DIAGONAL TYPE CONDITIONS ON GROUP C*-ALGEBRAS

NICO SPRONK AND PETER WOOD

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Abstract. Let $G$ be a locally compact group with $C^*(G)$ and $C^r(G)$ its enveloping and reduced C*-algebras respectively. We show that if $C^*(G)$ is residually finite dimensional, then $G$ is maximally almost periodic, and $C^r(G)$ is residually finite dimensional if and only if $G$ is both amenable and maximally almost periodic. Letting $\lambda_G$ be the left regular representation of $G$, we show that a certain quasidiagonality condition on $\{\lambda_G(s) : s \in G\}$ implies that $G$ is amenable.

This paper summarizes a few results relating the structures of locally compact groups to diagonal type structures for some of their C*-algebras, namely residual finite dimensionality and quasidiagonality.

The main results in the first section regard such a group $G$ having a residually finite dimensional enveloping C*-algebra, $C^*(G)$. We show that if $C^*(G)$ is residually finite dimensional, then $G$ is maximally almost periodic, and mention a case where the converse does not hold. We obtain a partial converse, though: $G$ is maximally almost periodic exactly when its reduced C*-algebra $C^r(G)$ is the quotient of a residually finite dimensional C*-algebra which is a quotient of $C^*(G)$. We then obtain the result that $G$ is maximally almost periodic and amenable exactly when $C^r(G)$ is residually finite dimensional. We conclude the section by exploring the class of groups having residually finite dimensional enveloping C*-algebras. In particular, we examine the stability of this class under free products and amalgams.

The second section records the result that a certain quasidiagonality condition for the unitaries of the left regular representation of $G$ implies that $G$ is amenable. We also extend, to a larger class than discrete groups, Rosenberg’s result that if $C^r(G)$ is quasidiagonal, then $G$ is amenable.

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1. Maximal almost periodicity and residual finite dimensionality

A locally compact group $G$ is called \textit{maximally almost periodic} if there exists a separating family of continuous finite dimensional unitary representations of $G$. A $C^*$-algebra $\mathcal{A}$ is called \textit{residually finite dimensional} if there is a separating family of finite dimensional representations of $\mathcal{A}$. If $\mathcal{A}$ admits a countable separating family of finite dimensional representations, then we could say that $\mathcal{A}$ may be faithfully represented as a block diagonal $C^*$-algebra.

We recall that there is a bijective correspondence between the collection of equivalence classes of continuous unitary representations of a locally compact group $G$ and the collection of equivalence classes of non-degenerate involutive representations of its enveloping $C^*$-algebra $C^*(G)$. If $\sigma : C^*(G) \to B(H)$ is an involutive representation on a Hilbert space $H$, the associated continuous unitary representation of $G$ will be denoted by $\sigma_G$; if $\sigma_G$ is a continuous unitary representation of $G$, then denote the corresponding representation of $C^*(G)$ by $\sigma$. Note that $C^*(G)$ may be viewed as the completion of $\mathcal{L}_c(G)$, the convolution algebra of continuous functions with compact support on $G$, in an appropriate norm. If for $s$ in $G$ and $f$ in $\mathcal{L}_c(G)$ we write $s \cdot f(t) = f(s^{-1} t)$ for $t$ in $G$, then $\sigma(s \cdot f) = \sigma_G(s) \sigma(f)$.

Let $\Sigma_G$ denote the collection of equivalence classes of continuous unitary representations of $G$. By abuse of notation we will treat subsets of $\Sigma_G$ as sets of representatives of elements of $\Sigma_G$. Given $\sigma_G, \tau_G$ in $\Sigma_G$, we say that $\sigma_G$ is \textit{weakly contained} in $\tau_G$, denoted $\sigma_G \prec \tau_G$, if $\sigma$ is weakly contained in $\tau$ as a representation of $C^*(G)$, i.e. $\ker \sigma \supset \ker \tau$. If $\sigma_G \in \Sigma_G$ and $S$ is a family in $\Sigma_G$, write $\sigma_G \prec S$ if $\ker \sigma \supset \bigcap_{\pi_G \in S} \ker \pi$. Let $\hat{G}$ denote the set of all irreducible representations in $\Sigma_G$, and $\hat{G}_f$ denote the set of all finite dimensional representations in $\hat{G}$. Then $G$ maximally almost periodic if and only if $\hat{G}_f$ is a separating family for $G$.

\textbf{Theorem 1.1.} For a locally compact group $G$, if $C^*(G)$ is residually finite dimensional, then $G$ is maximally almost periodic.

\textbf{Proof.} Let $F$ be a separating family of finite dimensional non-degenerate involutive representations of $C^*(G)$. Then $F_G$ is separating for $G$. Indeed, if there were $s \in \bigcap_{\pi \in F} \ker \pi_G$, then for all $f$ in $\mathcal{L}_c(G)$ we would have $s \cdot f - f \in \bigcap_{\pi \in F} \ker \pi$, a contradiction, unless $s = e$. 

Although categorically suggestive, the converse to Theorem 1.1 is false. In fact, it fails for a residually finite discrete group! In [8] p. 137 it is shown that for the residually finite group $G = SL_2(\mathbb{Z}[1/p])$, $C^*(G)$ is not residually finite dimensional.

We can answer the question posed in [8] p. 137 of whether for a discrete group $G$, $C^*(G)$ being residually finite dimensional implies that $G$ is residually finite.

\textbf{Corollary 1.2.} If $G$ is a finitely generated (hence discrete) group and $C^*(G)$ is residually finite dimensional, then $G$ is residually finite. However, there exist countable groups for which $C^*(G)$ is residually finite dimensional but $G$ is not residually finite.

\textbf{Proof.} If $C^*(G)$ is residually finite dimensional, then $G$ is maximally almost periodic. If $\pi \in \hat{G}_f$, then $\pi(G)$ is a finitely generated linear group and so is residually finite by [2] Cor. 1. This shows that $G$ is residually finite.

If $G$ is a non-trivial divisible group, i.e. $G$ is Abelian and $G^n = \{s^n : s \in G\}$ is all of $G$ for $n \in \mathbb{N}$, then any quotient of $G$ is also divisible, so can be finite only if...
Proof. It is trivial. Hence $G$ cannot be residually finite. However, any divisible group $G$ has that its commutative $C^*$-algebra $C^*(G)$ is residually finite dimensional, since its characters separate points. In particular, the additive group of rationals $\mathbb{Q}$ has a residually finite dimensional enveloping $C^*$-algebra, but is not residually finite. Much of this proof is contained in [9, Prop. 4].

For a locally compact group $G$, let $\lambda_G$ denote the left regular representation of $G$ on $L^2(G)$, given for $s \in G$ and $\xi$ in $L^2(G)$ by $\lambda_G(s)\xi(t) = \xi(st^{-1})$ for almost all $t$ in $G$. Let $C^*_r(G) = \lambda(C^*(G))$ denote the reduced $C^*$-algebra of $G$.

**Theorem 1.3.** If $G$ is a locally compact group, then the following are equivalent:

(i) $G$ is maximally almost periodic,

(ii) $\lambda_G \prec \mathbb{G}_f$,

(iii) $\lambda : C^*(G) \to C^*_r(G)$ factors through a residually finite dimensional $C^*$-algebra.

Proof. (i)$\Rightarrow$(ii) This is [1, Example 1.11 (ii)].

(ii)$\Rightarrow$(iii) Let $\rho_G = \bigoplus_{x \in G} \pi_G$ and $C^*_f(G) = \rho(C^*(G))$. Let $\sigma : C^*_f(G) \to C^*_r(G)$ be given by $\sigma(\rho(a)) = \lambda(a)$ for $a$ in $C^*(G)$. Since $\lambda_G \prec \rho_G$, $\sigma$ is well-defined, and is clearly an involutive representation of $C^*_f(G)$. Hence the following diagram commutes:

\[
\begin{array}{ccc}
C^*(G) & \xrightarrow{\lambda} & C^*_r(G) \\
\downarrow{\rho} & & \downarrow{\sigma} \\
C^*_f(G) & & 
\end{array}
\]

(iii)$\Rightarrow$(i) Let $\rho : C^*(G) \to \mathcal{B}(\mathcal{H})$ be any representation factoring $\lambda$, where $\rho(C^*(G))$ is residually finite dimensional with separating family $\mathcal{F}$ of finite dimensional representations. If $s \in \ker \rho$, then for all $f \in \mathcal{C}_r(G)$, $s \cdot f - f \in \ker \rho \subseteq \ker \lambda$. Hence $s \cdot f - f = 0$ for all $f$ in $\mathcal{C}_r(G)$; since $\lambda|_{\mathcal{C}_r(G)}$ is injective, and thus $s = e$. This means that $\mathcal{F}_G$ separates points of $G$ so $G$ is maximally almost periodic.

We can abstractly characterize all locally compact groups $G$ for which $C^*_r(G)$ is residually finite dimensional.

**Theorem 1.4.** A locally compact group $G$ is maximally almost periodic and amenable if and only if $C^*_r(G)$ is residually finite dimensional.

Proof. If $G$ is maximally almost periodic and amenable, then it is immediate that $C^*(G) \cong C^*_r(G) \cong C^*_f(G)$ by [15, Theo. 4.21] and Theorem 1.3 (iii) above, so $C^*_r(G)$ is residually finite dimensional.

If $C^*_r(G)$ is residually finite dimensional, let $\{\pi_\alpha\}_{\alpha \in A}$ be a separating family of finite dimensional involutive representations of $C^*_r(G)$. For $\alpha$ in $A$, $\pi_\alpha \circ \lambda$ is a finite dimensional representation on $C^*(G)$ for which $(\pi_\alpha \circ \lambda)_G \prec \lambda_G$. Hence $G$ is amenable by [16, Theo. 3]. Then $C^*(G) \cong C^*_r(G)$ is residually finite dimensional, so $G$ is maximally almost periodic by Theorem 1.1.

For countable groups the above theorem follows from [6, Cor. 4].
We have some interest in characterising the class of locally compact groups $G$ for which $C^*(G)$ is residually finite dimensional. We know that all such amenable groups are the maximally almost periodic ones. There are, however, non-amenable groups which are in this class.

If $G$ and $H$ are two discrete groups, denote their free product (or coproduct in the category of groups) by $G \star H$. Provided that at least one of $G$ or $H$ is not a 2 element group and both are non-trivial, it is known that $G \star H$ contains a copy of the free group $F_2$, and hence is not amenable.

**Proposition 1.5.** If $G$ and $H$ are discrete groups such that $C^*(G)$ and $C^*(H)$ are residually finite dimensional, then $C^*(G \star H)$ is residually finite dimensional.

**Proof.** There is an obvious isomorphism $C^*(G \star H) \cong C^*(G) \ast_C C^*(H)$, where $C^*(G) \ast_C C^*(H)$ is the unital $C^*$-algebra free product of $C^*(G)$ and $C^*(H)$. This $C^*$-algebra is residually finite dimensional by [11, Theo. 3.2].

We remark that for $F_2 \cong \mathbb{Z} \ast \mathbb{Z}$, the above result is due to Choi [5] (also see [7, VII.6.1]).

If $G$ and $H$ are groups and there is a group $\Gamma$ which is embedded in each of $G$ and $H$ via homomorphisms $i$ and $j$ respectively, then the amalgam of $G$ and $H$ over $\Gamma$ is $G \ast H = (G \star H)/N$, where $N$ is the smallest normal subgroup in $G \star H$ containing $\{i(a)j(a)^{-1} : a \in \Gamma\}$. It is interesting to try to determine if the class of discrete groups having residually finite dimensional $C^*$-algebras is closed under amalgams over common subgroups. We can conclude that this is not the case in general. Note that from [19, p. 35], $\text{SL}_2(\mathbb{Z}) \cong (\mathbb{Z}/4\mathbb{Z}) \ast_{\mathbb{Z}/2\mathbb{Z}} (\mathbb{Z}/6\mathbb{Z})$. We do not know if $C^*(\text{SL}_2(\mathbb{Z}))$ is residually finite dimensional. However, by [19, p. 80], for a prime $p$, there is a subgroup $\Gamma(p)$ of $\text{SL}_2(\mathbb{Z})$ such that $\text{SL}_2(\mathbb{Z}[1/p]) \cong \text{SL}_2(\mathbb{Z}) \ast_{\Gamma(p)} \text{SL}_2(\mathbb{Z})$. Hence, either $\text{SL}_2(\mathbb{Z})$ or $\text{SL}_2(\mathbb{Z}[1/p])$ shows that this class is not closed under amalgams by arbitrary subgroups. We are unaware of whether this class is closed for amalgams over finite subgroups.

If $G$ is a locally compact group for which $C^*(G)$ is residually finite dimensional, then the same is true for any open subgroup $H$ by the fact that $C^*(H)$ may be realized as a subalgebra of $C^*(G)$ by [17, Prop. 1.2]. We are not aware if this property holds for closed subgroups in general.

2. **Quasidiagonality and Amenability**

For our definitions and preliminary results, we follow [14], but suitably generalize them to non-separable Hilbert spaces. A set $S$ of bounded operators on a Hilbert space $\mathcal{H}$ is called quasidiagonal if for any set $\mathcal{M} \subset \mathcal{H}$ which spans a dense subspace of $\mathcal{H}$, any finite sets $S_0 \subset S$, $\mathcal{M}_0 \subset \mathcal{M}$ and $\varepsilon > 0$, there is a finite rank projection $p$ on $\mathcal{H}$ such that

\[(*) \quad \|pa - ap\| \leq \varepsilon \text{ for } a \in S_0 \text{ and } \|p\xi - \xi\| < \varepsilon \text{ for } \xi \in \mathcal{M}_0.\]

The following is the non-separable analogue of [14, Lem. 1 (4)].

**Proposition 2.1.** A set $S$ of bounded operators on a Hilbert space $\mathcal{H}$ is quasidiagonal if and only if there exists an increasing net of finite rank projections $\{p_\alpha\}_{\alpha \in \Lambda}$ tending strongly to the identity operator such that $\lim_{\alpha \in \Lambda} \|p_\alpha a - ap_\alpha\| = 0$ for $a$ in $S$. 

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Proof. If $S$ is quasidiagonal, let $M$ be any subset of $H$ which spans a dense subspace and let $A = \{(S_0, M_0, \varepsilon) : S_0 \subset S$ and $M_0 \subset M$ are finite, and $\varepsilon > 0\}$. There is a natural ordering under which $A$ is a directed set. For $\alpha = (S_0, M_0, \varepsilon)$ in $A$, let $p_\alpha$ be the projection satisfying (*) for the triple $\alpha$. The rest of the proof is straightforward.

A representation $\rho : A \to B(\mathcal{H})$ of a $C^*$-algebra $A$ is quasidiagonal if $\rho(A)$ is quasidiagonal. We say that a $C^*$-algebra $A$ is weakly quasidiagonal if it has a faithful quasidiagonal representation.

It is clear from the above proposition that a residually finite dimensional $C^*$-algebra is weakly quasidiagonal.

We now define two algebras associated with the left regular representation. Let $C_0^*(G) = \text{span}\{\lambda_G(s) : s \in G\}$ and $\text{VN}(G)$ be the weak operator closure of $C_0^*(G)$, or equivalently the weak operator closure of $C_0^*(G)$, in $B(L^2(G))$. Note that if $G$ is discrete, then $C^*_0(G) \cong C^*_\alpha(G)$, and if $G$ is Abelian, then $C^*_0(G) \cong \mathcal{A}(G)$, where $\mathcal{A}(G)$ is the commutative $C^*$-algebra of almost periodic functions on the dual group $\hat{G}$. It is well known that $\text{VN}(G)$ is the commutant of $\{\lambda'_G(s) : s \in G\}$, where $\lambda'_G$ is the right regular representation of $G$ on $L^2(G)$, given for $s \in G$ and $\xi$ in $L^2(G)$ by $\lambda'_G(s)\xi(t) = \Delta(s)^{1/2}\xi(ts)$ for almost all $t \in G$.

**Lemma 2.2.** If $G$ is not compact, then $\text{VN}(G)$ contains no non-trivial compact operators.

**Proof.** Fix $a$ in $\text{VN}(G) \setminus \{0\}$. Find $\xi_a$ in $C_c(G)$ such that $||\xi_a||_2 = 1$ and $\langle a\xi_a, \xi_a \rangle \neq 0$. (By a numerical range argument we may obtain that $||\langle a\xi_a, \xi_a \rangle || > (1 - \varepsilon)^{1/2}||a||$, for any $\varepsilon > 0$.) Let $K = \text{supp}(\xi_a)$. Then we may inductively find a sequence $\{t_n\}_{n \in \mathbb{N}}$ in $G$ such that $K_t \ldots t_n$ is disjoint from $K$ and $K_t \ldots t_k$ for any $k = 1, \ldots, n-1$. Put $\xi_{a,0} = \xi_a$ and for $n \in \mathbb{N}$, $\xi_{a,n} = \lambda'_G(t_1 \ldots t_n)\xi_a$. Then $\{\xi_{a,n}\}_{n \in \mathbb{N} \cup \{0\}}$ is an orthonormal sequence in $L^2(G)$ and $$\langle a\xi_{a,n}, \xi_{a,n} \rangle = \langle a\lambda'_G(t_1 \ldots t_n)\xi_{a,0}, \lambda'_G(t_1 \ldots t_n)\xi_{a,0} \rangle = \langle a\xi_{a,0}, \xi_{a,0} \rangle \neq 0$$ for all $n$, so $a$ is not compact; for if $a$ were compact, then $\lim_{n \to \infty} ||a\xi_{n,0}|| = 0$.

The problem of knowing the conditions under which $C^*_0(G)$ is (weakly) quasidiagonal has attracted some attention (14, 18, 21, 22). It is proved in 18 that for a discrete group $G$, $C^*_0(G)$ is quasidiagonal (as a concrete family of operators) only if $G$ is amenable. We have found that the most natural extension of this result is not an extension to $C^*_\alpha(G)$ for locally compact $G$, but to any space of operators containing $C^*_0(G)$.

**Theorem 2.3.** If $G$ is a locally compact group for which $C^*_\alpha(G)$ is weakly quasidiagonal, then $G$ is amenable.

**Proof.** We may assume that $G$ is not compact. Let $H$ be a countable subgroup of $G$ and $C^*_\alpha(H) = \text{span}\{\lambda_G(s) : s \in H\}$. Then $C^*_\alpha(H)$ is weakly quasidiagonal. Let $D \subset C^*_\alpha(H)$ be a countable dense subset, and for $d \in D \setminus \{0\}$, create $\{\xi_{d,n}\}_{n \in \mathbb{N} \cup \{0\}}$ as in the lemma above. Then for any separable, $C^*_\alpha(H)$-invariant subspace $\mathcal{H}$ of $L^2(G)$ containing $\{\xi_{d,n} : n \in \mathbb{N} \cup \{0\}$ and $d \in D\}$, $C^*_\alpha(H)|_{\mathcal{H}}$ intersects the compacts on $\mathcal{H}$ trivially. If $\rho$ is a faithful quasidiagonal $*$-representation of $C^*_\alpha(H)$ on a separable Hilbert space $\mathcal{K}$ (i.e. take $\rho$ to be the restriction of some faithful
quasidiagonal as a set of operators on $L^1(H) \to B(K^1)$, and

$$\ker \text{id}|_H = \ker \pi \text{id}|_H = \ker \pi \rho^{(\infty)} = \ker \rho^{(\infty)}$$

where $\pi$ is the map onto the Calkin algebra. Hence $\text{id}|_H$ and $\rho^{(\infty)}$ are approximately unitarily equivalent by Voiculescu’s “Weyl-von Neumann” Theorem [20] (see [2, Cor. II.5.6]), so $\text{id}|_H$ is quasidiagonal by [14] Cor. 3. Hence $C^*_\alpha(H)|_H$ is quasidiagonal as a set of operators on $H$. Since $H$ is an arbitrary separable subspace of $L^2(G)$ satisfying the conditions above, it follows that $C^*_\alpha(H)$ is quasidiagonal as a set of operators on $L^2(G)$.

Let $\{p_\alpha\}_{\alpha \in A}$ be a net of finite rank projections on $L^2(G)$ strongly increasing to the identity such that $\lim_{\alpha \in A} \|p_\alpha a - a p_\alpha\| = 0$ for $a$ in $C^*_\alpha(H)$. Let $U$ be any ultrafilter on $A$ containing the sets $\{\alpha \in A : \alpha \geq \alpha_0\}$ for each $\alpha_0$ in $A$. For any $f$ in $L^\infty(G)$ let $M_f$ be the multiplication operator by $f$ on $L^2(G)$. Let $tr$ be the canonical trace defined on finite rank operators on $L^2(G)$. Then we may define a mean (state) on $L^\infty(G)$ by setting

$$m(f) = \lim_{\alpha \in U, U \in U} \frac{1}{\text{tr}(p_\alpha)} \text{tr}(p_\alpha M_f p_\alpha)$$

for $f$ in $L^\infty(G)$. We can now follow [22] Prop. 4.2 (see also [7] Prop. VII.7.8) to see that $m$ is invariant for the left action of $H$ on $L^\infty(G)$. First see that for $t$ in $G$ and $f$ in $L^\infty(G)$, $\lambda_G(t) M_f \lambda_G(t)^* = M_t f$. Then for such $f$ and $t$ in $H$ we have

$$|m(t \cdot f) - m(f)| = \lim_{\alpha \in U, U \in U} \frac{1}{\text{tr}(p_\alpha)} |\text{tr}(p_\alpha \lambda_G(t) M_f \lambda_G(t)^* p_\alpha) - \text{tr}(\lambda_G(t) p_\alpha M_f \lambda_G(t)^* p_\alpha)|$$

$$= \lim_{\alpha \in U, U \in U} \frac{1}{\text{tr}(p_\alpha)} |\text{tr}(p_\alpha \lambda_G(t) - \lambda_G(t) p_\alpha) M_f \lambda_G(t)^* p_\alpha + \text{tr}(\lambda_G(t) p_\alpha M_f \lambda_G(t)^* p_\alpha - p_\alpha \lambda_G(t)^*)|$$

$$\leq \lim_{\alpha \in U, U \in U} 2 \|p_\alpha \lambda_G(t) - \lambda_G(t) p_\alpha\| \|f\|_\infty = 0.$$

For each finite subset $F$ in $G$, let $m_F$ be the mean defined above which is invariant for the action of $(F)$ on $L^\infty(G)$. Then any weak* cluster point of the net $\{m_F\}_{F \in F}$, indexed over the collection of all finite subsets of $G$, is a left invariant mean for the action of $G$ on $L^\infty(G)$. Hence $G$ is amenable.

We note that the method used above, of reducing to countable subgroups and applying Voiculescu’s theorem, gives a generalization of [18] Theo. A1.

Corollary 2.4. For a discrete group $G$, if $C_r^*(G)$ is weakly quasidiagonal, then $G$ is amenable.

For a locally compact group $G$, let $G_d$ denote the same group but with the discrete topology. Note that in general $C_r^*(G_d)$ is a quotient of $C_r^*(G)$ by [10] Theo. 2.5] (see also [2] Lem. 2)].

Corollary 2.5. Suppose $G$ is a locally compact group which contains an open subgroup $H$ such that $H_d$ is amenable (for example, if $H$ is solvable) and $C_r^*(G)$ is weakly quasidiagonal. Then $G_d$ is amenable.
Proof. The condition that $G$ has an open subgroup $H$ for which $H_d$ is amenable is equivalent to the existence of an isomorphism $C^*_r(G) \cong C^*_r(G_d)$ by [3, Theo. 2]. Then $C^*_r(G_d)$ is weakly quasidiagonal so $G_d$ is amenable. □

We have another extension of Rosenberg’s result of a slightly different nature.

**Theorem 2.6.** If $G$ is a locally compact group with an open normal compact subgroup $K$ and $C^*_r(G)$ is weakly quasidiagonal, then $G$ is amenable.

**Proof.** Let $C^*_r(K)(G) = \text{span}\{\lambda(\chi_t K) : t \in G\}$, where $\chi_t K$ is the characteristic function of $tK$. As in [12] p. 217 let $L^2(G)$ be the closed subspace of elements of $L^2(G)$ which are almost everywhere constant on (left) cosets of $K$, and let $j : \ell^2(G/K) \to L^2(G)$ be the unitary given by $j\xi = \xi_* q$, where $q : G \to G/K$ is the quotient map. Note that we may assume $\mu(K) = 1$, where $\mu$ is the Haar measure on $G$, so that $j$ is in fact a unitary. Then the map $a \mapsto j^* aj$ is an isomorphism identifying $C^*_r(K)(G)$ with $C^*_r(G/K)$. Since $C^*_r(K)(G)$ is weakly quasidiagonal, so too is $C^*_r(G/K)$. Since $G/K$ is discrete, this implies that $G/K$ is amenable; hence so too is $G$, by [16] Prop. 1.13. □

We remark that the class of groups having a compact open normal subgroup includes all $[IN]$ groups that have a compact connected component of the identity. (See [15] for a description of $[IN]$ groups.) This class also includes all Lie groups which have a compact connected component of the identity.

**Added in proof**

A.-T. Lau has kindly pointed out to us that Lemma 2.2 can be found in [23].

**References**

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Department of Pure Mathematics, University of Waterloo, Ontario, Canada N2L 3G1

E-mail address: nsproun@math.uwaterloo.ca

Department of Pure Mathematics, University of Waterloo, Ontario, Canada N2L 3G1

E-mail address: pwood@math.uwaterloo.ca