ABSOLUTELY CONTINUOUS MEASURES 
ON NON QUASI-ANALYTIC CURVES 
WITH INDEPENDENT POWERS

MATS ANDERS OLOFSSON

(Communicated by Christopher D. Sogge)

Abstract. We prove that every non quasi-analytic Carleman class contains 
functions whose graph supports measures that are absolutely continuous with 
respect to arc length measure and yet they have independent convolution pow-
ers in the measure algebra $M(R^2)$. The proof relies on conditions which ensure 
that the canonical map between two Cantor sets can be extended to a function 
in an arbitrary prescribed non quasi-analytic Carleman class.

1. Introduction

Our main technical object with this note is to prove certain sufficient conditions 
on pairs of Cantor sets $K_0$ and $K$ to ensure that the so-called canonical map 
$\psi : B_{K_0} \to B_K$ (see Definition 1 in Section 3) can be extended to a function 
in $C\{M_j\}$, where $C\{M_j\}$ is an arbitrary prescribed non quasi-analytic Carleman 
class of $C^\infty$-functions (see below). In Section 3 our extension result for Cantor set 
maps is given in three different forms in Theorem 3, Corollary 1 and Theorem 4. 
The content of those results is that extension is possible if the image Cantor set 
$K$ is sufficiently small relative to the domain Cantor set $K_0$. In some sense these 
extension results are rather sharp (see Remark 3 in Section 3). Our proofs of those 
results rely on the methodology used to to prove Whitney’s extension theorem (see 
Stein [10], Chapter VI).

The motivation for this work comes from Björk [1], which was inspired by [2] 
and [11]. Let $A$ be a closed subalgebra (with unit) of the measure algebra $M(R^n)$. 
The Fourier transform allows us to consider $R^n$ as a subset of the maximal ideal 
space of $A$. $A$ is called a Wiener algebra if $R^n$ is dense in the maximal ideal space 
of $A$; this is known to imply that every measure in $A$ whose Fourier transform 
is bounded away from zero is invertible in $A$. It is proved in [11] that if $A$ is the 
closed subalgebra of $M(R^n)$ generated by all Dirac measures and all measures that 
are absolutely continuous with respect to Lebesgue surface area measure on some 
real analytic submanifold of $R^n$, then $A$ is a Wiener algebra. See also Remark 8

Received by the editors April 29, 1999.

2000 Mathematics Subject Classification. Primary 43A10; Secondary 26E10.

Key words and phrases. Measure algebras, Wiener-Pitt phenomenon, independent powers.

The author was supported by the G. S. Magnusson Fund of the Royal Swedish Academy of 
Sciences.

©2000 American Mathematical Society

515
in Section 4. In contrast to this we shall prove the following Wiener-Pitt type theorem:

**Theorem 1.** Let $C\{M_j\}$ be non quasi-analytic. Then there exists a curve

$$\Gamma = \{(x, y) \in \mathbb{R}^2 : y = f(x)\},$$

where $f \in C\{M_j\}$, and a compactly supported measure $\nu \in M(\mathbb{R}^2)$ absolutely continuous with respect to arc length measure on $\Gamma$ such that $\nu$ has independent powers (Remark 6 Section 4) and is not an element in some Wiener subalgebra $A$ of $M(\mathbb{R}^2)$.

This result sharpens the analogous result in [1] stating the same as Theorem 1 except that the curve was only required to be of the class $C^1$ there. The proof of Theorem 1 is given in Section 4. In the construction used to prove Theorem 1 our extension result Theorem 4 is used in an essential way.

**Quasi-analytic classes.** Let $\{M_j\}_{j=0}^\infty$ be a sequence of positive numbers. We denote by $C\{M_j\}$ the Carleman class consisting of all $C^1$-functions $f$ on $\mathbb{R}$ satisfying the estimates

$$|f^{(j)}(x)| \leq \alpha A^j M_j, \quad x \in \mathbb{R}, \ j \geq 0,$$

for some constants $0 < \alpha, A < \infty$. It is known that every such class $C\{M_j\}$ can be defined by means of a logarithmically convex sequence. In fact, $C\{M_j\} = C\{M_j'\}$, where $\{M_j\}_{j=0}^\infty$ is the largest logarithmically convex minorant sequence of $\{M_j\}_{j=0}^\infty$ (see [6], Chapter V, Section E). Using logarithmic convexity of $\{M_j\}_{j=0}^\infty$ one proves that $C\{M_j\}$ is an algebra of functions (see [6], Chapter IV, Section E).

The class $C\{M_j\}$ is said to be quasi-analytic if the zero function $f \equiv 0$ is the only function $f$ in $C\{M_j\}$ satisfying $f^{(j)}(0) = 0$ for all $j \geq 0$. The celebrated theorem of Denjoy-Carleman (see [6], Chapter IV, Section B) states that $C\{M_j\}$ is quasi-analytic if and only if

$$\sum_{j=1}^\infty \frac{M_{j-1}}{M_j} = \infty.$$

A closely related result which will also be referred to as the theorem of Denjoy-Carleman is that

$$\sum_{j=1}^\infty \frac{M_{j-1}}{M_j} < \infty$$

implies that $C\{M_j\}$ is non quasi-analytic. In fact, this implication follows from the well-known construction of test functions by means of repeated convolutions of normalized characteristic functions (see [6], Theorem 1.3.2).

2. **Whitney extension of a flat map**

The aim with this section is to prove a variant of Whitney’s extension theorem (see [10], Chapter VI, Section 2 or [4], Theorem 2.3.6), Theorem 2 below. The novelty of Theorem 2 is that it is formulated in terms of flatness (see assumption (3) below) and Carleman classes. See also Remark 1 below.

**Lemma 1.** Let $F \subset \mathbb{R}$, $F \neq \mathbb{R}$, be a non-empty closed set and $C\{M_j\}$ a non quasi-analytic Carleman class. Then there exists a locally finite partition of unity $\{\varphi_k\}_{k=1}^\infty$ on $\mathbb{R} \setminus F$ such that:
1) Every point in $\mathbb{R} \setminus F$ has a neighborhood that intersects at most two of the $	ext{supp}(\varphi_k)$’s.

2) There exists a positive real constant $C$ not depending on $k$ such that:
$$\text{diam}(\text{supp}(\varphi_k)) \leq C \text{dist}(\text{supp}(\varphi_k), F).$$

3) There exist constants $0 < \alpha, A < \infty$ such that
$$|\varphi_k^{(j)}(x)| \leq \alpha A^j M_j \text{dist}(x; F)^{-j}, \quad x \in \mathbb{R}, \quad j = 0, 1, 2, \ldots,$$
holds for all $k$.

We mention that partitions of unity similar to that in Lemma 1 are constructed in much more generality (arbitrary finite dimensional vector space) by Hörmander in [4] (see Theorem 1.4.10). For the sake of completeness we include a sketch of the proof of Lemma 1.

**Sketch of the proof of Lemma 1.** Write $\mathbb{R} \setminus F = \bigcup_{k=1}^{\infty} I_k$, where the $I_k$’s are closed intervals whose interiors are pairwise disjoint and such that
$$c_1 \text{diam}(I_k) \leq \text{dist}(I_k, F) \leq c_2 \text{diam}(I_k), \quad k = 1, 2, \ldots,$$
for some constants $c_1$ and $c_2$. Let $I_k^*$ be the interval with the same center as $I_k$ but expanded by a factor $1 + \varepsilon$ for some $\varepsilon > 0$. We now choose $\varepsilon > 0$ sufficiently small to ensure that $I_k^*$ and $I_j^*$ intersect only if the intervals $I_k$ and $I_j$ have a common endpoint.

Let $I = [-1, 1]$ and $I^* = [-1 + \varepsilon, 1 + \varepsilon]$. Let $\psi \in C\{M_j\}$ be such that $\psi = 1$ on $I$, $\text{supp}(\psi) \subset I^*$ and $0 \leq \psi \leq 1$. (Such a test function $\psi$ can be constructed by convolving a non-negative compactly supported function in $C\{M_j\}$ with integral $= 1$ and small support with the characteristic function for an interval.) We now transfer $\psi$ to $I_k^*$ by defining $\psi_k(x) = \psi((x - a_k)/s_k)$, where $I_k = [a_k - s_k, a_k + s_k]$. We now define the $\varphi_k$’s by
$$\varphi_1(x) = \psi_1(x), \quad \varphi_k(x) = \psi_k(x) \prod_{j=1}^{k-1} (1 - \psi_j(x)), \quad k \geq 2.$$

One checks that $\{\varphi_k\}$ is a partition of unity on $\mathbb{R} \setminus F$. 1 and 2 hold by the construction of the $I_k$ and $I_k^*$’s. By the construction there are constants $\beta, B$ such that
$$|\psi_k^{(j)}(x)| \leq \beta B^j M_j \text{dist}(x; F)^{-j}, \quad x \in \mathbb{R}, \quad j = 0, 1, \ldots,$$
holds for all $k \geq 1$. Since locally $\varphi_k$ is a product of at most two functions satisfying (2) it follows from the Leibniz formula for the derivative of a product that (1) holds.

**Lemma 2.** Let $F$ and $\{\varphi_k\}$ be as in Lemma 1. For every $k$, let $p_k$ be a point in $F$ closest to $\text{supp}(\varphi_k)$. Then
$$|p_k - y| \leq (2 + C)|x - y|$$
for all $x \in \text{supp}(\varphi_k)$ and $y \in F$, where $C$ is as in Lemma 1.

**Proof.** Choose $x^* \in \text{supp}(\varphi_k)$ such that $|x^* - p_k| = \text{dist}(\text{supp}(\varphi_k), F)$. By part 2) of Lemma 1 we have
$$|p_k - y| \leq |p_k - x^*| + |x^* - x| + |x - y| \leq (2 + C)|x - y|.$$
We conclude that
\[ \text{Proof.} \]

By (5) we have
\[ \text{Proof.} \]

By (3) and Lemma 2, \( x \in R, \ j = 1, 2, \ldots, \)
\[ \text{Proof.} \]

Then \( \psi \) has an extension to a \( C^\infty \)-function \( \Psi \) on \( R \) satisfying the estimates
\[ \text{Proof.} \]

(4) \[ |\Psi^{(j)}(x)| \leq 2\alpha A^j (2 + C)^j \Lambda_j M_j, \quad x \in R, \ j = 1, 2, \ldots, \]
\[ \text{Proof.} \]

where \( \alpha, A \) and \( C \) are as in Lemma 2. If \( \psi \) is bounded, then \( \Psi \in C\{\Lambda_j M_j\} \).
\[ \text{Proof.} \]

Proof. Choose \( \{\varphi_k\}_{k=1}^\infty \) and \( \{p_k\}_{k=1}^\infty \) as in the preceding lemmas. The extension \( \Psi \)
\[ \text{Proof.} \]

of \( \psi \) is now defined by
\[ \text{Proof.} \]

\[ \Psi(x) = \left\{ \begin{array}{ll}
\sum_{k=1}^\infty \psi(p_k) \varphi_k(x), & x \not\in R, \\
\psi(x), & x \in R.
\end{array} \right. \]
\[ \text{Proof.} \]

It is clear that \( \Psi \in C^\infty \) in \( R \setminus F \) and that \( \Psi^{(j)} \) exists and is \( = 0 \) in the interior of \( F \)
\[ \text{Proof.} \]

for all \( j \geq 1 \). For proving that \( \Psi \) is \( C^\infty \) it suffices to check that every \( \Psi^{(j)} \) extends continuously to \( R \).
\[ \text{Proof.} \]

The case \( j = 0 \). It suffices to prove that \( F \not\ni x \rightarrow y \in F \) implies \( \Psi(x) \rightarrow \psi(y) \).
\[ \text{Proof.} \]

We have that
\[ \text{Proof.} \]

\[ \Psi(x) - \psi(y) = \sum_k (\psi(p_k) - \psi(y)) \varphi_k(x). \]
\[ \text{Proof.} \]

By (3) and Lemma 2
\[ \text{Proof.} \]

\[ |\Psi(x) - \psi(y)| \leq \sum_k \Lambda_1 |p_k - y| \varphi_k(x) \leq (2 + C) \Lambda_1 |x - y| \rightarrow 0 \]
\[ \text{Proof.} \]

as \( x \rightarrow y \).
\[ \text{Proof.} \]

The case \( j > 0 \). Assume \( x \not\in R \). Choose \( y \in R \) as close to \( x \) as possible. Since
\[ \text{Proof.} \]

\[ \sum_k \varphi_k^{(j)}(x) = 0 \] we have that
\[ \text{Proof.} \]

\[ \Psi^{(j)}(x) = \sum_k \psi(p_k) \varphi_k^{(j)}(x) = \sum_k (\psi(p_k) - \psi(y)) \varphi_k^{(j)}(x). \]
\[ \text{Proof.} \]

By (3) and Lemma 2
\[ \text{Proof.} \]

\[ |\Psi^{(j)}(x)| \leq \sum_k \Lambda_{j+1} |p_k - y| |x| \varphi_k^{(j)}(x) | \leq \Lambda_{j+1} (2 + C)^j |x - y|^{j+1} |x| \sum_k \varphi_k^{(j)}(x)|. \]
\[ \text{Proof.} \]

By (1) and part (b) of Lemma 1 we can estimate \( \sum_k |\varphi_k^{(j)}(x)| \) by \( 2\alpha A^j M_j |x - y|^{-j} \).
\[ \text{Proof.} \]

We conclude that
\[ \text{Proof.} \]

\[ |\Psi^{(j)}(x)| \leq 2\alpha A^j (2 + C)^j |x - y|^{j+1} \Lambda_{j+1} M_j |x - y| \rightarrow 0 \]
\[ \text{Proof.} \]

as \( x \rightarrow F \).
\[ \text{Proof.} \]

We now prove of Lemma 1. Assume \( x \not\in R \). Choose \( y \in R \) as close to \( x \) as possible.
\[ \text{Proof.} \]

By (3) we have
\[ \text{Proof.} \]

\[ |\Psi^{(j)}(x)| \leq \sum_k \Lambda_j |p_k - y| |x| |x| \varphi_k^{(j)}(x) | \leq \Lambda_j (2 + C)^j |x - y|^{j} |x| \sum_k |\varphi_k^{(j)}(x)|. \]
\[ \text{Proof.} \]

Estimating \( \sum_k |\varphi_k^{(j)}(x)| \) by \( 2\alpha A^j M_j |x - y|^{-j} \) we get (3). \qed
Remark 1. The case of interest in Theorem 2 is when the set $F$ is highly irregular. For instance, if $F = [0,1]$, then the assumption of $\delta_j = 2$ implies that $\psi$ is constant. In the proof of Theorem 3 in Section 3, we apply Theorem 2 to the case when $F$ is a Cantor set and $\psi$ is a so-called canonical map (Definition 1: Section 3).

3. Extension of maps between Cantor sets

A class of Cantor sets on $[0,1]$. For every $n \geq 1$ let there be given $2^{n-1}$ open intervals

$$\Delta_j^{(n)} = (\alpha_j^{(n)}, \beta_j^{(n)}), \quad j = 1, \ldots, 2^{n-1},$$

such that

$$0 < \alpha_1^{(n)} < \beta_1^{(n)} < \alpha_2^{(n)} < \beta_2^{(n)} < \cdots < \alpha_{2^{n-1}}^{(n)} < \beta_{2^{n-1}}^{(n)} < 1.$$

Write $S^{(0)} = [0,1] = S_1^{(0)}$ and set

$$S^{(n)} = S^{(n-1)} \setminus \bigcup_{j=1}^{2^{n-1}} \Delta_j^{(n)} = \bigcup_{j=1}^{2^n} S_j^{(n)}, \quad n \geq 1,$$

where the $S_j^{(n)}$, $1 \leq j \leq 2^n$, are pairwise disjoint closed intervals. We demand further in the construction that the intervals $\Delta_j^{(n)}$ are such that

$$\Delta_j^{(n+1)} \subset S_j^{(n)}, \quad 1 \leq j \leq 2^n, \quad n \geq 1.$$

The corresponding Cantor set is now defined by

$$K = \bigcap_{n=0}^{\infty} S^{(n)} = [0,1] \setminus \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{2^n} \Delta_j^{(n)}.$$

We define

$$\lambda_K(n) = \max_{1 \leq j \leq 2^n} |S_j^{(n)}| \quad (n \geq 0),$$

$$\delta_K(n) = \min_{1 \leq j \leq 2^{n-1}} |\Delta_j^{(n)}| \quad (n \geq 1)$$

and

$$B_K = \left\{ 0,1, \alpha_j^{(n)}, \beta_j^{(n)} : 1 \leq j \leq 2^{n-1}, \quad n \geq 1 \right\}.$$ 

We call $K$ an admissible Cantor set if $K$ is constructed as above and $\lambda_K(n) \to 0$ as $n \to \infty$.

Remark 2. Note that $B_K$ is dense in $K$ if $K$ is an admissible Cantor set. Let us also notice that there exist admissible Cantor sets with positive Lebesgue measure. For example when every $\Delta_j^{(n)}$, $1 \leq j \leq 2^{n-1}$, has length $1/4^n$ and the $S_j^{(n)}$, $1 \leq j \leq 2^n$, are of equal length, then the above construction gives an admissible Cantor set of measure $1/2$.

For easy reference we record the following lemma whose proof is trivial.

Lemma 3. Let $K$ be an admissible Cantor set on $[0,1]$. Let $x, y \in K$, $x \neq y$, and denote by $n(x,y)$ the largest integer $n \geq 0$ such that both $x \in S_j^{(n)}$ and $y \in S_j^{(n)}$ for some $j$. Then

$$\delta_K(n(x,y) + 1) \leq |x - y| \leq \lambda_K(n(x,y)).$$
Definition 1. Let \( K_0 \) and \( K \) be admissible Cantor sets on \([0,1]\), where \( K \) is constructed as above and \( K_0 \) is obtained at step \( n \geq 1 \) by removing the intervals
\[
(a_j^{(n)}, b_j^{(n)}), \quad j = 1, \ldots, 2^{n-1}.
\]
The canonical map \( \psi : B_{K_0} \to B_K \) is defined by \( \psi(0) = 0, \psi(1) = 1, \psi(a_j^{(n)}) = \alpha_j^{(n)}, \psi(b_j^{(n)}) = \beta_j^{(n)} \).

Note that \( \psi \) is strictly increasing and thereby preserves the construction.

Proposition 1. Let \( K_0 \) and \( K \) be admissible Cantor sets on \([0,1]\) and let \( \psi : B_{K_0} \to B_K \) be the corresponding canonical map (Definition 4). Then the following estimate holds:
\[
|\psi(x) - \psi(y)| \leq \Lambda_j |x - y|^{j}, \quad x, y \in B_{K_0}, \quad j = 0, 1, 2, \ldots,
\]
where
\[
\Lambda_j = \sup_{n \geq 0} \frac{\lambda_K(n)}{\delta_{K_0}(n + 1)^j}.
\]

Proof. Since \( \psi \) is strictly increasing, \( n(\psi(x), \psi(y)) = n(x, y) \), where \( n(\cdot, \cdot) \) is as in Lemma 3. For \( x, y \in B_{K_0}, x \neq y \), we have
\[
|\psi(x) - \psi(y)| \leq \lambda_K(n(\psi(x), \psi(y))) = \frac{\lambda_K(n(x,y))}{|x - y|^j} |x - y|^j
\]
\[
\leq \frac{\lambda_K(n(x,y))}{\delta_{K_0}(n(x,y) + 1)^j} |x - y|^j \leq \Lambda_j |x - y|^j.
\]

We are mainly interested in the case when \( \Lambda_j < \infty \) for all \( j \). Note however that \( \psi \) extends uniquely to a Lipschitz continuous function \( \psi : K_0 \to K \) if \( \Lambda_1 < \infty \).

Remark 3. Let the notation be as in Proposition 1 and assume that \( \Lambda_j < \infty \) for every \( j \). If \( \Psi \) is a \( C^\infty \)-extension of \( \psi \), then \( \Psi^{(j)} \equiv 0 \) on \( K_0 \) for every \( j \geq 1 \). In particular, such an extension \( \Psi \) (if it exists) cannot be in some quasi-analytic class. In Theorem 4 below we show that if \( \lambda_K(n) \to 0 \) sufficiently fast, then there exists an extension \( \Psi \in C\{M_j\} \) of \( \psi \), where \( C\{M_j\} \) is an arbitrary prescribed non quasi-analytic Carleman class.

Also, since \( \Psi^{(j)}(0) = 0 \) for all \( j \geq 0 \), \( \Psi \) can be glued together with any \( C^\infty \)-function, all of whose derivatives vanish at 0 to form a \( C^\infty \)-extension of \( \psi \).

Applying Theorem 2 to the canonical map between two Cantor sets we obtain:

Theorem 3. Let \( C\{M_j\} \) be non quasi-analytic and let \( \{\Lambda_j\} \) be a sequence of positive real numbers with \( \lim_{j \to \infty} \Lambda_j^{1/j} = \infty \). Let \( K_0 \) and \( K \) be admissible Cantor sets on \([0,1]\) and let \( \psi : B_{K_0} \to B_K \) be the corresponding canonical map (Definition 4).

If the set \( K \) is constructed in such a way that
\[
\lambda_K(n) \leq \min_{j \geq 0} \delta_{K_0}(n + 1)^j \Lambda_j
\]
for all \( n \) sufficiently large, then \( \psi \) has an extension to a function \( \Psi \) in \( C\{\Lambda_j M_j\} \).
Proof. By (8) and Proposition 1 we can uniquely extend to a continuous function
\( \psi : K_0 \to K \) such that for some \( C > 0 \),
\[
|\psi(x) - \psi(y)| \leq C^{j+1} \Lambda_j |x - y|, \quad j \geq 0,
\]
for all \( x, y \in K_0 \). By Theorem 2, \( \psi \) has an extension to a function \( \Psi \) in the class \( C\{C^{j+1} \Lambda_j M_j\} = C\{\Lambda_j M_j\} \).

Remark 4. The assumption on \( \{ \Lambda_j \}_{j=0}^{\infty} \) that \( \lim_{j \to \infty} \Lambda_j^{1/j} = \infty \) is needed to guarantee that the minima in (8) are attained.

Corollary 1. Let \( C\{M_j\} \) be non quasi-analytic. Let \( K_0, K \) and \( \psi \) be as in Theorem 3. Let
\[
\Lambda_j = \sup_{n \geq 0} \frac{\lambda_K(n)}{\delta_{K_0}(n+1)^j}.
\]

If \( C\{M_j/\Lambda_j\} \) is non quasi-analytic, then \( \psi \) has an extension to a function \( \Psi \) in \( C\{M_j\} \).

Proof. Let \( L_j = M_j/\Lambda_j \). By our choice of \( \Lambda_j \),
\[
\lambda_K(n) \leq \inf_{j \geq 0} \delta_{K_0}(n+1)^j \Lambda_j.
\]
By Theorem 3 we see that \( \psi \) extends to a function in \( C\{\Lambda_j L_j\} = C\{M_j\} \).

The assumption in Corollary 1 that \( C\{M_j/\Lambda_j\} \) is non quasi-analytic is, by the Denjoy-Carleman theorem, satisfied if
\[
\sum_{j=1}^{\infty} \frac{\Lambda_j}{\Lambda_{j-1}} \frac{M_{j-1}}{M_j} < \infty.
\]
This condition is fulfilled if \( \lambda_K(n) \to 0 \) sufficiently fast. We state this explicitly in the next theorem.

Theorem 4. Let \( C\{M_j\} \) be non quasi-analytic and let \( K_0 \) be an admissible Cantor set on \([0,1]\).

If \( K \) is an admissible Cantor set on \([0,1]\) such that \( \lambda_K(n) \to 0 \) sufficiently fast (the rate of convergence depends only on \( \{\delta_{K_0}(n)\}_{n=1}^{\infty} \) and \( \{M_j\}_{j=0}^{\infty} \)), then the canonical map \( \psi : \mathcal{B}_{K_0} \to \mathcal{B}_K \) can be extended to a function \( \Psi \) in \( C\{M_j\} \).

Proof. We may assume that \( \{M_j\}_{j=0}^{\infty} \) is logarithmically convex. Since, by the theorem of Denjoy-Carleman, \( \sum M_{j-1}/M_j < \infty \), we can choose \( 0 < \omega_j \to \infty \) such that
\[
\sum_{j=1}^{\infty} \omega_j \frac{M_{j-1}}{M_j} < \infty.
\]
Set \( \Lambda_0 = 1 \) and
\[
\Lambda_j = \prod_{l=1}^{j} \omega_l, \quad j \geq 1.
\]
By the theorem of Denjoy-Carleman (the non log-convex case, see the introductory remark on quasi-analytic classes), \( \Psi \) implies that the class \( C\{M_j/\Lambda_j\} \) is non quasi-analytic.
If $K$ is such that
$$\lambda_K(n) \leq \min_{j \geq 0} \delta_{K_0}(n + 1)^j A_j, \quad n \text{ large},$$
then by Theorem 4, $\psi$ extends to a function in $C\{A_j M_j / A_j\} = C\{M_j\}$. \hfill \Box

4. Application to Wiener algebras

In this section we give the proof of Theorem 1 in the introduction. First we state some preliminary results.

**Lemma 4.** There exists a Cantor set $K$ on $[0, 1]$ such that $K \setminus \{0\}$ is linearly independent over $\mathbb{Q}$ and $\lambda_K(n) \to 0$ faster than any prescribed quantity.

**Proof.** By translating and dilating, this lemma is an immediate consequence of the construction of a linearly independent Cantor set (see [3], pages 187-189 or [5], pages 20-21). \hfill \Box

**Lemma 5** (Theorem 5.3.2 in [8]). Let $G$ be a non-discrete locally compact abelian group and $E$ an independent (see Remark 5) compact subset of $G$. If $\mu \in M(G)$ is a continuous measure concentrated on $E \cup (-E)$, then the measures $\delta_0, \mu, \mu^2, \mu^3, \ldots$ are mutually singular.

**Remark 5.** A set $E \subset G$ is said to be independent if for every choice of distinct points $x_1, \ldots, x_k$ in $E$ and $n_1, \ldots, n_k \in \mathbb{Z}$, $n_1 x_1 + \cdots + n_k x_k = 0$ implies $n_1 x_1 = \cdots = n_k x_k = 0$.

**Remark 6.** Recall that $\mu \in M(G)$ is said to have independent powers if the measures $\delta_0, \mu, \mu^2, \ldots$ are mutually singular.

Below we use the notation $\sigma_A(x)$ for the spectrum of the element $x$ in the Banach algebra $A$.

**Lemma 6.** Let $A$ be a closed subalgebra of a Banach algebra $B$. Let $x \in A$. Then the boundary of $\sigma_A(x)$ is contained in $\sigma_B(x)$.

**Proof.** This result is well known and can be found as part of Theorem 10.18 in [9]. \hfill \Box

**Proof of Theorem 1** Let $C\{M_j\}$ be non quasi-analytic. Let $K_0$ be an admissible Cantor set on $[0, 1]$ with positive Lebesgue measure (see Remark 2 in Section 3). By Lemma 4 we can find an admissible Cantor set $K$ on $[0, 1]$ such that $K \setminus \{0\}$ is linearly independent over $\mathbb{Q}$ and $\lambda_K(n) \to 0$ sufficiently fast in the sense of Theorem 4. By Theorem 4 the canonical map $\psi : B_{K_0} \to B_K$ (Definition 1) has an extension to a function $\Psi$ in the class $C\{M_j\}$. Let $P = K_0 \times K$ and define

$$f(x) = \begin{cases} \Psi(x) & \text{if } x \geq 0, \\ -\Psi(-x) & \text{if } x \leq 0. \end{cases}$$

Then, by Remark 3, $f$ is a function in $C\{M_j\}$. Let

$$\Gamma = \{(x, y) \in \mathbb{R}^2 : y = f(x), \ x \in \mathbb{R}\}$$

be the graph of $f$. Let $\sigma$ denote the arc length measure on $\Gamma$. Define the measure $\nu$ by

$$\nu(E) = \sigma (E \cap (P \cup (-P)))$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
for every Borel set $E$ in $\mathbb{R}^2$. It is obvious that $\nu$ is absolutely continuous with respect to $\sigma$. The proof is now completed by the following theorem:

**Theorem 5.** In the above situation, $\nu$ has independent powers (Remark 6: Section 4) and is not an element in some Wiener subalgebra $A$ of $M(\mathbb{R}^2)$.

**Proof.** By Lemma 5 with $G = \mathbb{R}^2$ and $E = P \cap \Gamma = \{ (x, f(x)) : x \in K_0 \}$, $\nu$ has independent powers. Using Remark 5 a computation shows that the action of $\nu$ on test functions is given by

$$\int \varphi d\nu = \int_{K_0 \cup (-K_0)} \varphi(x, f(x)) dx.$$  

Since $K_0$ has positive Lebesgue measure, $\nu \neq 0$. Assume for reaching a contradiction that $\nu \in A$ for some Wiener algebra $A$ in $M(\mathbb{R}^2)$. Since $f$ is odd it follows from (10) that the Fourier transform of $\nu$ is real-valued. Since $A$ is Wiener, $\sigma_A(\nu) \subset \mathbb{R}$.

Let $A_0$ be the closed subalgebra of $M(\mathbb{R}^2)$ generated by $\nu$. Since $\nu$ has independent powers, $A_0$ consists of all measures of the form

$$\mu = \sum_{j=0}^{\infty} a_j \nu^j,$$

with

$$\|\mu\| = \sum_{j=0}^{\infty} |a_j| \|\nu\|^j < \infty.$$  

It is well known that the multiplicative linear functionals on such an algebra are of the form

$$\mu \mapsto \sum_{j=0}^{\infty} a_j \lambda^j,$$

where $\lambda$ is a complex number such that $|\lambda| \leq \|\nu\|$ and $\mu$ is given by (11). In particular,

$$\sigma_{A_0}(\nu) = \{ \lambda \in \mathbb{C} : |\lambda| \leq \|\nu\| \}.$$  

By Lemma 6 this implies that

$$\{ \lambda \in \mathbb{C} : |\lambda| = \|\nu\| \} \subset \sigma_A(\nu),$$

which contradicts $\sigma_A(\nu) \subset \mathbb{R}$.  

**Remark 7.** By (10), $\nu$ is the push-forward measure $F_*(dx|K_0)$, where $F : x \mapsto (x, f(x))$.

**Remark 8.** The strongest result in [1] states that if $A$ is the closed subalgebra of $M(\mathbb{R}^n)$ generated by all discrete measures and all measures that are absolutely continuous with respect to Lebesgue surface area measure on some $C^1$-submanifold of $\mathbb{R}^n$ in the so-called generic position, then $A$ is a Wiener algebra. A consequence of this result is that if $B$ is the closed subalgebra of $M(\mathbb{R}^2)$ generated by all discrete measures, $L^1(\mathbb{R}^2)$ and all measures that are absolutely continuous with respect to arc length measure on some quasi-analytic curve $\gamma$ in $\mathbb{R}^2$, then $B$ is a Wiener algebra. Here the assertion that $\gamma$ is a quasi-analytic curve is to be understood in the sense that the components of $\gamma$, considered as functions on the parameter interval, are both members in some linear quasi-analytic class of $C^\infty$-functions. An exposition of the above results by Björk and some related matters will appear in [7].
Acknowledgement

The author is grateful to Jan-Erik Björk and Jan Boman for their valuable suggestions regarding this work.

References