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NOTE ON A DIOPHANTINE INEQUALITY IN SEVERAL VARIABLES

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Abstract. We establish estimates for the number of points that belong to an aligned box in $(\mathbb{R}/\mathbb{Z})^N$ in terms of certain exponential sums. These generalize previous results that were known only in case $N = 1$.

1. Introduction

Let $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \dots, \boldsymbol{\xi}_M$ be a finite set of points in $(\mathbb{R}/\mathbb{Z})^N$. A basic problem in Diophantine approximation is to estimate the number of points in this set which belong to an aligned box in $(\mathbb{R}/\mathbb{Z})^N$ from knowledge of the exponential sums

$$\sum_{m=1}^{M} e(\boldsymbol{\ell} \cdot \boldsymbol{\xi}_m) \ ,$$

where $\boldsymbol{\ell}$ is restricted to a finite subset of $\mathbb{Z}^N$ and $e(x) = e^{2\pi i x}$. The Erdös-Turán inequality, as stated in [2], is a result of this sort, but it is generally not useful when the measure of the box is small. In the case of a small box the usual approach is Vinogradov's "method of little glasses", as discussed in [5], pp. 32-34. In the present note we establish inequalities that are generally sharper and easier to use in applications. For $N = 1$ this is described in [1], section 2.1, and in [3], section 1.2. Here we obtain the corresponding inequalities for arbitrary N.

Let $\mathcal{B}_1$ denote the collection of all normalized characteristic functions $\varphi_{u,v} : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ defined by

$$\varphi_{u,v}(x) = \begin{cases} 1 \text{ if } u < x - n < v \text{ for some } n \in \mathbb{Z} \ , \\ \frac{1}{2} \text{ if } u - x \in \mathbb{Z} \text{ or if } v - x \in \mathbb{Z} \ , \\ 0 \text{ otherwise,} \end{cases} \tag{1.1}$$

where $u < v < u + 1$. Then for each positive integer L let $\mathcal{B}_1(L) \subseteq \mathcal{B}_1$ be the subcollection of functions (1.1) such that $(v - u)(L + 1)$ is a positive integer. We write $\mathcal{B}_N$ for the collection of functions $\Phi_{\mathbf{u},\mathbf{v}} : (\mathbb{R}/\mathbb{Z})^N \to \mathbb{R}$ of the form

$$\Phi_{\mathbf{u},\mathbf{v}}(\mathbf{x}) = \prod_{n=1}^{N} \varphi_{u_n,v_n}(x_n) \ , \tag{1.2}$$

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where $\mathbf{u}$ and $\mathbf{v}$ are points in $\mathbb{R}^N$ with $u_n < v_n < u_n + 1$ in each coordinate. If $\mathbf{L}$ in $\mathbb{Z}^N$ has positive coordinates L_n, $n = 1, 2, \ldots, N$, we write $\mathcal{B}_N(\mathbf{L}) \subseteq \mathcal{B}_N$ for the subcollection of functions (1.2) such that $(v_n - u_n)(L_n + 1)$ is a positive integer for each $n = 1, 2, \ldots, N$. Given $\mathbf{L}$ and $\Phi_{\mathbf{u},\mathbf{v}}$ in $\mathcal{B}_N$ it will be convenient to set

$$(v_n - u_n)(L_n + 1) = w_n \ , \qquad n = 1, 2, \ldots, N \ , \tag{1.3}$$

so that $0 < w_n < L_n + 1$. Thus $\Phi_{\mathbf{u},\mathbf{v}}$ belongs to $\mathcal{B}_N(\mathbf{L})$ if and only if $w_n \in \{1, 2, \ldots, L_n\}$ for each n. Also, we use the lattice point $\mathbf{L}$ to determine the subset $\mathcal{L} = \mathcal{L}(\mathbf{L}) \subseteq \mathbb{Z}^N$ defined by

$$\mathcal{L} = \{\boldsymbol{\ell} \in \mathbb{Z}^N : |\ell_n| \le L_n \ , \ n = 1, 2, \ldots, N\} \ . \tag{1.4}$$

Now a precise form of the problem we consider in this note is as follows. If $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_M$ is a finite set of points in $(\mathbb{R}/\mathbb{Z})^N$, we wish to estimate sums of the type

$$\sum_{m=1}^{M} \Phi_{\mathbf{u},\mathbf{v}}(\boldsymbol{\xi}_m)$$

from knowledge of the exponential sums

$$\sum_{m=1}^{M} e(\boldsymbol{\ell} \cdot \boldsymbol{\xi}_m) \ ,$$

where $\boldsymbol{\ell}$ is in $\mathcal{L}$. Here we are concerned with the case where the measure

$$\int_{(\mathbb{R}/\mathbb{Z})^N} \Phi_{\mathbf{u},\mathbf{v}}(\mathbf{x})\, d\mathbf{x} = \prod_{n=1}^{N} (v_n - u_n)$$

of the aligned box is small. Thus our main result is a lower bound for the number of points in the box.

Theorem 1. *Let $\Phi_{\mathbf{u},\mathbf{v}}$ belong to $\mathcal{B}_N(\mathbf{L})$ with $w_1, w_2, \ldots, w_N$ determined by* (1.3). *Assume that $\delta > 0$ and $\eta > 0$ satisfy*

$$\sum_{n=1}^{N} w_n^{-1} \le \delta \tag{1.5}$$

and

$$\sum_{\substack{\boldsymbol{\ell} \in \mathcal{L} \\ \boldsymbol{\ell} \ne \mathbf{0}}} \Big| \sum_{m=1}^{M} e(\boldsymbol{\ell} \cdot \boldsymbol{\xi}_m) \Big| \le \eta M \ . \tag{1.6}$$

Then we have

$$M(1 - \delta - \eta - \delta\eta) \prod_{n=1}^{N} (v_n - u_n) \le \sum_{m=1}^{M} \Phi_{\mathbf{u},\mathbf{v}}(\boldsymbol{\xi}_m) \ . \tag{1.7}$$

As an application of Theorem 1, we obtain a generalization to $(\mathbb{R}/\mathbb{Z})^N$ of the inequality given in [1] as Theorem 2.2 and in [4] as Corollary 21. We write $\|x\|$ for the distance from the real number x to the nearest integer.

Corollary 2. *Let $0 < \varepsilon_n \leq \frac{1}{2}$ and set $L_n = [N\varepsilon_n^{-1}]$ for each $n = 1, 2, \ldots, N$. Assume that*

$$\max_{1\leq n\leq N} \frac{\|\xi_{nm}\|}{\varepsilon_n} \geq 1 \tag{1.8}$$

for each point $\boldsymbol{\xi}_m$ in $(\mathbb{R}/\mathbb{Z})^N$, $m = 1, 2, \ldots, M$. Then we have

$$M \leq 3 \sum_{\substack{\boldsymbol{\ell}\in\mathcal{L} \\ \boldsymbol{\ell}\neq\mathbf{0}}} \Big| \sum_{m=1}^{M} e(\boldsymbol{\ell}\cdot\boldsymbol{\xi}_m) \Big| \ . \tag{1.9}$$

There is an upper bound analogous to (1.7), but this is much easier to prove.

Theorem 3. *Let $\mathbf{L}$ in $\mathbb{Z}^N$ have positive coordinates, let $\Phi_{\mathbf{u},\mathbf{v}}$ belong to $\mathcal{B}_N$ with $w_1, w_2, \ldots, w_N$ determined by (1.3). Assume that $\delta > 0$ and $\eta > 0$ satisfy*

$$\prod_{n=1}^{N} (1 + w_n^{-1}) \leq (1 + \delta) \tag{1.10}$$

and

$$\sum_{\substack{\boldsymbol{\ell}\in\mathcal{L} \\ \boldsymbol{\ell}\neq\mathbf{0}}} \Big| \sum_{m=1}^{M} e(\boldsymbol{\ell}\cdot\boldsymbol{\xi}_m) \Big| \leq \eta M \ . \tag{1.11}$$

Then we have

$$\sum_{m=1}^{M} \Phi_{\mathbf{u},\mathbf{v}}(\boldsymbol{\xi}_m) \leq M(1 + \delta + \eta + \delta\eta) \prod_{n=1}^{N} (v_n - u_n) \ . \tag{1.12}$$

2. Preliminary lemmas

As in [4] we define entire functions H, J and K by

$$H(z) = \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{m=-\infty}^{\infty} \operatorname{sgn}(m)(z-m)^{-2} + 2z^{-1} \right\} , \tag{2.1}$$

$$J(z) = \frac{1}{2} H'(z) \ , \quad \text{and} \quad K(z) = \left(\frac{\sin \pi z}{\pi z}\right)^2 \ .$$

We note that each of these functions is real valued on the real axis and has exponential type 2π. The functions J and K are integrable on $\mathbb{R}$ and their Fourier transforms

$$\widehat{J}(t) = \int_{-\infty}^{\infty} J(x) e(-tx)\, dx \quad \text{and} \quad \widehat{K}(t) = \int_{-\infty}^{\infty} K(x) e(-tx)\, dx$$

are continuous functions supported on $[-1, 1]$. These Fourier transforms are given explicitly by

$$\widehat{J}(t) = \pi t(1 - |t|) \cot \pi t + |t| \quad \text{if} \quad 0 < |t| < 1 \ ,$$

$$\widehat{K}(t) = (1 - |t|) \quad \text{if} \quad 0 \leq |t| \leq 1 \ ,$$

$$\widehat{J}(0) = 1 \ , \ \text{and} \ \widehat{J}(t) = \widehat{K}(t) = 0 \quad \text{if} \quad 1 \leq |t| \ .$$

If L is a positive integer we write $J_{L+1}(z) = (L+1)J((L+1)z)$ so that $J_{L+1}(z)$ has exponential type $2\pi(L+1)$. Then the Fourier transforms $\widehat{J}$ and $\widehat{J}_{L+1}$ are related by the identity $\widehat{J}((L+1)^{-1}t) = \widehat{J}_{L+1}(t)$ for all real t. Similar remarks apply to K and K_{L+1}.

For each positive integer L we define trigonometric polynomials $j_L(x)$ and $k_L(x)$ by

$$j_L(x) = \sum_{m=-\infty}^{\infty} J_{L+1}(x+m) = \sum_{\ell=-L}^{L} \widehat{J}_{L+1}(\ell) e(\ell x) \tag{2.2}$$

and

$$k_L(x) = \sum_{m=-\infty}^{\infty} K_{L+1}(x+m) = \sum_{\ell=-L}^{L} \widehat{K}_{L+1}(\ell) e(\ell x)\ . \tag{2.3}$$

The identities (2.2) and (2.3) follow from the Poisson summation formula. We also define the periodic function $\psi(x)$ by

$$\psi(x) = x - [x] - \tfrac{1}{2} \ \text{ if } \ x \notin \mathbb{Z}\,, \quad \text{and} \quad \psi(x) = 0 \ \text{ if } \ x \in \mathbb{Z}\ .$$

The trigonometric polynomials

$$\begin{aligned} \psi * j_L(x) &= \int_{-1/2}^{1/2} \psi(x-y) j_L(y)\, dy \\ &= \sum_{\substack{\ell=-L \\ \ell \neq 0}}^{L} (-2\pi i \ell)^{-1} \widehat{J}\left(\frac{\ell}{L+1}\right) e(\ell x) \end{aligned}$$

and $k_L(x)$ satisfy the basic inequality

$$|\psi(x) - \psi * j_L(x)| \leq (2L+2)^{-1} k_L(x) \tag{2.4}$$

for all x in $\mathbb{R}/\mathbb{Z}$. A proof of (2.4) is given in [3], Chapter 1, and in [4], Theorem 18. If $u < v < u+1$, then the periodic functions $\varphi_{u,v}(x)$ and $\psi(x)$ are related by the elementary identity

$$\varphi_{u,v}(x) = (v-u) + \psi(u-x) + \psi(x-v)\ . \tag{2.5}$$

By combining (2.4) and (2.5) we obtain the inequality

$$\begin{aligned} &|\varphi_{u,v}(x) - \varphi_{u,v} * j_L(x)| \\ &\quad \leq |\psi(u-x) - \psi * j_L(u-x)| + |\psi(x-v) - \psi * j_L(x-v)| \\ &\quad \leq (2L+2)^{-1}\{k_L(u-x) + k_L(x-v)\} \end{aligned} \tag{2.6}$$

for all x in $\mathbb{R}/\mathbb{Z}$. Alternatively, (2.6) follows directly from [4], Theorem 19.

We now establish some new inequalities.

Lemma 4. *Let α and β be real numbers such that $\beta - \alpha = M$ is a positive integer. Then*

$$0 \leq H(x-\alpha) + H(\beta - x) \tag{2.7}$$

for all real x.

Proof. From (2.1) we have

$$\begin{aligned}
H(x)+H(1-x) &= \left(\frac{\sin \pi x}{\pi}\right)^2 \left\{ \sum_{m=-\infty}^{\infty} \operatorname{sgn}(m)(x-m)^{-2} + 2x^{-1} \right. \\
&\qquad \left. - \sum_{n=-\infty}^{\infty} \operatorname{sgn}(n-1)(x-n)^{-2} + 2(1-x)^{-1} \right\} \\
&= \left(\frac{\sin \pi x}{\pi}\right)^2 \{x^{-2} + 2(1-x)^{-1} + 2x^{-1} + (1-x)^{-2}\} \\
&= \left(\frac{\sin \pi x}{\pi}\right)^2 \{x^{-1} + (1-x)^{-1}\}^2 \\
&\geq 0
\end{aligned}$$

for all real x. As H is an odd function, we conclude that

$$H(x) + H(M-x) = \sum_{m=0}^{M-1} \{H(x-m) + H(1+m-x)\} \geq 0 \ . \tag{2.8}$$

The lemma follows from (2.8) by replacing x with $x-\alpha$.

Lemma 5. *Assume that the periodic function $\varphi_{u,v}(x)$ belongs to $\mathcal{B}_1(L)$. Then the trigonometric polynomial*

$$\varphi_{u,v} * j_L(x) = \int_{-1/2}^{1/2} \varphi_{u,v}(x-y) j_L(y)\, dy \tag{2.9}$$

satisfies the inequality

$$0 \leq \varphi_{u,v} * j_L(x) \leq 1 \tag{2.10}$$

for all x in $\mathbb{R}/\mathbb{Z}$.

Proof. Write

$$\chi_{u,v}(x) = \tfrac{1}{2}\{\operatorname{sgn}(x-u) + \operatorname{sgn}(v-x)\}$$

for the normalized characteristic function of the real interval having endpoints u and v. As $u < v < u+1$ we have the obvious identity

$$\varphi_{u,v}(x) = \sum_{m=-\infty}^{\infty} \chi_{u,v}(x+m) \ . \tag{2.11}$$

Next we apply (2.7) with $\alpha = u(L+1)$, $\beta = v(L+1)$, and conclude that

$$\begin{aligned}
0 &\leq \tfrac{1}{2}\Big\{ H((L+1)(x-u)) + H((L+1)(v-x)) \Big\} \\
&= \tfrac{1}{2}(L+1) \int_u^v H'((L+1)(x-y))\, dy \\
&= \int_{-\infty}^{\infty} J_{L+1}(x-y)\chi_{u,v}(y)\, dy
\end{aligned} \tag{2.12}$$

for all real x. Then we use (2.2), (2.11), (2.12) and the fact that J_{L+1} is integrable, to establish the inequality

$$\begin{aligned} 0 &\le \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} J_{L+1}(x+n-y)\chi_{u,v}(y)\,dy \\ &= \int_{-\infty}^{\infty} j_L(x-y)\chi_{u,v}(y)\,dy \\ &= \sum_{m=-\infty}^{\infty} \int_{m-1/2}^{m+1/2} j_L(x-y)\chi_{u,v}(y)\,dy \\ &= \int_{-1/2}^{1/2} j_L(x-y)\left\{\sum_{m=-\infty}^{\infty} \chi_{u,v}(y+m)\right\}dy \\ &= \varphi_{u,v} * j_L(x)\ . \end{aligned} \tag{2.13}$$

Now let

$$\varphi_{v,u+1}(x) = (u+1-v) + \psi(v-x) + \psi(x-u-1)$$

be the normalized characteristic function of the complimentary interval in $\mathbb{R}/\mathbb{Z}$. Then

$$\varphi_{u,v} * j_L(x) + \varphi_{v,u+1} * j_L(x) = \int_{-1/2}^{1/2} j_L(y)\,dy = 1 \tag{2.14}$$

and we have just proved that

$$0 \le \varphi_{v,u+1} * j_L(x) \tag{2.15}$$

for all x in $\mathbb{R}/\mathbb{Z}$. Therefore (2.14) and (2.15) verify the inequality on the right of (2.10).

Lemma 6. *For each integer $n = 1, 2, \ldots, N$, let α_n, β_n and ε_n be real numbers such that $0 \le \alpha_n \le 1$, $0 \le \beta_n$, $\alpha_n - \beta_n \le \varepsilon_n$, and either $\varepsilon_n = 0$ or $\varepsilon_n = 1$. Then we have*

$$\prod_{n=1}^{N} \alpha_n - \sum_{n=1}^{N} \beta_n \prod_{\substack{m=1\\ m\ne n}}^{N} \alpha_m \le \prod_{n=1}^{N} \varepsilon_n\ . \tag{2.16}$$

Proof. If $\varepsilon_n = 1$ for each $n = 1, 2, \ldots, N$, then (2.16) is obvious. Assume that $\varepsilon_\ell = 0$ for some index ℓ, $1 \le \ell \le N$. It follows that $0 \le \alpha_\ell \le \beta_\ell$ and therefore

$$\prod_{n=1}^{N} \alpha_n - \sum_{\substack{n=1\\ n\ne \ell}}^{N} \beta_n \prod_{\substack{m=1\\ m\ne n}}^{N} \alpha_m - \beta_\ell \prod_{\substack{m=1\\ m\ne \ell}}^{N} \alpha_m \le -\sum_{\substack{n=1\\ n\ne \ell}}^{N} \beta_n \prod_{\substack{m=1\\ m\ne n}}^{N} \alpha_m \le 0\ .$$

This proves the lemma.

Let $\mathbf{L}$ be a point in $\mathbb{Z}^N$ with positive coordinates and $\Phi_{\mathbf{u},\mathbf{v}}$ a function in $\mathcal{B}_N$ having the representation (1.2). For each integer $n = 1, 2, \ldots, N$, we define trigonometric polynomials

$$\alpha_n(x_n) = \varphi_{u_n,v_n} * j_{L_n}(x_n)$$

and

$$\beta_n(x_n) = (2L_n+2)^{-1}\{k_{L_n}(x_n-u_n) + k_{L_n}(x_n-v_n)\}\ .$$

We assemble these into multiple trigonometric polynomials

$$A(\mathbf{x}) = \prod_{n=1}^{N} \alpha_n(x_n) \ , \tag{2.17}$$

$$B(\mathbf{x}) = \sum_{n=1}^{N} \beta_n(x_n) \prod_{\substack{m=1\\ m\neq n}}^{N} \alpha_m(x_m) \ , \tag{2.18}$$

and

$$C(\mathbf{x}) = \prod_{n=1}^{N} \{\alpha_n(x_n) + \beta_n(x_n)\} \ . \tag{2.19}$$

Here A, B and C depend on $\mathbf{u}$ and $\mathbf{v}$, but we drop reference to these points so as to simplify our notation. It is clear that the Fourier coefficients of A, B and C are supported on $\mathcal{L} \subseteq \mathbb{Z}^N$. In particular, we find that

$$\widehat{A}(\mathbf{0}) = \int_{(\mathbb{R}/\mathbb{Z})^N} A(\mathbf{x})\, d\mathbf{x} = \prod_{n=1}^{N} (v_n - u_n) \ , \tag{2.20}$$

$$\begin{aligned} \widehat{B}(\mathbf{0}) &= \sum_{n=1}^{N} (L_n+1)^{-1} \prod_{\substack{m=1\\ m\neq n}}^{N} (v_m - u_m) \\ &= \left\{ \sum_{n=1}^{N} w_n^{-1} \right\} \prod_{m=1}^{N} (v_m - u_m) \ , \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} \widehat{C}(\mathbf{0}) &= \prod_{n=1}^{N} \{(v_n - u_n) + (L_n+1)^{-1}\} \\ &= \left\{ \prod_{n=1}^{N} (1 + w_n^{-1}) \right\} \prod_{m=1}^{N} (v_m - u_m) \ . \end{aligned} \tag{2.22}$$

In case A, B and C take nonnegative values, we also get the estimates

$$|\widehat{A}(\boldsymbol{\ell})| \le \widehat{A}(\mathbf{0}) \ , \quad |\widehat{B}(\boldsymbol{\ell})| \le \widehat{B}(\mathbf{0}) \ \text{ and } \ |\widehat{C}(\boldsymbol{\ell})| \le \widehat{C}(\mathbf{0}) \tag{2.23}$$

for all $\boldsymbol{\ell}$ in $\mathbb{Z}^N$. Lemma 5 shows that A and B take nonnegative values if $\Phi_{\mathbf{u},\mathbf{v}}$ belongs to $\mathcal{B}_N(\mathbf{L})$, while (2.6) implies that C always takes nonnegative values.

Theorem 7. *We have*

$$\Phi_{\mathbf{u},\mathbf{v}}(\mathbf{x}) \le C(\mathbf{x}) \tag{2.24}$$

for all $\mathbf{x}$ *in* $(\mathbb{R}/\mathbb{Z})^N$*, and if* $\Phi_{\mathbf{u},\mathbf{v}}$ *belongs to* $\mathcal{B}_N(\mathbf{L})$*, then*

$$A(\mathbf{x}) - B(\mathbf{x}) \le \Phi_{\mathbf{u},\mathbf{v}}(\mathbf{x}) \tag{2.25}$$

for all $\mathbf{x}$ *in* $(\mathbb{R}/\mathbb{Z})^N$*.*

Proof. The inequality (2.24) is obvious from (2.6) and the definition of C.

In order to verify (2.25) let

$$E_{\mathbf{u},\mathbf{v}} = \{\mathbf{x} \in (\mathbb{R}/\mathbb{Z})^N : x_n = u_n \text{ or } x_n = v_n \text{ for some } n \ , \ 1 \le n \le N\} \ .$$

Then either $\varphi_{u_n,v_n}(x_n) = 0$ or $\varphi_{u_n,v_n}(x_n) = 1$ for each point $\mathbf{x}$ in $(\mathbb{R}/\mathbb{Z})^N \setminus E_{\mathbf{u},\mathbf{v}}$. From Lemma 5 we know that

$$0 \le \alpha_n(x_n) \le 1 \quad \text{and} \quad 0 \le \beta_n(x_n) \tag{2.26}$$

for all x_n in $\mathbb{R}/\mathbb{Z}$. And (2.6) implies that

$$\alpha_n(x_n) - \beta_n(x_n) \le \varphi_{u_n,v_n}(x_n) \tag{2.27}$$

for all x_n in $\mathbb{R}/\mathbb{Z}$. It follows using (2.26), (2.27) and Lemma 6 that

$$A(\mathbf{x}) - B(\mathbf{x}) \le \Phi_{\mathbf{u},\mathbf{v}}(\mathbf{x}) \tag{2.28}$$

for all points $\mathbf{x}$ in $(\mathbb{R}/\mathbb{Z})^N \setminus E_{\mathbf{u},\mathbf{v}}$. As the left hand side of (2.28) is a continuous function of $\mathbf{x}$, we have

$$A(\mathbf{x}) - B(\mathbf{x}) \le 0 \le \Phi_{\mathbf{u},\mathbf{v}}(\mathbf{x})$$

when $\mathbf{x}$ is in $E_{\mathbf{u},\mathbf{v}}$.

We note that the entire functions H and K satisfy the basic inequality

$$|\operatorname{sgn}(x) - H(x)| \le K(x) \tag{2.29}$$

for all real x. This is established in [4], Lemma 5. If we use (2.29) in place of (2.4) and apply Lemma 6, then it is possible to construct an entire function of N complex variables having exponential type and such that its restriction to $\mathbb{R}^N$ minorizes the characteristic function of an aligned box in $\mathbb{R}^N$. We do not pursue these ideas here as we require only the periodic version of this construction.

3. Proof of Theorems 1 and 3

Assume, as in the statement of Theorem 1, that $\Phi_{\mathbf{u},\mathbf{v}}$ belongs to $\mathcal{B}_N(\mathbf{L})$. Then we apply (2.20), (2.21), (2.23) and (2.25). In this way we obtain the inequality

$$\begin{aligned}
&M\prod_{n=1}^{N}(v_n - u_n) - \sum_{m=1}^{M}\Phi_{\mathbf{u},\mathbf{v}}(\boldsymbol{\xi}_m) \\
&\quad\le M\prod_{n=1}^{N}(v_n - u_n) + \sum_{m=1}^{M}\{B(\boldsymbol{\xi}_m) - A(\boldsymbol{\xi}_m)\} \\
&\quad= \sum_{\boldsymbol{\ell}\in\mathcal{L}}\widehat{B}(\boldsymbol{\ell})\sum_{m=1}^{M}e(\boldsymbol{\ell}\cdot\boldsymbol{\xi}_m) - \sum_{\substack{\boldsymbol{\ell}\in\mathcal{L}\\ \boldsymbol{\ell}\neq\mathbf{0}}}\widehat{A}(\boldsymbol{\ell})\sum_{m=1}^{M}e(\boldsymbol{\ell}\cdot\boldsymbol{\xi}_m) \\
&\quad= M\Big\{\sum_{k=1}^{N}w_k^{-1}\Big\}\prod_{n=1}^{N}(v_n - u_n) + \sum_{\substack{\boldsymbol{\ell}\in\mathcal{L}\\ \boldsymbol{\ell}\neq\mathbf{0}}}\{\widehat{B}(\boldsymbol{\ell}) - \widehat{A}(\boldsymbol{\ell})\}\sum_{m=1}^{M}e(\boldsymbol{\ell}\cdot\boldsymbol{\xi}_m) \\
&\quad\le M\delta\prod_{n=1}^{N}(v_n - u_n) + \{\widehat{B}(\mathbf{0}) + \widehat{A}(\mathbf{0})\}\sum_{\substack{\boldsymbol{\ell}\in\mathcal{L}\\ \boldsymbol{\ell}\neq\mathbf{0}}}\Big|\sum_{m=1}^{M}e(\boldsymbol{\ell}\cdot\xi_m)\Big| \\
&\quad\le M(\delta + \eta + \delta\eta)\prod_{n=1}^{N}(v_n - u_n)\ .
\end{aligned} \tag{3.1}$$

The inequality (1.7) plainly follows from (3.1).

The proof of Theorem 3 is essentially the same but uses (2.22), (2.23) and (2.24).

4. Proof of Corollary 2

Select $\mathbf{L}$ in $\mathbb{Z}^N$ so that $L_n = [N\varepsilon_n^{-1}]$ and note that

$$2 \leq L_n \quad \text{and} \quad \frac{N}{L_n + 1} < \varepsilon_n \tag{4.1}$$

for each $n = 1, 2, \ldots, N$. Then select $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^N$ by setting

$$u_n = -N(L_n + 1)^{-1} \quad \text{and} \quad v_n = N(L_n + 1)^{-1}$$

for each $n = 1, 2, \ldots, N$. From (1.8) and (4.1) we conclude that

$$\Phi_{\mathbf{u},\mathbf{v}}(\boldsymbol{\xi}_m) = 0 \tag{4.2}$$

for each $m = 1, 2, \ldots, M$. Now let $0 < \delta$ and $0 < \eta$ satisfy (1.5) and (1.6) in the statement of Theorem 1. In view of (4.2) and the conclusion (1.7) of Theorem 1, we must have

$$1 - \delta - \eta - \delta\eta \leq 0 \ . \tag{4.3}$$

As $w_n = (v_n - u_n)(L_n + 1) = 2N$ for each $n = 1, 2, \ldots, N$, we can take $\delta = 1/2$. Then $1/3 \leq \eta$ follows immediately from (4.3). This verifies the corollary.

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