POSITIVE SOLUTIONS OF A DEGENERATE ELLIPTIC EQUATION WITH LOGISTIC REACTION

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Abstract. The degenerate elliptic equation \( \lambda \Delta_p u + u^{q-1}(1 - u^r) = 0 \) with zero Dirichlet boundary condition, where \( \lambda \) is a positive parameter, \( 2 < p < q \) and \( r > 0 \), is studied in three aspects: existence of maximal solution, \( \lambda \)-dependence of maximal solution and multiplicity of solutions.

1. Introduction and Results

Let \( \Omega \) be a connected, bounded open subset of \( \mathbb{R}^N, N \geq 2 \), with \( C^{2,\alpha} \)-boundary \( \partial \Omega \) for some \( \alpha \in (0,1) \). We consider the following degenerate elliptic equation:

\[
(P)_{\lambda, \Omega}
\begin{cases}
\lambda \Delta_p u + f(u) = 0 & \text{in } \Omega, \\
u \geq 0, \neq 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( \lambda \) is a positive parameter and \( \Delta_p \) is the \( p \)-Laplace operator defined by

\[
\Delta_p u = \text{div}(\nabla |\nabla u|^{p-2}\nabla u)
\]

with \( p > 2 \) and \( f \) is given by

\[
f(u) = u^{q-1}(1 - u^r)
\]

with \( q \geq 2 \) and \( r > 0 \). We often write \( (P)_\lambda \) instead of \( (P)_{\lambda, \Omega} \). A function \( u = u_\lambda \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \) is called a solution of \( (P)_\lambda \) if \( u \geq 0 \) a.e. in \( \Omega \), \( u \) does not vanish in a set of positive measure, and

\[
-\lambda \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla \varphi dx + \int_\Omega f(u)\varphi dx = 0
\]

for all \( \varphi \in W^{1,p}_0(\Omega) \). A solution \( u \) of \( (P)_\lambda \) is called a maximal solution of \( (P)_\lambda \) if \( u \geq v \) a.e. in \( \Omega \) for all solutions \( v \) of \( (P)_\lambda \). Obviously, a maximal solution is decided uniquely. If a function \( u \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \) satisfies \( u \geq 0 \) (resp. \( u \leq 0 \)) on \( \partial \Omega \) and

\[
-\lambda \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla \varphi dx + \int_\Omega f(u)\varphi dx \leq 0 \quad (\text{resp. } \geq 0)
\]

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for all \( \varphi \in W^{1,p}_0(\Omega) \) satisfying \( \varphi \geq 0 \) a.e. in \( \Omega \), then it is called an upper (resp. a lower) solution of (P)\(_\lambda\).

With respect to (P)\(_\lambda\), there are a few works on the equidiffusive case \( p = q \) as follows. Let \( \lambda_1 \) be the first eigenvalue of \(-\Delta_\rho\) under zero Dirichlet boundary condition. In the one-dimensional case \( N = 1 \), Guedda and Véron \cite{10} have shown by phase-plane analysis that if \( \lambda < 1/\lambda_1 \), then (P)\(_\lambda\) has a unique solution \( u_\lambda \), and that a set called the flat core of \( u_\lambda \),

\[
O_\lambda = O_\lambda(u_\lambda) := \{ x \in \Omega; u_\lambda(x) = 1 \}, \tag{1.2}
\]
is non-empty for sufficiently small \( \lambda \). Since the length of \( O_\lambda \) can be indicated explicitly, we can see that as \( \lambda \to 0 \), \( O_\lambda \) spreads out toward the whole of \( \Omega \) with the growth as

\[
\lim_{\lambda \to 0} \lambda^{-1/p} \text{dist}(O_\lambda, \partial \Omega) = C(f, p), \tag{1.3}
\]
where \( C(f, p) = (p_1^{-1/p})^{1/p} \int_1^1 (F(1) - F(s))^{-1/p} ds \) and \( F(s) = \int_s^1 f(t) dt \). In the higher-dimensional case \( N \geq 2 \), phase-plane analysis is no longer useful and one has to use other methods. Constructing a suitable lower solution by using the eigenfunction for \( \lambda_1 \), Kamin and Véron \cite{12} have proved that the unique solution of (P)\(_\lambda\) has a flat core for sufficiently small \( \lambda \) and extended the results of \cite{10}. However, they have given only an estimate \( \text{dist}(O_\lambda, \partial \Omega) \leq C\lambda^{1/p} \) as \( \lambda \to 0 \), where \( C \) is a constant independent of \( \lambda \), without explicit information about \( C \) and any estimate of \( \text{dist}(O_\lambda, \partial \Omega) \) from below. In virtue of an exact estimate for \( O_\lambda \), García-Melián and Sabina de Lis \cite{9} have utilized the solutions for \( N = 1 \), whose dependence on \( \lambda \) is understood well, to make upper and lower solutions and concluded that \( \text{dist}(O_\lambda, \partial \Omega) \) also holds true in the case \( N \geq 2 \). The subdiffusive case \( p > q \) can also be investigated in the same way as the equidiffusive case. One can observe that there exists a unique solution \( u_\lambda \) for every \( \lambda > 0 \) and that as the equidiffusive case, \( O_\lambda(u_\lambda) \) is nonempty for sufficiently small \( \lambda > 0 \) and it grows as in \( \text{dist}(O_\lambda, \partial \Omega) \). See the author and Yamada \cite{19} for \( N = 1 \) and \cite{9} with its Remarks 2.2 b for \( N \geq 2 \). For uniqueness, see also Diaz and Saa \cite{5}.

On the other hand, the structure of solution set in the superdiffusive case \( p < q \) is essentially different from those in the other cases. For \( N = 1 \), using time-map, the author and Yamada \cite{19} have shown that (P)\(_\lambda\) produces a spontaneous bifurcation for \( \lambda \). That is, there exists \( \Lambda > 0 \) such that if \( \lambda > \Lambda \), then (P)\(_\lambda\) has no solution; if \( \lambda = \Lambda \), then (P)\(_\lambda\) has a unique solution; if \( \lambda < \Lambda \), then (P)\(_\lambda\) has exactly two distinct solutions \( u_\lambda \) and \( \overline{u}_\lambda \) satisfying \( u_\lambda > \overline{u}_\lambda \) in \( \Omega \). It also follows from our analysis that as \( \lambda \to 0 \), \( O_\Lambda(u_\lambda) \) spreads out toward the whole of \( \Omega \) with \( \text{dist}(O_\lambda, \partial \Omega) \) and \( u_\lambda \to 0 \) uniformly in \( \Omega \). For \( N \geq 2 \), Guo \cite{11} has studied the case that there exists \( \beta > 0 \) such that \( f(0) = f(\beta) = 0 \), \( (\beta - x) f(x) > 0 \) in \((0, \beta) \cup (\beta, +\infty)\), \( \lim_{s \to 0} f(s) / s^{p-1} = 0 \) and \( f(s) / s^{p-1} \) \( < 0 \) in \((0, \beta)\) (the condition \( f''(x) < 0 \) in \cite{11} Theorem 3.3) is a misprint and should be replaced by \( f''(x)/x^{p-1} < 0 \), and have found two distinct solutions. This is the case \( p < q < p + 1 \) in our problem and no information about the shape of solutions is given. In the present paper, we will discuss (P)\(_\lambda\) in the case \( 2 < p < q \), \( N \geq 2 \), and study (P)\(_\lambda\) in three aspects: (a) existence of solution, especially maximal solution; (b) \( \lambda \)-dependence of maximal solution; and (c) multiplicity of solutions. As for (a), we can prove the following theorem by the method of upper and lower solutions:
Theorem 1.1. Let $2 < p < q$ and $r > 0$. Then there exists a positive number $\lambda > 0$ such that

(i) if $\lambda > \lambda$, then $(P)_\lambda$ has no solution;
(ii) if $\lambda \leq \lambda$, then $(P)_\lambda$ has a maximal solution $\overline{u}_\lambda$;
(iii) if $\lambda_1 < \lambda_2 \leq \lambda$, then $\overline{u}_{\lambda_2} \leq \overline{u}_{\lambda_1}$;
(iv) the mapping $\lambda \mapsto \overline{u}_\lambda$ is left-continuous on $(0, \lambda]$ in $C^{1,\beta'}(\overline{\Omega})$ for any $\beta' \in (0, \beta)$, where $\beta$ is the constant appearing in Proposition 2.1.

Remark 1.1. Theorem 1.1 (i) has been obtained by Véron [21, Theorem 3] for the $p$-Laplace operator on a compact Riemannian manifold without boundary.

We will state our result on (b). The proof essentially consists of constructing suitable upper and lower solutions by the idea of García-Melián and Sabina de Lis [8, 9] and the one-dimensional result in [19].

Theorem 1.2. Let $2 < p < q$ and $r > 0$. There exists a positive number $\lambda^* \in (0, \lambda]$ such that

(i) if $\lambda \leq \lambda^*$, then $\mathcal{O}_\lambda = \mathcal{O}_\lambda(\overline{u}_\lambda)$ is non-empty;
(ii) if $\lambda_1 < \lambda_2 \leq \lambda^*$, then $\mathcal{O}_{\lambda_1} \subset \mathcal{O}_{\lambda_2}$;
(iii) for sufficiently small $\varepsilon > 0$, there exists $\lambda \leq \lambda^*$ such that $\Omega \setminus \Omega_\varepsilon \subset \mathcal{O}_\lambda$, where $\Omega_\varepsilon := \{x \in \Omega; \text{dist}(x, \partial \Omega) < \varepsilon\}$.

Furthermore, $\mathcal{O}_\lambda$ satisfies (1.5) as $\lambda \to 0$.

Remark 1.2. From the last assertion of Theorem 1.2, we can see that the growth order of maximal solution of $(P)_\lambda$ when $\lambda \to 0$ is same as that of case $p \geq q$.

To mention (c), we define the functional $\Phi$ on $W^{1, p}_0(\Omega)$ corresponding with $(P)_\lambda$:

$$\Phi(u) = \frac{\lambda}{p} \|\nabla u\|_p^p - \int_{\Omega} F(u)ds,$$

where $F(u) = \int_0^u f(s)ds$ and $\overline{f}(s) := f(s)$ in $[0, 1]$, $\bar{f} := 0$ in $\mathbb{R} \setminus [0, 1]$. By the Mountain Pass Theorem (cf. [11, 17]) for $\Phi$, we can find a distinct solution from $\overline{u}_\lambda$ for small $\lambda (< \lambda)$, and consequently deduce the multiplicity of solutions. At this time, it plays an important role that $\Phi(\overline{u}_\lambda)$ becomes negative if $\lambda$ is sufficiently small. In other words, the larger $\mathcal{O}_\lambda(\overline{u}_\lambda)$ spreads out, the more $\Phi(\overline{u}_\lambda)$ decreases.

Theorem 1.3. Let $2 < p < q$ and $r > 0$. There exists a positive number $\Lambda \in (0, \lambda]$ such that if $\lambda < \Lambda$, then $(P)_\lambda$ has another solution $u_\lambda \leq \overline{u}_\lambda$, $\neq \overline{u}_\lambda$.

Remark 1.3. We expect that a solution distinct from $\overline{u}_\lambda$ exists for all $\lambda \in (0, \lambda)$. Theorem 1.3 whose proof directly utilizes the growth of flat hat, gives a partial result for this problem. It will be discussed in the forthcoming paper [18]. (See also Remark 2.2). In connection with multiplicity for the $p$-Laplace operator, we can refer to Ambrosetti, Garcia Azorero and Peral [2], Drábek and Pohozaev [7].

2. Proofs of results

The following proposition is fundamental in this paper.

Proposition 2.1. Let $u$ be a solution of $(P)_\lambda$. Then $u \in C^{1,\beta}(\overline{\Omega}) \cap C^{2,\alpha}(\Omega_\varepsilon)$ for some $\beta \in (0, 1)$ and sufficiently small $\varepsilon > 0$, and $0 < u(x) \leq 1$ for all $x \in \Omega$. 


Proof of Proposition 2.1. Let \( u \) be any solution of \((P)_\lambda\). Putting \( \varphi = (u - 1)^+ \) := max\{u - 1, 0\} \( \in W^{1,p}_0(\Omega) \) in (1.1), we have

\[
\lambda \int_{\Omega} |(u - 1)^+|^p \, dx = \int_{(u > 1)} f(u)(u - 1) \, dx \leq 0.
\]

Hence \((u - 1)^+ = 0 \) a.e. in \( \Omega \); so \( u(x) \leq 1 \) a.e. in \( \Omega \). This boundedness and a regularity result of Lieberman [13, Theorem 1] deduce that \( u \in C^{1,\beta}(\overline{\Omega}) \) for some \( \beta \in (0, 1) \). Thus, it follows from Vázquez’s maximum principle [20, Theorem 5] that \( 0 < u \leq 1 \) in \( \Omega \) and

\[
\begin{align*}
\partial u / \partial n(x) &< 0 \quad \text{on } \partial \Omega, \\
\end{align*}
\]

where \( n \) denotes an outer normal at \( \partial \Omega \). By (2.1) and the fact that \( |\nabla u| \in C^{0,\beta}(\overline{\Omega}) \), there exists \( \varepsilon_0 > 0 \) such that \( |\nabla u| \geq \delta > 0 \) in \( \Omega_{\varepsilon_0} \) for some \( \delta > 0 \). Therefore, since the equation of \((P)_\lambda\) in \( \Omega_{\varepsilon_0} \) becomes a strictly elliptic one, we can conclude from classical theory that \( u \in C^{2,\alpha}(\Omega_{\varepsilon}) \) for all \( \varepsilon \in (0, \varepsilon_0) \).

Lemma 2.1. For sufficiently small \( \lambda > 0 \), there exists a maximal solution \( \overline{w}_\lambda \) such that \( \mathcal{O}_\lambda \) is non-empty and

\[
\limsup_{\lambda \to 0} \lambda^{-1/p} \text{dist}(\mathcal{O}_\lambda, \partial \Omega) \leq C(f, p).
\]

Proof. Take \( R > 0 \) and \( x_0 \in \Omega \) satisfying \( B_R(x_0) \subset \Omega \), where \( B_R(x_0) \) is the ball with radius \( R \) and center at \( x_0 \). To obtain a lower solution of \((P)_\lambda \), we will construct a lower solution \( v_{R,x_0} \) of \((P)_\lambda \). It suffices to find a radially symmetric one, i.e., \( v(\rho) = v_{R,x_0}(x) \) satisfying

\[
\begin{align*}
\lambda(\rho^{N-1} |v_\rho|^{p-2} v_\rho)_\rho + \rho^{N-1} f(v) &\geq 0 \quad \text{in } (0, R), \\
v_\rho(0) = v(R) &= 0,
\end{align*}
\]

where \( \rho = |x - x_0| \). By a change of variable \( \xi = g(\rho) \) such that

\[
\xi = g(\rho) = \begin{cases} 
\frac{R^{1-\theta} - \rho^{1-\theta}}{\theta} & \text{if } \theta \neq 1, \\
\log \frac{R}{\rho} & \text{if } \theta = 1,
\end{cases}
\]

where \( \theta := (N - 1)/(p - 1) \), (2.3) can be rewritten as follows:

\[
\begin{align*}
\lambda(|w_\xi|^p - w_\xi)_\xi + g^{-1}(\xi)^p f(w) &\geq 0 \quad \text{in } (0, T), \\
w(0) = w_\xi(T) &= 0,
\end{align*}
\]

where \( w(\xi) = v(g^{-1}(\xi)) \) and \( T = +\infty \) if \( \theta \geq 1 \), \( = \frac{R^{1-\theta}}{1-\theta} \) if \( \theta < 1 \). In order to find a function \( w \) satisfying (2.4), we take any \( b \in (0, T) \) and consider the following auxiliary boundary value problem:

\[
\begin{align*}
\lambda(\phi_\xi)^p - 2\phi_\xi)_\xi + g^{-1}(\xi)^p f(\phi) &= 0 \quad \text{in } (0, b), \\
\phi(0) = \phi(b) &= 0.
\end{align*}
\]

A change of scale \( \xi = b\eta \) gives

\[
\begin{align*}
\lambda(|\psi_\eta|^{p-2} \psi_\eta)_\eta + \{bg^{-1}(\eta)^p f(\psi) &= 0 \quad \text{in } (0, 1), \\
\psi(0) = \psi(1) &= 0,
\end{align*}
\]

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where $\psi(\eta) = \phi(b\eta)$. Take $\lambda$ sufficiently small as

$$\lambda \leq \left\{ \frac{bg^{-1}(b^\theta)}{2C(f,p)} \right\}^p.$$ 

Then, we already know from [19, Theorem 3.3] that (2.6) has a solution $\psi$ such that $\psi(x) = 1$ in $[C_{\lambda,b}/b, 1 - C_{\lambda,b}/b]$, $0 \leq \psi(x) < 1$ otherwise, where

(2.7) $C_{\lambda,b} = \frac{C(f,p)}{g^{-1}(b^\theta)} \lambda^{1/p} \leq b/2$.

Thus, (2.5) also has a solution $\phi$ such that $\phi(x) = 1$ in $[C_{\lambda,b}, b - C_{\lambda,b}]$ and $0 \leq \phi(x) < 1$ otherwise. Using $\phi$, we construct a function $w$ satisfying (2.4) as follows: $w = \phi$ in $[0, C_{\lambda,b})$, $w = 1$ in $[C_{\lambda,b}, T)$. Indeed, since $g^{-1}$ is monotone decreasing,

$$\lambda(\|w\|_{C_0}^p - \|w\|_{C_0}) + g^{-1}(\|w\|_{C_0})\phi(w(\frac{\pi}{\sin(\frac{\pi}{2})})) \geq g^{-1}(\|w\|_{C_0})\phi(\frac{\pi}{\sin(\frac{\pi}{2})})$$

and the boundary conditions are obviously satisfied. Therefore $v(\rho) = w(g(\rho))$ satisfies (2.8), hence the function

(2.8) $v_{R,x_0}(x) = \begin{cases} 1 & \text{if } 0 \leq |x - x_0| \leq g^{-1}(C_{\lambda,b}), \\ \phi(g(|x - x_0|)) & \text{if } g^{-1}(C_{\lambda,b}) < |x - x_0| \leq R \end{cases}$

is a lower solution of (P)$_{\lambda,B_R(x_0)}$. Now, we define $\tilde{v}_{R,x_0}(x) = v_{R,x_0}(x)$ in $B_R(x_0)$. Then, one can observe that $\tilde{v}$ is a lower solution of (P)$_{\lambda,\Omega}$. Taking the function $w \equiv 1$ as an upper solution, we obtain a maximal solution $\tau_\lambda$ of (P)$_{\lambda}$ such that $\tilde{v}_{R,x_0}(x) \leq \tau_\lambda(x) \leq 1$ for all $x \in \Omega$ by Diaz’s book [4, Theorem 4.14] (see also Deuel and Hess [3], and Puel [15, Theorem 4.2]). In particular, it follows from (2.5) that $\tau_\lambda(x) = 1$ in $B_{g^{-1}(C_{\lambda,b})}(x_0)$. By the arbitrariness of $x_0$ satisfying $B_R(x_0) \subset \Omega$ and the uniqueness of maximal solution, it holds that $\tau_\lambda(x) = 1$ in $\Omega \setminus \Omega_{R'}$, where $R' = R(\lambda, b) = R - g^{-1}(C_{\lambda,b})$. Thus $\text{dist}(O_{\lambda,\partial\Omega}) \leq R'$. It follows from (2.7) and l’Hospital’s theorem that $R'(\lambda, b) = R^\theta C_{\lambda,b} + o(\lambda^{1/p})$ as $\lambda \to 0$; so we obtain

$$\limsup_{\lambda \to 0} \lambda^{-1/p} \text{dist}(O_{\lambda,\partial\Omega}) \leq \lim_{\lambda \to 0} \lambda^{-1/p} R'(\lambda, b) = \left\{ \frac{R}{g^{-1}(b)} \right\}^\theta C(f,p).$$

Passing to the limit as $b \to 0$, we conclude (2.2).

\textbf{Proof of Theorem 4.14} Define

$$\bar{\lambda} = \sup\{\lambda > 0; (P)_\lambda \text{ has a solution}\}.$$ 

Since Lemma 2.1 implies $\bar{\lambda} > 0$, we will show $\bar{\lambda} < +\infty$ to see (i). Suppose that there exists a sequence $\{\lambda_m\}_{m=1}^\infty$ such that $\lambda_m \to \infty$ as $m \to \infty$ and $(P)_{\lambda_m}$ has a solution $u_m = u_{\lambda_m}$. Putting $\lambda = \lambda_m$ and $u = \varphi = u_m$ in (1.1), we have

$$\lambda_m \|\nabla u_m\|_p^p = \int_{\Omega} u_m f(u_m) dx.$$ 

Since $sf(s) \leq s^p$ for $s \in [0,1]$ if $p < q$, it follows that

$$\lambda_m \|\nabla u_m\|_p^p \leq \|\lambda_m u_m\|_p^p.$$ 

Combining this inequality and the Poincaré inequality, we obtain $C\lambda_m \|u_m\|_p^p \leq \|\nabla u_m\|_p^p$, where $C$ is a positive constant. Since $\|u_m\|_p^p > 0$, the inequality is a contradiction for sufficiently large $m$. Next, we will prove (ii) and (iii). Consider the case $\lambda < \bar{\lambda}$. From the definition of $\bar{\lambda}$, for $\lambda < \bar{\lambda}$ there exists $\mu \in (\lambda, \bar{\lambda})$ such that $(P)_\mu$ has a solution $u_\mu$. By an easy calculation, $u_\mu$ is a lower solution of $(P)_\lambda$. Since $u \equiv 1$ is an upper solution of $(P)_\lambda$, it follows from [4, Theorem 4.14] that $(P)_\lambda$ admits a maximal solution $\tau_\lambda$ satisfying $\tau_\lambda \geq u_\mu$. (Note that the same arguments give the proof of (iii).) The case $\lambda = \bar{\lambda}$ is treated as follows. Let $\{\lambda_m\}_{m=1}^\infty$ be a positive increasing sequence satisfying $0 < \lambda_m < \bar{\lambda}$ and $\lambda_m \to \bar{\lambda}$.
as \( m \to \infty \), and let \( \varpi_m \) be the maximal solution of \((P)_{\lambda_m}\). From \cite[Theorem 1]{[13]} we know that \( \{\varpi_m\} \) is uniformly bounded in \( C^{1,\beta}(\overline{\Omega}) \) for some \( \beta \in (0,1) \). Thus, Ascoli-Arzelà’s theorem assures that there exist \( \varpi_\infty \) and a subsequence of \( \{\varpi_m\} \), still denoted by \( \{\varpi_m\} \), such that \( \varpi_m \to \varpi_\infty \) in \( C^{1,\beta}(\overline{\Omega}) \) for each \( \beta \in (0,\beta) \). It is easy to see that \( \varpi_\infty \geq 0 \) in \( \Omega \) and that \( \varpi_\infty \) satisfies \((\text{iv})\). To observe that \( \varpi_\infty \neq 0 \), we assume \( \varpi_\infty \equiv 0 \). Since \( \{\varpi_m\} \) converges to 0 uniformly in \( \Omega \) as \( m \to \infty \), it follows from \( p < q \) that for sufficiently large \( m \)

\[
C||\varpi_m||_p^p \leq ||\nabla \varpi_m||_p^p = \frac{1}{\lambda_m} \int_\Omega \varpi_m f(\varpi_m) dx \leq \frac{C}{2}||\varpi_m||_p^p,
\]

which contradicts to \( ||\varpi_m||_p^p > 0 \). Therefore, \( \varpi_\infty \) is a solution of \((P)_{\lambda_\infty}\). We have to show the maximality of \( \varpi_\infty \). Suppose that \( \varpi_\infty \) is not maximal. Then, \((P)_{\lambda_\infty}\) has a maximal solution \( \varpi_\infty \geq \varpi_\infty \) (\( \neq \varpi_\infty \)) and there exists \( x_0 \in \Omega \) such that \( \varpi_\infty(x_0) < \varpi_\infty(x_0) \). By (iii), since \( \varpi_m \) decreases toward \( \varpi_\infty \) as \( m \to \infty \), it holds that \( \varpi_\infty(x_0) \leq \varpi_m(x_0) \) for sufficiently large \( m \). On the other hand, it follows from (iii) and the fact \( \lambda_m < \lambda_\infty \) that \( \varpi_\infty(x_0) \leq \varpi_m(x_0) \). These inequalities contradict each other; so \( \varpi_\infty \) is maximal, which can be written as \( \varpi_\infty \). Finally, one can observe (iv) in the similar way as the proof for maximality of \( \varpi_\infty \).

**Proof of Theorem \cite{[13]}** The existence of \( \lambda^* \) satisfying (i) is directly induced from Lemma \cite{[24]} and (ii) follows from (iii) of Theorem \cite{[11]}. From the proof of Lemma \cite{[24]} (iii) is obvious for sufficiently small \( \varepsilon > 0 \) such that \( \Omega \setminus \Omega_\varepsilon \neq \emptyset \). It remains to show \( \lambda \leq 0 \) near \( \partial \Omega \).

Take any \( x_0 \in \partial \Omega \). Let \( a > 0 \) (resp. \( R > 0 \)) be sufficiently small (resp. large) such that the annulus \( A := \{x \in \mathbb{R}^N; a < |x - y_0| < R\} \), where \( y_0 := x_0 + an \) and \( n \) denotes the outer normal at \( x_0 \), satisfies \( \Omega \subset A \). Define \( \hat{u}_\lambda \) by \( \hat{u}_\lambda := \varpi_\lambda \) in \( \Omega \), \( = 0 \) in \( A \setminus \Omega \). Then \( \hat{u}_\lambda \) is a lower solution of \((P)_{\lambda,A};\) so a maximal solution \( \varpi_{\lambda,A} \) of \((P)_{\lambda,A}\) exists, in particular

\[
\varpi_\lambda(x) \leq \varpi_{\lambda,A}(x) \quad \text{in} \quad \Omega.
\]

From the maximality, \( \varpi_{\lambda,A} \) is radially symmetric on \( A \); hence \( v(\rho) = \varpi_{\lambda,A}(x) \) satisfies

\[
\begin{align*}
\lambda(\rho^{N-1}|v_\rho|^{p-2}v_\rho)_\rho + \rho^{N-1}f(v) &= 0 \quad \text{in} \quad (a,R), \\
v(a) &= v(R) = 0,
\end{align*}
\]

where \( \rho = |x - y_0| \). As in the proof of Lemma \cite{[24]}, we introduce a change of variable

\[
\xi = h(\rho) = \begin{cases}
\frac{\rho^{1-\theta}}{1-\theta} & \text{if } \theta \neq 1, \\
\log \frac{\rho}{\rho_0} & \text{if } \theta = 1,
\end{cases}
\]

where \( \theta := (N-1)/(p-1) \); then \cite{[24]}, \cite{[24]} can be rewritten as

\[
\begin{align*}
\lambda(|w_\xi|^{p-2}w_\xi)_\xi + h^{-1}(\xi)^\rho f(w) &= 0 \quad \text{in} \quad (0,T), \\
w(0) &= w(T) = 0,
\end{align*}
\]

where \( w(\xi) = v(h^{-1}(\xi)) \) and \( T = h(R) \). It is easy to see that \( w \) is a lower solution of

\[
\begin{align*}
\lambda(|\phi_\xi|^{p-2}\phi_\xi)_\xi + h^{-1}(b)^\rho f(\phi) &= 0 \quad \text{in} \quad (0,b), \\
\phi(0) &= 0, \quad \phi(b) = 1.
\end{align*}
\]
for any \( b \in (0, T) \). Thus, (2.11) has a maximal solution \( \overline{\phi} \) such that
\[
(2.12) \quad w(\xi) \leq \overline{\phi}(\xi) \text{ in } (0, b).
\]
In fact, we know from [19] Theorem 3.3] that \( 0 < \overline{\phi}(\xi) < 1 \) in \((0, D_{\lambda, b})\), \( \overline{\phi}(\xi) = 1 \) otherwise, where \( D_{\lambda, b} = C(f, p)\lambda^{1/p}/h^{-1}(b)^\theta (\leq b/2) \). Hence, it follows from \((2.10)\) and \((2.12)\) that \( \overline{\pi}_\lambda(x) = \phi(h(|x - y_0|)) \leq 1 \) if \( x \in \Omega \) and \( a < |x - y_0| < h^{-1}(D_{\lambda, b}) \). This means that \( \text{dist}(x_0, \Omega) \geq h^{-1}(D_{\lambda, b}) - a \) for each \( x_0 \in \partial \Omega \). Making \( a > 0 \)
(resp. \( R > 0 \)) sufficiently small (resp. large), one can get an uniform estimate \( \text{dist}(\Omega, \partial \Omega) \geq h^{-1}(D_{\lambda, b}) - a \). Since \( h^{-1}(D_{\lambda, b}) - a = a^p D_{\lambda, b} + o(\lambda^{1/p}) \) as \( \lambda \to 0 \), it is possible to obtain that
\[
\liminf_{\lambda \to 0} \lambda^{-1/p} \text{dist}(\Omega, \partial \Omega) \geq \left\{ \frac{a}{h^{-1}(b)} \right\}^\theta C(f, p).
\]
Passing to the limit \( b \to 0 \), we have
\[
(2.13) \quad \liminf_{\lambda \to 0} \lambda^{-1/p} \text{dist}(\Omega, \partial \Omega) \geq C(f, p);
\]
so combining \((2.13)\) and \((2.2)\) of Lemma 2.1, we conclude (1.3).

Remark 2.1. From \((2.13)\) and more delicate analyses of \((2.2)\), we can see
\[
\lim_{\lambda \to 0} \lambda^{-1/p} \sup_{x \in \partial \Omega} \text{dist}(x, \Omega) = C(f, p),
\]
which implies that \( \Omega \) uniformly spreads out toward the whole of \( \Omega \) as the order of \( \lambda^{1/p} \).

Proof of Theorem 2.3. In virtue of Proposition 2.1 it is well known that \( u \) is a solution of (P)\( _\lambda \) if and only if \( u \) is a critical point of the \( C^1 \)-functional \( \Phi \), defined by (1.4) (cf. Rabinowitz’s book \cite{17} Proposition B.10). We will check all conditions of the Mountain Pass Theorem (cf. \cite{17}). Take any \( q^* \in (p, p^*) \), where \( p^* := Np/(N - p) \) if \( p < N \), := +\( \infty \) if \( p \geq N \), and fix it. Since \( p < q \), for any \( \delta > 0 \) there exists \( C_\delta > 0 \) such that \( |f(s)| \leq \delta s^{p-1} + C_\delta s^{q - 1} \). First, it is easy to see that \( \Phi \) satisfies the Palais-Smale condition. Indeed, let \( \{u_n\} \) be any sequence in \( W_0^{1,p}(\Omega) \) such that \( \Phi(u_n) \) is bounded and \( \Phi(u_n) \to 0 \) as \( n \to \infty \). Then, it follows from the boundedness of \( \overline{F} \) that \( \|\nabla u_n\|_p \) is bounded; namely \( \{u_n\} \) is bounded in \( W_0^{1,p}(\Omega) \). Thus, as a result of Dinca, Jebelean and Mawhin \cite{16} Lemma 2.1 yields the assertion. In addition, the Sobolev inequality assures that there exist constants \( \gamma, \rho > 0 \) such that \( \Phi(u) \geq \gamma \) if \( \|\nabla u\|_p = \rho \), because
\[
\Phi(u) \geq \frac{\lambda}{p} \|\nabla u\|_p^p - \frac{\delta}{p} \|u\|_p^p - \frac{C_\delta}{q^*} \|u\|_{q^*}^{q^*} \\
\geq \left( \frac{\lambda - C_1 \delta}{p} - \frac{C_2 C_\delta}{q^*} \|\nabla u\|_{q^*-p} \right) \|\nabla u\|_p^p \geq \gamma > 0,
\]
where \( C_1 \) and \( C_2 \) are positive constants, provided that \( \delta \in (0, \lambda/C_1) \) and \( \|\nabla u\|_p = \rho \) is sufficiently small. Next, clearly \( \Phi(0) = 0 \) and we will show that the maximal solution \( \overline{\pi}_\lambda \) of (P)\( _\lambda \) satisfies \( \Phi(\overline{\pi}_\lambda) < 0 \) for sufficiently small \( \lambda > 0 \). Since \( \lambda \|\overline{\pi}_\lambda\|_p = \int_\Omega \overline{\pi}_\lambda f(\overline{\pi}_\lambda) dx \), \( \Phi(\overline{\pi}_\lambda) \) can be expressed as
\[
\Phi(\overline{\pi}_\lambda) = \left( \frac{1}{p} - \frac{1}{q} \right) \|\overline{\pi}_\lambda\|_q^q - \left( \frac{1}{p} - \frac{1}{q + r} \right) \|\overline{\pi}_\lambda\|_{q + r}^{q + r}.
\]
with use of \( \overline{F}(u) = F(u) \), due to Proposition 2.1. Noting that \( \|u\| = |\lambda| + \int_{\Omega \setminus \Omega_{\lambda}} |\nabla u|^{\tau} \) for any \( \tau \geq 1 \), we have

\[
\Phi(\nabla) = \int_{\Omega \setminus \Omega_{\lambda}} \left( \frac{1}{p} - \frac{1}{q} \right) |\nabla|^q - \left( \frac{1}{p} - \frac{1}{q + r} \right) |\nabla|^{|q + r|} \, dx - \left( \frac{1}{q} - \frac{1}{q + r} \right) |\Omega_{\lambda}|
\]

\[
\leq C|\Omega \setminus \Omega_{\lambda}| - \left( \frac{1}{q} - \frac{1}{q + r} \right) |\Omega_{\lambda}|
\]

\[
= C|\Omega| - \left( C + \frac{1}{q} - \frac{1}{q + r} \right) |\Omega_{\lambda}|.
\]

Take \( \varepsilon > 0 \) so sufficiently small that

\[
|\Omega_{\varepsilon}| < \frac{\frac{1}{q} - \frac{1}{q + r}}{C + \frac{1}{q} - \frac{1}{q + r}} |\Omega|.
\]

We see from (iii) of Theorem 1.2 that there exists \( \Lambda \in (0, \lambda] \) such that if \( \lambda \leq \Lambda \), then \( |\Omega_{\lambda}| > |\Omega \setminus \Omega_{\varepsilon}| \). Thus, if \( \lambda \in (0, \Lambda) \), then

\[
\Phi(\nabla_{\lambda}) < C|\Omega| - \left( C + \frac{1}{q} - \frac{1}{q + r} \right) |\Omega \setminus \Omega_{\varepsilon}|
\]

\[
= \left( C + \frac{1}{q} - \frac{1}{q + r} \right) |\Omega_{\varepsilon}| - \left( \frac{1}{q} - \frac{1}{q + r} \right) |\Omega| < 0.
\]

Therefore, all conditions for the Mountain Pass Theorem hold; so we obtain a solution \( u_{\lambda} \) of (P), which is distinct from \( u \) and satisfies \( \Phi(u_{\lambda}) > 0 \). \( \square \)

Remark 2.2. In connection with multiplicity of solutions, we have known a number of results on the linear diffusion case. Rabinowitz \[16\] has studied the case \( p = 2 < q \) (for example, equations like \( \lambda u + u^2(1 - u) = 0 \)) by combining critical point theory and the Leray-Schauder degree theory, and proved there exists \( \Lambda > 0 \) such that if \( \lambda > \Lambda \), then (P) has no solution and if \( \lambda < \Lambda \), then (P) has at least two distinct solutions (see also Ambrosetti and Rabinowitz \[1\], and Rabinowitz \[17\]). Particularly, when \( \Omega \) is a ball, Ouyang and Shi \[14\] have obtained a precise global bifurcation diagram and concluded that there exist exactly two solutions for small \( \lambda \) by using a bifurcation theorem of Crandall and Rabinowitz.

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References

POSITIVE SOLUTIONS OF A DEGENERATE ELLIPTIC EQUATION


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