PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 2, Pages 621–622
S 0002-9939(00)05869-X
Article electronically published on September 27, 2000

NO n-POINT SET IS σ-COMPACT

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(Communicated by Alan Dow)

ABSTRACT. Let n be an integer greater than 1. We prove that there exist no F_σ-subsets of the plane that intersect every line in precisely n points.

Let $n \geq 2$ be some fixed integer. A subset of the plane $\mathbb{R}^2$ is called an *n-point set* if every line in the plane meets the set in precisely n points. The question of whether n-point sets can be Borel sets is a long standing open problem (see e.g. Mauldin [6] for details). Sierpinski [7, p. 447] has given a simple example of a closed set that meets every line in $\aleph_0$ points. It was shown by Baston and Bostock [1] and by Bouhjar, Dijkstra, and van Mill [2] that 2-point sets, respectively 3-point sets, cannot be F_σ in the plane. Both papers use a method suggested by Larman [5] for the case $n = 2$ which consists of proving on the one hand that 2-point sets cannot contain arcs and on the other hand that 2-point sets that are F_σ must contain arcs. Observe that to prove the result that is the subject of this note Larman's program cannot be followed because it was shown in [2] that n-point sets can contain arcs whenever $n \geq 4$.

Theorem. *Let $n \geq 2$. No n-point set is an F_σ-subset of the plane.*

The three authors of this note each, independently of each other, found a proof for this theorem. We decided to publish the shortest proof jointly.

Proof. Assume that A is an n-point set that is an F_σ-subset of the plane. Let xy be an *arbitrary* rectangular coordinate system for the plane and let λ be the Lebesgue measure on $\mathbb{R}$. According to [2, Proposition 3.2] there exists a nondegenerate interval $[a, b]$ on the x-axis and continuous functions $f_1 < f_2 < \cdots < f_n$ from $[a, b]$ into $\mathbb{R}$ such that A contains the graph of each f_i. Consider an f_i and its graph G_i. Since A is an n-point set each horizontal line intersects G_i in at most n points. So every fibre of f_i has cardinality at most n. Consequently, according to Banach [4, Exercise 17.34], the variation of f_i is bounded by $n(M - m)$, where m and M are the minimum and maximum values of f_i. According to Lebesgue [4, Theorem 17.17] the derivative of a function of bounded variation such as f_i exists almost everywhere. Select a Borel set $B \subset [a, b]$ such that $\lambda(B) = b - a$ and every f_i is differentiable at every point of B. By the Whitney Extension Theorem for C^1 functions [3, Theorem 3.1.16] there exists a set $C \subset B$ such that $\lambda(C) > 0$ and

Received by the editors November 30, 1999 and, in revised form, January 7, 2000.
2000 *Mathematics Subject Classification.* Primary 54H05, 57N05, 54G99.
The second author was supported in part by a grant from the Research Advisory Committee of the University of Alabama.
The third author was supported in part by NSF grant DMS-9801583.

continuously differentiable functions $g_i : [a, b] \to \mathbb{R}$ with $g_i|C = f_i|C$ for $1 \leq i \leq n$. The functions g_i satisfy the premises of Theorem 7 in [6] so we may conclude that the set A is bounded or intersects some line in $n + 1$ points. Either way, the result is inconsistent with A being an n-point set. □

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