

## THE BRAUER GROUP OF SWEEDLER'S HOPF ALGEBRA $H_4$

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(Communicated by Ken Goodearl)

ABSTRACT. We calculate the Brauer group of the four dimensional Hopf algebra  $H_4$  introduced by M. E. Sweedler. This Brauer group  $\text{BM}(k, H_4, R_0)$  is defined with respect to a (quasi-) triangular structure on  $H_4$ , given by an element  $R_0 \in H_4 \otimes H_4$ . In this paper  $k$  is a field. The additive group  $(k, +)$  of  $k$  is embedded in the Brauer group and it fits in the exact and split sequence of groups:

$$1 \longrightarrow (k, +) \longrightarrow \text{BM}(k, H_4, R_0) \longrightarrow \text{BW}(k) \longrightarrow 1$$

where  $\text{BW}(k)$  is the well-known Brauer-Wall group of  $k$ . The techniques involved are close to the Clifford algebra theory for quaternion or generalized quaternion algebras.

### 1. INTRODUCTION

In their search for general unified theory allowing them to deal with the Brauer group of a quantum group (cf. [2, 3]), the authors have introduced the Brauer group,  $\text{Br}(\mathcal{C})$ , of a braided monoidal category (see [10]). In loc. cit. it is shown that all known Brauer groups, algebraic as well as geometric ones, appear as examples of  $\text{Br}(\mathcal{C})$  for a suitable choice of  $\mathcal{C}$  or the braiding on  $\mathcal{C}$ . The theory of the Brauer group of a quantum group deals mainly with Yetter-Drinfel'd modules (YD modules) and YD module algebras. However, when a Hopf algebra  $H$  has a quasi-triangular structure, then the category of left  $H$ -modules is itself a braided monoidal category, say  ${}_H\mathbf{M}$ . The Brauer group  $\text{Br}({}_H\mathbf{M})$  is a subgroup of  $\text{BQ}(k, H)$  which is the Brauer group of the category YD modules for  $H$ , where  $k$  is the ground field (or a commutative ring). In [9], the authors have established that the Hopf automorphism group of  $H$  may be embedded into  $\text{BQ}(k, H)$  (up to factoring out some finite group eventually). Hence  $\text{BQ}(k, H)$  need not be a torsion group. The ruling idea was that  $\text{BQ}(k, H)$  modulo the image of the Hopf automorphism group would be a torsion group, at least for a finite dimensional  $H$ . The calculation in this paper contradicts that idea; even in such a low-dimensional case as  $H_4$ , many non-torsion elements do exist in the Brauer group. We provide precise structure and calculation for the Brauer group  $\text{BM}(k, H_4, R_0)$  of  $H_4$  with respect to a (quasi-) triangular structure on  $H_4$  given by an element  $R_0 \in H_4 \otimes H_4$ . Even if the Hopf automorphism group of  $H_4$ , the multiplicative group of  $k$ , does now not appear, the additive group  $(k, +)$  of the field  $k$  may be embedded in  $\text{BM}(k, H_4, R_0)$ . The structure results make it clear that the non-torsion part of  $\text{BM}(k, H_4, R_0)$  is represented by  $H_4$ -Azumaya

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Received by the editors February 22, 1999 and, in revised form, May 4, 1999.  
1991 *Mathematics Subject Classification*. Primary 16W30, 16H05, 16K50.

algebra structures that are non-trivial but that are defined on the trivial element of the Brauer-Wall group. This is contained in the fact that the following sequence of group homomorphism is exact and split:

$$1 \longrightarrow (k, +) \longrightarrow \text{BM}(k, H_4, R_0) \longrightarrow \text{BW}(k) \longrightarrow 1.$$

## 2. PRELIMINARIES

In this section, we recall the definition of the Brauer group of a Hopf algebra with a bijective antipode and some related notions. Let  $k$  be a commutative ring with unit, and  $H$  a Hopf algebra over  $k$  with a bijective antipode. A Yetter-Drinfel'd  $H$ -module ( simply, YD  $H$ -module)  $M$  is a crossed  $H$ -bimodule [13]. That is,  $M$  is a  $k$ -module which is at once a left  $H$ -module and a right  $H$ -comodule satisfying the following equivalent compatibility conditions [6, 5.1.1]:

- (i)  $\sum h_{(1)} \cdot m_{(0)} \otimes h_{(2)} m_{(1)} = \sum (h_{(2)} \cdot m)_{(0)} \otimes (h_{(2)} \cdot m)_{(1)} h_{(1)},$
- (ii)  $\sum (h \cdot m)_{(0)} \otimes (h \cdot m)_{(1)} = \sum (h_{(2)} \cdot m_{(0)}) \otimes h_{(3)} m_{(1)} S^{-1}(h_{(1)})$

where the sigma notations for comodules and for comultiplications can be found in the standard reference book [11]. An Yetter-Drinfel'd  $H$ -module algebra is a YD  $H$ -module  $A$  such that  $A$  is a left  $H$ -module algebra and a right  $H^{op}$ -comodule algebra. For the details of  $H$ -(co)module algebras we refer to [11].

In [2] we defined the Brauer group of a Hopf algebra  $H$  by considering isomorphism classes of  $H$ -Azumaya algebras. A YD  $H$ -module algebra  $A$  is called  $H$ -Azumaya if it is faithfully projective as a  $k$ -module and if the following YD  $H$ -module algebra are isomorphisms:

$$F : A \# \bar{A} \longrightarrow \text{End}(A), \quad F(a \# \bar{b})(c) = \sum a c_{(0)}(c_{(1)} \cdot b),$$

$$G : \bar{A} \# A \longrightarrow \text{End}(A)^{op}, \quad G(\bar{a} \# b)(c) = \sum a_{(0)}(a_{(1)} \cdot c)b,$$

where  $\bar{A}$  is the  $H$ -opposite YD  $H$ -module algebra of  $A$  (cf. [2]). For a faithfully projective YD  $H$ -module  $M$ , the endomorphism ring  $\text{End}_k(M)$  is a YD  $H$ -module algebra with  $H$ -structures given by

$$\begin{aligned} (h \cdot f)(m) &= \sum h_{(1)} \cdot f(S(h_{(2)}) \cdot m), \\ \sum f_{(0)}(m) \otimes f_{(1)} &= \sum f(m_{(0)})_{(0)} \otimes S^{-1}(m_{(1)})f(m_{(0)})_{(1)}. \end{aligned}$$

Two  $H$ -Azumaya algebras  $A$  and  $B$  are Brauer equivalent (denoted by  $A \sim B$ ) if there exist two faithfully projective YD  $H$ -modules  $M$  and  $N$  such that  $A \# \text{End}(M) \cong B \# \text{End}(N)$ . Note that  $A \sim B$  if and only if  $A$  is  $H$ -Morita equivalent to  $B$  (cf. [3, Theorem 2.10]). The Brauer group of the Hopf algebra  $H$  is denoted by  $\text{BQ}(k, H)$ . An element in  $\text{BQ}(k, H)$  represented by an  $H$ -Azumaya algebra  $A$  is indicated by  $[A]$ . The unit in  $\text{BQ}(k, H)$  is represented by  $\text{End}(M)$  for any faithfully projective YD  $H$ -module  $M$ .

If  $A$  is an  $H$ -Azumaya algebra, the left and right  $H$ -centers defined by

$$\{a \in A \mid ax = \sum x_{(0)}(x_{(1)} \cdot a), \forall x \in A\},$$

resp.

$$\{a \in A \mid xa = \sum a_{(0)}(a_{(1)} \cdot x), \forall x \in A\},$$

are trivial.

Now let  $H$  be a quasi-triangular Hopf algebra, that is,  $H$  is a Hopf algebra with an invertible element  $R = \sum R^{(1)} \otimes R^{(2)}$  in  $H \otimes H$  satisfying several axioms (e.g.

see [7]). If  $A$  is a left  $H$ -module algebra, then  $A$  is simultaneously a Yetter-Drinfel'd  $H$ -module algebra with the right  $H^{op}$ -comodule structure given by

$$A \longrightarrow A \otimes H^{op}, \quad a \mapsto \sum R^2 \cdot a \otimes R^1$$

for  $a \in A$ . The subset of  $\text{BQ}(k, H)$  consisting of the elements represented by the  $H$ -Azumaya algebras with right  $H^{op}$ -comodule structures stemming from left  $H$ -module structures in the above way, turns out to be a subgroup of  $\text{BQ}(k, H)$ , denoted by  $\text{BM}(k, H, R)$ . It is obvious that  $\text{BM}(k, H, R)$  contains the Brauer group  $\text{Br}(k)$ .

The Brauer group  $\text{BQ}(k, H)$  is a special case of the Brauer group  $\text{Br}(\mathcal{C})$  of a braided monoidal category  $\mathcal{C}$  as introduced in [10]. The fact that  $\text{BM}(k, H, R)$  is a subgroup of  $\text{BQ}(k, H)$  when  $(H, R)$  is a quasi-triangular Hopf algebra can be explained in a categorical way. If  $\mathcal{D}$  is a closed braided monoidal subcategory of a braided monoidal category  $\mathcal{C}$ , then the Brauer group  $\text{Br}(\mathcal{D})$  is a subgroup of  $\text{Br}(\mathcal{C})$ . This fact allows us to consider various subgroups of the Brauer group  $\text{Br}(\mathcal{C})$  of a braided monoidal category  $\mathcal{C}$  whenever  $\mathcal{C}$  contains certain closed braided subcategories. For example, if  $(H, R)$  is a quasi-triangular Hopf algebra, then the category  ${}_H\mathbf{M}$  of left  $H$ -modules is a closed braided subcategory of the braided category of YD  $H$ -modules with braiding  $\phi$  given by:

$$M \otimes N \longrightarrow N \otimes M, \quad m \otimes n \mapsto \sum R^2 \cdot n \otimes R^1 \cdot m,$$

where  $m \in M, n \in N$ . The Brauer group  $\text{Br}({}_H\mathbf{M})$  of  ${}_H\mathbf{M}$  is indeed  $\text{BM}(k, H)$ . When  $H$  is a finite dimensional Hopf algebra, it is well-known that the category of YD  $H$ -modules is equivalent to the category of left  $D(H)$ -modules, where  $D(H)$  is the Drinfel'd double of  $H$ . So we have that  $\text{BQ}(k, H)$  is equal to  $\text{BM}(k, D(H), R)$ , where  $R$  is the canonical quasi-triangular structure on  $D(H)$ .

### 3. ACTIONS ON AZUMAYA ALGEBRAS

In this section, we will consider actions of  $H_4$  on Azumaya algebras. Throughout  $k$  is a field with characteristic different from two. Let  $H_4$  be the Sweedler 4-dimensional Hopf algebra over  $k$ . That is,  $H_4$  is generated by two elements  $g$  and  $h$  such that

$$g^2 = 1, \quad h^2 = 0, \quad gh + hg = 0.$$

The comultiplication, the antipode and the counit of  $H_4$  are given by

$$\begin{aligned} \Delta(g) &= g \otimes g, & \Delta(h) &= 1 \otimes h + h \otimes g, \\ S(g) &= g, & S(h) &= gh, \\ \epsilon(g) &= 1, & \epsilon(h) &= 0. \end{aligned}$$

It is well-known that  $H_4$  is a quasi-triangular Hopf algebra with a family of quasi-triangular structures with respect to parameter  $t$  varying over  $k$ :

$$R_t = \frac{1}{2}(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g) + \frac{t}{2}(1 \otimes 1 + g \otimes g + 1 \otimes g - g \otimes 1)(h \otimes h).$$

When  $t$  is zero, then  $(H_4, R_0)$  is a triangular Hopf algebra. In this paper we will mainly concentrate on the case  $(H_4, R_0)$ .

An algebra  $A$  is called a *generalized quaternion algebra* if  $A$  is a four dimensional algebra generated by two elements  $u, v$  subject to the relations:

$$(1) \quad u^2 = \alpha, \quad v^2 = \beta, \quad uv + vu = \gamma$$

for some  $\alpha, \beta$  and  $\gamma$  in  $k$ .  $A$  is said to be *non-singular* if  $\gamma^2 - 4\alpha\beta \neq 0$  (in this case, it is isomorphic to a quaternion algebra). Otherwise  $A$  is *singular*.  $d = \gamma^2 - 4\alpha\beta$  is called the *determinant* of the generalized quaternion algebra  $A$ . We denote  $A$  by  $\left[\frac{\alpha, \beta, \gamma}{k}\right]$ .

There is a natural action of  $H_4$  on  $\left[\frac{\alpha, \beta, \gamma}{k}\right]$  given by

$$(2) \quad \begin{aligned} g \rightharpoonup u &= -u, & g \rightharpoonup v &= -v, & g \rightharpoonup (uv) &= uv, \\ h \rightharpoonup u &= 0, & h \rightharpoonup v &= 1, & h \rightharpoonup (uv) &= u \end{aligned}$$

such that  $\left[\frac{\alpha, \beta, \gamma}{k}\right]$  is a (left)  $H_4$ -module algebra (cf. [4]). We call the action (2) the *standard* action of  $H_4$  on  $\left[\frac{\alpha, \beta, \gamma}{k}\right]$ . This standard action makes  $\left[\frac{\alpha, \beta, \gamma}{k}\right]$  into an  $H_4^*$ -Galois extension of  $k$  (cf. [4]).

Now let us recall from [1] that an action of a Hopf algebra  $H$  on an algebra  $A$  is called *inner* if there is an invertible element  $\pi \in \text{Hom}(H, A)$ , the convolution algebra, such that

$$h \rightharpoonup a = \sum \pi(h_{(1)})a\pi^{-1}(h_{(2)})$$

whenever  $h$  is in  $H$  and  $a$  is in  $A$ . If, in addition,  $\pi$  is an algebra map, then the action is called a *strongly inner* action.

If  $A$  is an Azumaya algebra, then any action of  $H_4$  on  $A$  is an inner action by the Skolem-Noether theorem (cf. [8, 5]). If the inner action is induced by an invertible element  $\pi \in \text{Hom}(H_4, A)$ , then the two elements  $u = \pi^{-1}(g)$  and  $v = \pi^{-1}(h)$  in  $A$  satisfy the relation (1) for some  $\alpha \neq 0, \beta$  and  $\gamma$  in  $k$ . One may check that, in this case, the action of  $H_4$  on  $A$  is determined by  $u$  and  $v$  and reads as follows:

$$(3) \quad g(a) = u^{-1}au, \quad h(a) = av - vg(a)$$

for any  $a \in A$ . We call the subalgebra generated by the elements  $u$  and  $v$  the *induced subalgebra with respect to the action*. The induced subalgebra is unique though the elements  $u$  and  $v$  are not.

**Lemma 1.** *Let  $H_4$  act non-trivially on an Azumaya algebra  $A$ . Let  $u, v$  be the generators of the induced subalgebra of  $A$  such that  $u$  and  $v$  satisfy (3) and the relations in (1) for some  $\alpha \neq 0, \beta$  and  $\gamma$  in  $k$ . Then the action is strongly inner if and only if  $d = \gamma^2 - 4\alpha\beta$  is zero and  $\alpha$  is a square of some non-zero element in  $k$ .*

*Proof.* Assume that the action of  $H_4$  on  $A$  is strongly inner. Then there is an algebra map  $\delta : H_4 \rightarrow A$  such that

$$\begin{aligned} g \rightharpoonup a &= \delta(g)a\delta(g), \\ h \rightharpoonup a &= a\delta(h) + \delta(h)a\delta(g) = au'w + wau' \end{aligned}$$

for any  $a \in A$ , where  $\delta(g) = u'$ ,  $\delta(h) = w$  and

$$u'^2 = 1, \quad w^2 = 0, \quad u'w + wu' = 0.$$

On the other hand, we have that

$$g \rightharpoonup a = u^{-1}au, \quad h \rightharpoonup a = av - v(g \rightharpoonup a)$$

(see (3)) for any  $a \in A$ . Comparing these two actions, we obtain that

$$(4) \quad u = su', \quad w = \frac{1}{2}(u'v - vu')$$

where  $s \neq 0$  is in  $k$ . It follows that  $\alpha = s^2$  and that

$$0 = w^2 = \frac{1}{4s^2}(uv - vu)^2 = \frac{1}{4s^2}(\gamma^2 - 4\alpha\beta).$$

Therefore  $d = \gamma^2 - 4\alpha\beta = 0$ .

Now one may easily see that if  $d = 0$  and  $\alpha$  is a square of some non-zero element in  $k$ , then the elements  $u'$  and  $w$  given by (4) give a strongly inner action of  $H_4$  on  $A$ . □

As a consequence of Lemma 1, we obtain that when an inner action of  $H_4$  on an Azumaya algebra  $A$  is not strong and the action of  $g$  is strongly inner, then the induced subalgebra is a non-singular generalized quaternion algebra.

**Corollary 2.** *If an action of  $H_4$  on an Azumaya algebra  $A$  is not strongly inner and the action of  $g$  is strongly inner, then there are an Azumaya subalgebra  $B$  and a non-singular generalized quaternion algebra  $\left[\frac{\alpha, \beta, \gamma}{k}\right]$  in  $A$  such that*

$$A = \left[\frac{\alpha, \beta, \gamma}{k}\right] \otimes B$$

as  $H_4$ -module algebras, where  $H_4$  acts in a non-strongly inner way on  $\left[\frac{\alpha, \beta, \gamma}{k}\right]$  and acts trivially on  $B$ .

*Proof.* By assumption and Lemma 1,  $A$  contains a generalized quaternion algebra  $\left[\frac{\alpha, \beta, \gamma}{k}\right]$  which is non-singular. Since  $\left[\frac{\alpha, \beta, \gamma}{k}\right]$  is an Azumaya algebra, by the commutator theorem there is an Azumaya subalgebra  $B$  in  $A$  such that

$$A = \left[\frac{\alpha, \beta, \gamma}{k}\right] \otimes B.$$

Since  $B$  commutes with  $\left[\frac{\alpha, \beta, \gamma}{k}\right]$  and the inner action of  $H_4$  on  $A$  is induced by the two generators  $u, v$  of  $\left[\frac{\alpha, \beta, \gamma}{k}\right]$ ,  $H_4$  acts trivially on  $B$ . Thus the action on  $A$  is uniquely determined by the restricted action on the subalgebra  $\left[\frac{\alpha, \beta, \gamma}{k}\right]$ . □

#### 4. QUATERNION ALGEBRAS

In this section, we work with the non-singular generalized quaternion algebras which, over a field  $k$ , turn out to be quaternion algebras.

**Lemma 3.** *Let  $A$  be an  $H_4$ -module quaternion algebra. If the induced subalgebra is equal to  $A$ , then we can choose a basis for  $A$  such that the action of  $H_4$  is the standard action on  $A$ .*

*Proof.* Let  $u, v$  be the two generators of the induced subalgebra  $\left[\frac{\alpha, \beta, \gamma}{k}\right]$  which is equal to  $A$  by assumption. Set  $u' = uv - vu$  and  $v' = -\alpha d^{-1}u^{-1}u'$ , where  $d = \gamma^2 - 4\alpha\beta$ . Then

$$u'^2 = d, \quad v'^2 = -\alpha d^{-1}, \quad u'v' + v'u' = 0.$$

It follows that  $A \cong \left[\frac{d, -\alpha d^{-1}, 0}{k}\right]$ . Now one may check that, with respect to the basis  $\{1, u', v', u'v'\}$  of  $A$ , the action of  $H_4$  is the standard action, that is,

$$g(u') = -u', \quad g(v') = -v', \quad h(u') = 0, \quad h(v') = 1. \quad \square$$

**Lemma 4.** *Let  $A, B$  be two  $H_4$ -module algebras. Then the braided product  $A\#B$  is the same as the graded product of the graded algebras  $A$  and  $B$ .*

*Proof.* The braided product is determined by the (quasi-) triangular structure  $R_0$  in which the skew derivation  $h$  disappears. If we write  $A = A_0 + A_1$  and  $B = B_0 + B_1$ , where  $g(x_0) = x_0, g(x_1) = -x_1$  if  $x_i \in A_i, B_i, i = 0, 1$ , then  $R_0(a_i \otimes b_j) = (-1)^{ij} b_j \otimes a_i$ . By definition,  $R_0$  exactly induces the classic graded product  $A \hat{\otimes} B$  (see [12]).  $\square$

In the sequel, we use  $\left\langle \frac{\alpha, \beta, \gamma}{k} \right\rangle$  to indicate the generalized quaternion algebra  $\left[ \frac{\alpha, \beta, \gamma}{k} \right]$  with the standard  $H_4$ -action. The following proposition says that a generalized quaternion algebra with the standard  $H_4$ -action is  $H_4$ -Azumaya only if it is a quaternion algebra.

**Proposition 5.** *Let  $A = \left\langle \frac{\alpha, \beta, \gamma}{k} \right\rangle$  be a generalized quaternion algebra. Then  $A$  is an  $H_4$ -Azumaya algebra if and only if the determinant  $d = \gamma^2 - 4\alpha\beta$  is not zero.*

*Proof.* Assume that  $d$  is not equal to zero. Then  $A$  is an Azumaya algebra in the usual sense. Consider the  $H_4$ -opposite algebra  $\bar{A}$  which is generated by  $\bar{u}$  and  $\bar{v}$  subject to the relations:

$$\bar{u}^2 = -u^2 = -\alpha, \quad \bar{v}^2 = -v^2 = -\beta, \quad \bar{u}\bar{v} + \bar{v}\bar{u} = -\gamma.$$

It turns out that  $\bar{A}$  is equal to  $\left\langle \frac{-\alpha, -\beta, -\gamma}{k} \right\rangle$ . Since  $A$  and  $\bar{A}$  are graded Clifford algebras and  $A\#\bar{A}$  is the graded product,  $A\#\bar{A}$  is still a graded Clifford algebra and hence an Azumaya algebra in the usual sense. Now the canonical  $H_4$ -linear map

$$F : A\#\bar{A} \longrightarrow \text{End}(A)$$

given by  $F(a\#\bar{b})(x) = \sum a(R^2 \cdot x)(R^1 \cdot b)$ , where  $a, x \in A$  and  $\bar{b} \in \bar{A}$ , is an isomorphism because  $F$  sends the center  $k$  of  $A\#\bar{A}$  onto the center of  $\text{End}(A)$ . Similarly, the other canonical  $H_4$ -linear map

$$G : \bar{A}\#A \longrightarrow \text{End}(A)^{op}$$

is an isomorphism too. So we have proved that  $A = \left\langle \frac{\alpha, \beta, \gamma}{k} \right\rangle$  is an  $H_4$ -Azumaya algebra.

Conversely, if  $\left\langle \frac{\alpha, \beta, \gamma}{k} \right\rangle$  is an  $H_4$ -Azumaya algebra, then its left (or right)  $H_4$ -center is trivial. Let  $x = s_1 + s_2u + s_3v + s_4w$  be an element in the  $H_4$ -center of  $A$ , where  $w = uv, s_i \in k$ . By definition, we have

$$xy = \sum (R^2 \cdot y)(R^1 \cdot x)$$

for any  $y \in A$ . Choose  $y$  to be the element  $u$ . We obtain

$$(\alpha s_2 + \gamma s_3) + (s_1 + \gamma s_4)u - \alpha s_4v - s_3w = -\alpha s_2 + s_1u + s_4v - s_3w.$$

This amounts to the linear equations:

$$\begin{cases} 2\alpha s_2 + \gamma s_3 = 0, \\ \gamma s_4 = 0, \\ \alpha s_4 = 0. \end{cases}$$

Since  $\alpha$  and  $\gamma$  cannot be zero at the same time, we have  $s_4 = 0$ . Similarly,  $xv = \sum(R^2 \cdot v)(R^1 \cdot x)$  implies that

$$\gamma s_2 + 2\beta s_3 = 0.$$

Now the uniqueness of the solution for  $s_2$  and  $s_3$  yields that the determinant  $d = \gamma^2 - 4\alpha\beta$  is not equal to zero.  $\square$

As a consequence of Corollary 2 and Proposition 5, we obtain the following:

**Corollary 6.** *If  $A$  is an Azumaya algebra on which  $H_4$  acts in a non-strongly inner way and  $g$  acts in a strongly inner way, then  $A$  is an  $H_4$ -Azumaya algebra.*

### 5. THE BRAUER GROUP $\text{BM}(k, H_4, R_0)$

In this section, we calculate the Brauer group  $\text{BM}(H_4, R_0, k)$ . We first consider the  $2 \times 2$  matrix algebra  $M_2$  which is isomorphic to the quaternion algebra  $\left[\frac{\alpha, -\alpha, 0}{k}\right]$  for any non-zero element  $\alpha \in k$ . However, when we endow  $\left[\frac{\alpha, -\alpha, 0}{k}\right]$  with the standard  $H_4$ -action, the  $H_4$ -module algebra  $\left\langle \frac{\alpha, -\alpha, 0}{k} \right\rangle$  is no longer isomorphic to  $\left\langle \frac{\beta, -\beta, 0}{k} \right\rangle$  as  $H_4$ -module algebras whenever  $\alpha$  is not equal to  $\beta$ . This is the case when  $\left\langle \frac{\alpha, -\alpha, 0}{k} \right\rangle$  and  $\left\langle \frac{\beta, -\beta, 0}{k} \right\rangle$  represent different elements in the Brauer group  $\text{BM}(H_4, R_0, k)$ . Actually, we have a group embedding of the abelian group  $(k, +)$  into  $\text{BM}(k, H_4, R_0)$ . We let  $M_2$  stand for the  $2 \times 2$  matrix algebra with the trivial  $H_4$ -action; in other words,  $[M_2]$  represents the unit in  $\text{BM}(k, H_4, R_0)$ .

Let  $\Phi$  be a map from  $(k, +)$  to  $\text{BM}(k, H_4, R_0)$  sending 0 to  $[M_2]$  and  $\alpha \neq 0$  to  $\left[\left\langle \frac{\alpha^{-1}, -\alpha^{-1}, 0}{k} \right\rangle\right]$ .

**Proposition 7.**  $\Phi$  is a group monomorphism from  $(k, +)$  into  $\text{BM}(k, H_4, R_0)$ .

*Proof.* We first compute the action of  $H_4$  on the braided product  $\left\langle \frac{\alpha^{-1}, -\alpha^{-1}, 0}{k} \right\rangle \# \left\langle \frac{\beta^{-1}, -\beta^{-1}, 0}{k} \right\rangle$ . Let  $\{u, v\}$  and  $\{u', v'\}$  be the generators of  $\left\langle \frac{\alpha^{-1}, -\alpha^{-1}, 0}{k} \right\rangle$  and of  $\left\langle \frac{\beta^{-1}, -\beta^{-1}, 0}{k} \right\rangle$  respectively. One may take a while to check that the inner action of  $H_4$  on  $\left\langle \frac{\alpha^{-1}, -\alpha^{-1}, 0}{k} \right\rangle \# \left\langle \frac{\beta^{-1}, -\beta^{-1}, 0}{k} \right\rangle$  is induced by two elements:

$$U = sw\#w', \quad V = -\frac{\alpha}{2}v\#1 - \frac{\beta}{2}1\#v' + tw\#w'$$

where  $w = uv, w' = u'v', s (\neq 0)$  and  $t$  vary over  $k$ . Choose  $s = \alpha\beta$  and  $t = 0$ . Then  $U^2 = 1$  and  $V^2 = -\frac{1}{4}(\alpha + \beta)$ . We may assume that  $\alpha \neq -\beta$ . Otherwise, we have that  $\left\langle \frac{\beta^{-1}, -\beta^{-1}, 0}{k} \right\rangle = \left\langle \frac{\alpha^{-1}, -\alpha^{-1}, 0}{k} \right\rangle$  and  $\left\langle \frac{\alpha^{-1}, -\alpha^{-1}, 0}{k} \right\rangle \# \left\langle \frac{\beta^{-1}, -\beta^{-1}, 0}{k} \right\rangle \cong \text{End}\left(\left\langle \frac{\alpha^{-1}, -\alpha^{-1}, 0}{k} \right\rangle\right)$  which represents the unit in  $\text{BM}(k, H_4, R_0)$ .

Assume that  $\alpha + \beta \neq 0$ . Let  $U' = 2UV$  and  $V' = -2(\alpha + \beta)^{-1}V$ . Then  $U'$  and  $V'$  generate the quaternion algebra  $\left\langle \frac{\sigma^{-1}, -\sigma^{-1}, 0}{k} \right\rangle$  with the standard action, where  $\sigma = \alpha + \beta$ . By Corollary 2, we obtain

$$\left\langle \frac{\alpha^{-1}, -\alpha^{-1}, 0}{k} \right\rangle \# \left\langle \frac{\beta^{-1}, -\beta^{-1}, 0}{k} \right\rangle \cong \left\langle \frac{\sigma^{-1}, -\sigma^{-1}, 0}{k} \right\rangle \otimes M_2$$

where  $\sigma = \alpha + \beta$ . It follows that

$$\left\langle \frac{\alpha^{-1}, -\alpha^{-1}, 0}{k} \right\rangle \# \left\langle \frac{\beta^{-1}, -\beta^{-1}, 0}{k} \right\rangle = \left[ \left\langle \frac{(\alpha + \beta)^{-1}, -(\alpha + \beta)^{-1}, 0}{k} \right\rangle \right]$$

in the Brauer group  $\text{BM}(k, H_4, R_0)$ . So we have proved that  $\Phi$  is a group morphism. Finally, since the standard  $H_4$ -action on a quaternion algebra is not strongly inner, we have  $\left[ \left\langle \frac{\alpha^{-1}, -\alpha^{-1}, 0}{k} \right\rangle \right] \neq 1$  whenever  $\alpha$  is not equal to zero.  $\square$

Now we are ready to compute the Brauer group  $\text{BM}(k, H_4, R_0)$ . Let  $A$  be an  $H_4$ -module algebra. As mentioned in the proof of Lemma 4,  $A$  has a natural  $\mathbb{Z}_2$ -gradation:

$$A_0 = \{x \in A \mid g(x) = x\}, \quad A_1 = \{x \in A \mid g(x) = -x\}.$$

If  $A$  is an  $H_4$ -Azumaya algebra,  $A = A_0 + A_1$  is  $\mathbb{Z}_2$ -graded Azumaya algebra in the sense of Wall. This is so because the canonical isomorphism

$$F : A \# \overline{A} \longrightarrow \text{End}(A)$$

is automatically a  $\mathbb{Z}_2$ -graded isomorphism. Thus the forgetful map

$$\Psi : [A] \longrightarrow [A]$$

yields a homomorphism from  $\text{BM}(k, H_4, R_0)$  to the Brauer-Wall group  $\text{BW}(k)$  (for details on  $\text{BW}(k)$ , we refer to [12]). Here  $\Psi$  is a homomorphism because of Lemma 4.

On the other hand, if  $B$  is a  $\mathbb{Z}_2$ -graded Azumaya algebra in the sense of Wall, then we can endow  $B$  with a natural  $H_4$ -module structure in the following way:

$$g \curvearrowright x_i = (-1)^i x_i, \quad h \curvearrowright x_i = 0$$

where  $x_i$  is a homogeneous element in  $B_i$ ,  $i = 0, 1$ . One may easily check that  $B$  is an  $H_4$ -Azumaya algebra. So we obtain that  $\text{BW}(k)$  is a subgroup of  $\text{BM}(k, H_4, R_0)$ . In fact, the split map  $\Psi$  can be easily explained in a categorical way as follows. Since  $R_0$  is a (quasi-) triangular structure of  $k\mathbb{Z}_2$ , we have (quasi-) triangular Hopf algebra maps:

$$(k\mathbb{Z}_2, R_0) \xrightarrow{\text{incl.}} (H_4, R_0) \xrightarrow{\text{proj.}} (k\mathbb{Z}_2, R_0)$$

which induce braided monoidal functors:

$$(5) \quad k\mathbb{Z}_2 \mathbf{M} \longleftarrow_{H_4} \mathbf{M} \longleftarrow_{k\mathbb{Z}_2} \mathbf{M}$$

whose composite functor is the identity functor. Since the Brauer groups are functorial (see [10]), the monoidal functors of (5) induce group morphisms:

$$\text{BM}(k, k\mathbb{Z}_2, R_0) \longleftarrow \text{BM}(k, H_4, R_0) \longleftarrow \text{BM}(k, k\mathbb{Z}_2, R_0)$$

whose composite map is the identity map. One may identify  $\text{BM}(k, k\mathbb{Z}_2, R_0)$  with  $\text{BW}(k)$  by Lemma 4. Now we are able to state our main theorem.

**Theorem 8.**

$$(6) \quad 1 \longrightarrow (k, +) \xrightarrow{\Phi} \text{BM}(k, H_4, R_0) \xrightarrow{\Psi} \text{BW}(k) \longrightarrow 1$$

is a split and exact sequence of group morphisms.

*Proof.* It suffices to prove that  $\Phi(k, +)$  is the kernel of  $\Psi$ . Since a  $\mathbb{Z}_2$ -graded matrix algebra represents the unit in the Brauer-Wall group, we have that  $\Phi(k, +) \subseteq \text{Ker}(\Psi)$ . Conversely, assume that  $A$  is an  $H_4$ -Azumaya algebra such that  $\Psi([A]) = 1$  in  $\text{BW}(k)$  and  $[A] \neq 1$  in  $\text{BM}(k, H_4, R_0)$ . Since  $\Psi([A]) = 1$ , the action of  $g$  on  $A$  is strongly inner, and we may identify  $A$  with a graded matrix algebra when we forget the action of the skew-derivation  $h$ , say,  $\langle M_n \rangle$ . Since  $[A] \neq 1$  in  $\text{BM}(k, H_4, R_0)$ , we know that  $H_4$  acts in a non-strongly inner way on  $A$ . Thus we may apply Corollary 2 to obtain:

$$\langle M_n \rangle = \left[ \frac{\alpha, \beta, \gamma}{k} \right] \otimes B$$

for some Azumaya algebra  $B$  with the trivial  $H_4$ -action. This implies that  $\left[ \frac{\alpha, \beta, \gamma}{k} \right]$  is not a non-trivial quaternion algebra because  $\alpha$  is a square of some non-zero element in  $k$ . So  $\left[ \frac{\alpha, \beta, \gamma}{k} \right]$  must be the  $2 \times 2$ -matrix algebra. Hence  $B$  is a matrix algebra as well and represents the unit in  $\text{BM}(k, H_4, R_0)$ .

Now we may choose  $\alpha$  equal to  $d^2$ . By applying Lemma 3 we obtain that

$$\left[ \frac{1, \beta, \gamma}{k} \right] \cong \left\langle \frac{d, -d, 0}{k} \right\rangle$$

where  $d$  is the determinant. So we get  $[A] = \left[ \left\langle \frac{d, -d, 0}{k} \right\rangle \right] = \Phi(d^{-1})$  for some non-zero element  $d$ .  $\square$

From the exact sequence (6) we get the whole Brauer group  $\text{BM}(k, H_4, R_0)$  which is the direct product of  $(k, +)$  with  $\text{BW}(k)$  that is easily computed (see [12]). Finally we remark that the Brauer group  $\text{BM}(k, H_4, R_t)$ ,  $t \neq 0$ , is much more complicated than  $\text{BM}(k, H_4, R_0)$ . The full Brauer group  $\text{BQ}(k, H_4)$  is a very rich group containing subgroups like  $\text{BM}(k, H_4, R_0)$ ,  $\text{BM}(k, H_4, R_1)$ ,  $BC(k, H_4, R'_t)$  and  $k^*/\mathbb{Z}_2$ , where  $\{R'_t\}$  is a family of coquasi-triangular structures of  $H_4$ . The investigation of  $\text{BQ}(k, H_4)$  will be carried out in the forthcoming paper.

#### ACKNOWLEDGMENT

We thank the referee very much for many valuable comments.

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