ON CONTRACTIBLE \(n\)-DIMENSIONAL COMPACTA, NON-EMBEDDABLE INTO \(\mathbb{R}^{2n}\)

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Abstract. We present a very short proof of a well-known result, that for each \(n\) there exists a contractible \(n\)-dimensional compactum, non-embeddable into \(\mathbb{R}^{2n}\).

We present a very short proof of the following well-known result, which answers a question from [DD] and was first proved in [RSS, Corollary 1.5] (later an alternative proof appeared in [KR]).

Theorem. For each \(n \geq 1\) there exists a contractible \(n\)-dimensional compactum which does not embed into \(\mathbb{R}^{2n}\).

We shall use a construction and an idea from [RSS] (see also [CRS, §4], [RS1] and [RSSp]). However, instead of using the main result of [We], we shall apply its corollary, to the effect that for every \(n\) there exists a contractible \(n\)-polyhedron \(X\), for which there is no equivariant map \(\tilde{X} \to S^{2n-1}\). A simple proof of this corollary was presented in [Sc, p. 223]. Our proof also makes it possible to avoid referring to a (not difficult) result in [CF, Theorem 2.5] and [Hu].

Proof of Theorem. There exist a contractible \(n\)-polyhedron \(X\) and a map \(\varphi : S^{2n-1} \to X\) which does not identify antipodal points [Sc, p. 223]. (Notice that the map \(\varphi_n = p^{n,\partial(D^2)} : \partial(D^2)^n \to T^n\) also has this property, where \(T\) is the triod and \(p\) is the map defined in [KR, §2]. Indeed, \(\varphi_1\) does not identify antipodal points [KR, §2], hence neither does \(\varphi_n\).) Let \(X' = X \times (0 \cup \{\frac{1}{k}\}) \cup x \times [0,1]\), where \(x \in X\). Clearly, \(X'\) is contractible.

Suppose that there existed an embedding \(f : X' \to \mathbb{R}^{2n}\). Then we could define a map \(\psi : S^{2n-1} \to X \times X\) by \(\psi(s) = (\varphi(s), \varphi(-s))\). Since \(\varphi\) does not identify antipodal points, it would follow that \(\psi(S^{2n-1}) \cap \text{diag} X = \emptyset\). Hence the maps \(g_0 : \psi(S^{2n-1}) \to S^{2n-1}\) and \(g_k : X \times X \to S^{2n-1}\) given by

\[
g_0(x,y) = \frac{f(x,0) - f(y,0)}{|f(x,0) - f(y,0)|} \quad \text{and} \quad g_k(x,y) = \frac{f(x,0) - f(y,\frac{1}{k})}{|f(x,0) - f(y,\frac{1}{k})|}
\]
would be well-defined. The maps \( \psi, g_0 \) and \( g_k \) would be equivariant with respect to involutions on \( \psi(S^{2n-1}) \subset X \times X \) and \( S^{2n-1} \), exchanging factors and antipodal points, respectively.

Since \( \text{dist}(\psi(S^{2n-1}), \text{diag } X) > 0 \), it would follow that for sufficiently large \( k \) and any point \( (x, y) \in \psi(S^{2n-1}) \), the points \( g_0(x, y) \) and \( g_k(x, y) \) would be close and hence could not be antipodal. Therefore \( g_0 \simeq_{eq} g_k|_{\psi(S^{2n-1})} \). But \( g_k|_{\psi(S^{2n-1})} \) extends to a contractible space \( X \times X \) and therefore is null-homotopic. Hence \( g_0 : \psi(S^{2n-1}) \to S^{2n-1} \) is null-homotopic. Thus the map \( g_0 \circ \psi : S^{2n-1} \to S^{2n-1} \) is equivariant and null-homotopic, which contradicts the Borsuk-Ulam Theorem. So \( X' \) cannot embed into \( \mathbb{R}^{2n} \).

By attaching \( k \)-dimensional cells to \( X' \) we can make \( X' \) locally \((k-1)\)-connected, hence our compactum can even be made to be locally \((n-1)\)-connected. This observation (due to R. J. Daverman) is interesting because the Borsuk Conjecture states that every contractible locally \( n \)-connected \( n \)-dimensional compactum embeds into \( \mathbb{R}^{2n} \).

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**References**


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