Remark 8. The strongest result in [1] states that if A is the closed subalgebra of $M(\mathbb{R}^n)$ generated by all discrete measures and all measures that are absolutely continuous with respect to Lebesgue surface area measure on some C^1-submanifold of $\mathbb{R}^n$ in the so-called generic position, then A is a Wiener algebra. A consequence of this result is that if B is the closed subalgebra of $M(\mathbb{R}^2)$ generated by all discrete measures, $L^1(\mathbb{R}^2)$ and all measures that are absolutely continuous with respect to arc length measure on some quasi-analytic curve γ in $\mathbb{R}^2$, then B is a Wiener algebra. Here the assertion that γ is a quasi-analytic curve is to be understood in the sense that the components of γ, considered as functions on the parameter interval, are both members in some linear quasi-analytic class of C^∞-functions. An exposition of the above results by Björk and some related matters will appear in [7].

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