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A NOTE ON BRANCHING THEOREMS

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Abstract. Let G be a complex, simply connected semisimple analytic group with K a closed connected reductive subgroup. Suppose V is an irreducible holomorphic G-module and W an irreducible holomorphic K-module. We prove that $\mathrm{Hom}_K(W,V)$ possesses the structure of an irreducible $U(\mathfrak{g})^K$-module whenever $\mathrm{Hom}_K(W,V)$ is $\neq (0)$. Moreover, $\dim \mathrm{Hom}_K(W,V) \leq 1$ for all W and V if and only if $U(\mathfrak{g})^K$ is commutative.

1. Introduction

Suppose G is a complex, semisimple, simply connected, analytic group with K a closed, reductive, complex, analytic subgroup. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the respective Lie algebras of G and K with $U(\mathfrak{g})$ and $U(\mathfrak{k})$ their corresponding enveloping algebras. Set $M(G)$ equal to the set of equivalence classes of irreducible holomorphic G-modules, and similarly, let $M(K)$ be the corresponding set of equivalence classes of K-modules (as usual we identify elements of $M(G)$ or $M(K)$ with representations of the class). Finally, let $Z = U(\mathfrak{g})^K$ be the ring of AdK-invariant elements of $U(\mathfrak{g})$.

Given $V \in M(G)$ we have that as a K-module

$$V = \bigoplus_{W \in M(K)} i(W,V)W$$

where $i(W,V) = \dim \mathrm{Hom}_K(W,V)$. We now state the main result of this note.

Theorem. *The ring Z is commutative if and only if $i(W,V) \leq 1$ for all $W \in M(K)$ and $V \in M(G)$.*

Although this theorem is folklore to anyone who has studied branching results, a proof does not appear in the literature. Our purpose here is to provide a simple proof of this theorem.

Weyl's branching theorems tell us that

$$i(W,V) \leq 1 \text{ if } (G,K) = (SL(n+1,\mathbb{C}), GL(n,\mathbb{C})) \text{ or}$$

$$(G,K) = (\mathrm{Spin}(n+1,\mathbb{C}),\ \mathrm{Spin}(n,\mathbb{C})).$$

A result of Knop [K] tells us that Z is commutative only when (G,K) is one of these two cases or a product of them.

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2. The proof

Given $V \in M(G)$ and $W \in M(K)$, $\mathrm{Hom}_K(W,V)$ possesses the structure of a Z-module. If $D \in Z, T \in \mathrm{Hom}_K(W,V)$, then $D(T)(w) = D(Tw)$ for $w \in W$. Hence $\mathrm{Hom}_K(W,V) \otimes W$ becomes a (Z,K)-module, and thus

$$\widetilde{V} = \bigoplus_{W \in M(K)} \mathrm{Hom}_K(W,V) \otimes W$$

has the structure of a (Z,K)-module.

Consider the linear evaluation map $E : \widetilde{V} \to V$ such that $E(T \otimes w) = T(w)$ if $T \in \mathrm{Hom}_K(W,V)$ and $w \in W$.

Proposition 2.1. *The map E is a (Z,K)-isomorphism.*

Proof. It is easily seen that E intertwines the action of Z, and K, and by definition E is an isomorphism.

We now see that $\widetilde{V}$ has the structure of a G-module compatible with the given (Z,K)-structure. Moreover, $\widetilde{V}$ is equivalent to V as a G-module. Let π denote the corresponding representation of $G, \mathfrak{g}$ and $U(\mathfrak{g})$ on $\widetilde{V}$. $\square$

Proposition 2.2. $\mathrm{End}_K(\widetilde{V}) = \pi(Z)$.

Proof. Clearly, $\pi(Z) \subset \mathrm{End}_K(\widetilde{V})$. Suppose $S \in \mathrm{End}_K(\widetilde{V})$. By Burnside's theorem $\pi(U(\mathfrak{g})) = End(\widetilde{V})$. Hence there is a $D \in U(\mathfrak{g})$ such that $\pi(D) = S$. Since K is reductive, there is a compact analytic group $K_0 \subset K$ whose Lie algebra is a real form of $\mathfrak{k}$. Then integrating with respect to the normalized Haar integral of K_0, we have

$$D_0 = \int_{K_0} Adk(D)dk$$

is in Z and $\pi(D_0) = \pi(D) = S$. $\square$

Proposition 2.3. $\mathrm{End}_K(\widetilde{V}) = \mathrm{End}(\mathrm{Hom}_K(W,V)) \otimes I_W$.

Proof. Clearly

$$\mathrm{End}_K(\widetilde{V}) = \bigoplus_{W \in M(K)} \mathrm{End}_K(\mathrm{Hom}_K(W,V) \otimes W).$$

Since $\mathrm{End}_K(\mathrm{Hom}_K(W,V) \otimes W) = \mathrm{End}(\mathrm{Hom}_K(W,V)) \otimes I_W$, we are done. $\square$

Proof of Theorem 1. If Z is commutative, $\mathrm{End}(\mathrm{Hom}_K(W,V))$ is commutative for any $V \in M(G)$ and any $W \in M(K)$. Thus $i(W,V) \leq 1$ for any $V \in M(G)$ and any $W \in M(K)$. On the other hand suppose $i(W,V) \leq 1$ for all W and V. Since $i(W,V) \leq 1$ for all $V \in M(G)$ and all $W \in M(K), \mathrm{End}(\mathrm{Hom}_K(W,V))$ is abelian and so $\mathrm{End}_K(V)$ is abelian. Thus $[Z,Z]$ is 0 on V for any $V \in M(G)$. From 2.5.7 of [D], Z maps injectively into the direct product of the $\mathrm{End}_K(V)$ $(V \in M(G))$. Hence $[Z,Z] = 0$, and Z is abelian. $\square$

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