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ON THE LENGTH OF THE SPECTRAL SEQUENCE OF A LIE ALGEBRA EXTENSION

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Abstract. The length of the spectral sequence of a Lie algebra extension is at most 1+ the dimension of the quotient algebra. We show that this bound can be attained for arbitrarily large quotient algebras even when the algebra is nilpotent and the extension splits.

1. Introduction

The spectral sequences $\{(E^r, d^r) \mid r = 0, 1, \dots\}$ considered in this paper all terminate, that is, there exists t such that $E^r = E^t$ for all $r \geq t$. This stable value is denoted by E^∞. We define the length l of the spectral sequence to be the smallest t for which $E^t = E^\infty$. Note that $d^r = 0$ for $r \geq l$, but $d^{l-1} \neq 0$.

Let L be a Lie algebra over the field F, M an ideal of L and let V be an L-module. We consider the Hochschild-Serre spectral sequence of this extension L of M by L/M, working with homology rather than cohomology to simplify the notation. The terms (E^0, d^0) and (E^1, d^1) depend on the choices of resolutions used in the construction of the spectral sequence, and

$$E^2_{pq} = H_p(L/M, H_q(M, V)).$$

Thus for $r \geq 2$, non-zero terms E^r_{pq} are confined to the rectangle $0 \leq p \leq \dim(L/M)$ and $0 \leq q \leq \dim(M)$. Since d^r_{pq} maps E^r_{pq} into $E^r_{p-r,q+r-1}$, for it to be non-zero, we must have $r \leq \dim(L/M)$ and $r - 1 \leq \dim(M)$. Thus the length l of the spectral sequence always satisfies

$$l \leq 1 + \dim(L/M) \quad \text{and} \quad l \leq 2 + \dim(M).$$

The Hochschild-Serre spectral sequence is discussed at length in Barnes [3], with the particular resolution and filtration used here appearing in section IV.4 of [3]. Like most references, this concentrates on cohomology. The homology version is mentioned briefly in Cartan and Eilenberg [4, p. 351] and in Barnes [1]. The dual $\bar{V} = \mathrm{Hom}_F(V, F)$ of a right L-module V is a left L-module, with the action of $x \in L$ on $f \in \bar{V}$ given by $(xf)(v) = f(vx)$ for all $v \in V$. For a left L-module projective resolution $P_\bullet$ of F,

$$\mathrm{Hom}_F(V \otimes_L P_\bullet, F) \simeq \mathrm{Hom}_L(P_\bullet, \mathrm{Hom}_F(V, F)) = \mathrm{Hom}_L(P_\bullet, \bar{V}).$$

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Thus dualisation converts a complex for the homology of V into a complex for the cohomology of $\bar{V}$ and $\mathrm{Hom}_F(H_n(L,V)) \simeq H^n(L,\bar{V})$. Any ascending filtration of $V \otimes_L P_\bullet$ gives a descending filtration of the cochain complex, and the dual of the homology spectral sequence for V is the cohomology spectral sequence for $\bar{V}$. Thus the results of this paper apply also to the cohomology spectral sequence.

2. Sharper bounds

In this section, we impose conditions on the algebra L, the ideal M and the module V to obtain sharper bounds on the length l. We suppose that L splits over M. We have a subalgebra U of L which complements M. We choose a basis $\{u_1, \dots, u_n\}$ of U and a basis $\{x_1, \dots, x_m\}$ of M. We work with the complex $C_\bullet = V \otimes \Lambda(L)$, where

$$d(v \otimes \langle \alpha_1, \dots, \alpha_r\rangle) = \sum_{i=1}^{r} (-1)^{i-1} v\alpha_i \otimes \langle \alpha_1, \dots \widehat{i} \dots, \alpha_r\rangle + \sum_{i<j} (-1)^{i+j} v \otimes \langle [\alpha_i, \alpha_j], \alpha_1, \dots \widehat{i} \dots \widehat{j} \dots, \alpha_r\rangle$$

with the filtration Φ given by

$$v \otimes \langle \alpha_1, \dots, \alpha_r\rangle \in \Phi^p C_r \text{ if at most } p \text{ of the } \alpha_i \text{ lie outside } M.$$

Lemma 1. *Suppose L splits over M and that M acts trivially on V. Then $d^r_{p0} = 0$ for $r \geq 2$, and $l \leq 1 + \dim(M)$.*

Proof. If $d^r_{p0} \neq 0$, then there exists $c \in \Phi^p C_p$ with $dc \in \Phi^{p-r} C_{p-1}$ and $dc \notin d\Phi^{p-1}C_p + \Phi^{p-r-1}C_{p-1}$. We express c as the sum $c = c' + c''$ where c' is the sum of the terms of the form $v \otimes \langle u_{i_1}, \dots, u_{i_p}\rangle$ and c'' is the sum of the terms of the form $v \otimes \langle u_{i_1}, \dots, u_{i_{p-t}}, x_{j_1}, \dots, x_{j_t}\rangle$ with $t \geq 1$. For each term of this latter type, every non-zero term in its boundary involves at least one of the x_j. It follows that $dc'' \in \Phi^{p-2}C_{p-1}$, so also $dc' \in \Phi^{p-2}C_{p-1}$. But dc' consists of terms which do not involve any of the x_j, so $dc' = 0$ and $dc = dc'' \in d\Phi^{p-1}C_p$ contrary to assumption. Since a non-zero arrow cannot start on the lower edge of the rectangle, we have $l \leq 1 + \dim(M)$.

Theorem 2. *Suppose L splits over M, and that M is nilpotent and acts trivially on V. Then the spectral sequence has length at most the greater of 2 and $\dim(M)$.*

Proof. By Lemma 1, we have that, for $r \geq 2$, a non-zero arrow d^r_{pq} cannot start on the lower edge of the rectangle. We show that, under our additional assumption that M is nilpotent, it also cannot end on the upper edge of the rectangle. Suppose $d^r_{pq} \neq 0$, $r \geq 2$ and that d^r_{pq} ends on the upper edge of the rectangle. Then $q = m - r + 1$. There exists $c \in \Phi^p C_{p+q}$ such that $dc \in \Phi^{p-r}C_{p+q-1}$ and $dc \notin d\Phi^{p-1}C_{p+q} + \Phi^{p-r-1}C_{p+q-1}$. Now $c = c' + c''$ where c' is a sum of terms of the form $v \otimes \langle\ u_{i_1}, \dots, u_{i_s}, x_{j_1}, \dots, x_{j_{m-t}}\rangle$ where $s = p + q - m + t$ and $t \geq 1$, and c'' is a sum of terms of the form $v \otimes \langle u_{i_1}, \dots, u_{i_{p-r+1}}, x_1, \dots, x_m\rangle$. Since M is nilpotent, the x_j may be chosen such that for any $\alpha \in M$, $[\alpha, x_j]$ is in the space spanned by $x_{j+1}, \dots, x_m$. It follows that the only non-zero terms in $dv \otimes \langle u_{i_1}, \dots, u_{i_{p-r+1}}, x_1, \dots, x_m\rangle$ involve all m of the x_j. But dc' consists of terms involving fewer than m of the x_j, while dc by assumption, consists only of terms involving all the x_j. Thus $dc' = 0$, and $dc = dc'' \in d\Phi^{p-1}C_{p+q}$ contrary to assumption.

If M is nilpotent, any L-module V is the direct sum $V_0 \oplus V_1$ of L-submodules with M acting nilpotently on V_0 and with $V_1^M = 0$. As $H_q(M, V_1) = 0$ for all q, (see Dixmier [5, Theorème 1], see also Barnes [1, Theorem 1] and [2]), $E^2_{pq}(V_1) = 0$ for all p, q, and $E^2(V_1) = E^\infty(V_1)$ trivially. Only the null component V_0 can give a spectral sequence of length greater than 2. Our next result gives a bound which does not depend on the dimensions of M or L/M.

Theorem 3. *Suppose L splits over M, that M is abelian and acts trivially on V. Then the spectral sequence has length at most* 2.

Proof. Every non-zero term in $dv \otimes \langle u_{i_1}, \dots, u_{i_p}, x_{j_1}, \dots, x_{j_q}\rangle$ involves exactly q of the x_j. Thus $C_\bullet$ is in fact *graded* by Φ, with $v \otimes \langle u_{i_1}, \dots, u_{i_p}, x_{j_1}, \dots, x_{j_q}\rangle$ a homogeneous element of degree p, and d reducing grade by 1. It follows that d^1 is the only possibly non-zero differential in the spectral sequence.

3. Examples

In both the following examples, L is nilpotent of class 2, splits over M and acts on the 1-dimensional trivial module. The trick used to make the calculation of the length reasonably simple is to arrange that each $d\langle \alpha_1, \dots, \alpha_t\rangle$ which must be calculated has at most two non-zero terms. The first example, unfortunately of low dimension, achieves both the bounds $l = 1 + \dim(L/M)$ and $l = \dim(M)$.

Example 1. Let U and M have bases $\{u, v\}$ and $\{x, y, z\}$ respectively, with the multiplication given by $[u, x] = [v, y] = [x, y] = z$ and all other products 0. Then the spectral sequence has length $l = 3$.

Proof. We have $d(\langle u, v, x\rangle + \langle v, x, y\rangle) = \langle z, x\rangle \in \Phi^0 C_2$. Since for any $\alpha, \beta \in L$, $d\langle \alpha, \beta, z\rangle = 0$, $d\Phi^1 C_3$ is spanned by $d\langle u, x, y\rangle = \langle y, z\rangle + \langle u, z\rangle$ and $d\langle v, x, y\rangle = \langle v, z\rangle - \langle x, z\rangle$, so $\langle z, x\rangle \notin d\Phi^1 C_3$. Thus $d^2_{21} : E^2_{21} \to E^2_{02}$ is non-zero and $l = 3 = 1 + \dim(L/M) = \dim(M)$.

Example 2. Let M have basis $\{v_1, \dots, v_n, w_1, \dots, w_{n-1}, z_1, \dots, z_n\}$ and let U have basis $\{u_1, \dots, u_n\}$, with the multiplication given by $[u_i, v_i] = z_i = [w_i, v_{i+1}]$ and all other products 0. Then the spectral sequence has length $l = n + 1$.

Proof. We put $\alpha_0 = 0$,

$$\alpha_i = \langle z_i, w_1, \dots, w_{i-1}, u_{i+1}, \dots, u_n\rangle \text{ and } c_i = \langle w_1, \dots, w_{i-1}, v_i, u_i, \dots, u_n\rangle.$$

Then $dc_i = -\alpha_{i-1} + \alpha_i$. Observe that $c_1 + \dots + c_n \in \Phi^n C_{n+1}$ and that

$$d(c_1 + \dots + c_n) = \alpha_n \in \Phi^0 C_n.$$

We show that $\alpha_n \notin d\Phi^{n-1} C_{n+1}$.

C_n has a basis consisting of the n-fold wedge products of members of our basis of L. Let A be the subspace of C_n spanned by the α_i, and let B be the subspace spanned by the other n-fold wedge products. Let $\pi : C_n \to A$ be the projection onto A defined by the direct decomposition $C_n = A \oplus B$. We similarly decompose C_{n+1} into the direct sum $C \oplus D$ where C is spanned by the c_i and D by the other $(n+1)$-fold wedge products. Of the factors in α_i, only z_i is expressible as a product of basis elements, and that, only in two ways. Thus c_i and c_{i+1} are the only $(n+1)$-fold wedge products of basis elements whose boundaries have α_i as a term. If $\gamma \in D$, then $d\gamma \in B$ and $\pi d\gamma = 0$.

If $c \in \Phi^{n-1}C_{n+1}$, then $c = \sum_{i=2}^{n} \lambda_i c_i + \gamma$ where $\gamma \in D$ and λ_i are elements of the field. We then have

$$dc = \sum_{i=2}^{n} \lambda_i(-\alpha_{i-1} + \alpha_i) + d\gamma$$

and

$$\pi dc = \sum_{i=2}^{n} \lambda_i(-\alpha_{i-1} + \alpha_i) \neq \alpha_n.$$

Thus $dc \neq \alpha_n$ and $\alpha_n \notin d\Phi^{n-1}C_{n+1}$. Hence $d_{n1}^n : E_{n1}^n \to E_{0n}^n$ is non-zero and $l = n + 1$.

4. Questions

Several interesting questions remain unanswered. In Theorem 2, is $\dim(M)$ the best possible bound? Example 2 only achieved $l = 2 +$ integral part of $\dim(M)/3$. What happens in Lemma 1 and Theorems 2 and 3 if we weaken the condition that M acts trivially to M acts nilpotently? If V has a submodule W, is $l(V)$ bounded by some function of $l(W)$ and $L(V/W)$?

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