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ON THE NUMBER OF GENERATORS OF THE TORSION MODULE OF DIFFERENTIALS

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Abstract. In this paper we study the (minimum) global number of generators of the torsion module of differentials of affine hypersurfaces with only isolated singularities. We show that for reduced plane curves the torsion module of differentials can be generated by at most two elements, whereas for higher codimensions there is no universal upper bound. We then proceed to give explicit examples. In particular (when $N \geq 5$) , we give examples of a reduced hypersurface with a single isolated singularity at the origin in $\mathbf{A}_K^N$ that require

$$\frac{N!}{2} + N(N-1)/2$$

generators for the torsion module, $\mathrm{Torsion}(\Omega_{A/K}^{N-1})$.

1. Introduction

In this paper we consider the torsion modules of differentials for affine isolated hypersurface singularities, extending earlier results of the author [5] and previous results of G.M. Greuel [2] in the local analytic quasi-homogeneous case. Throughout K is an algebraically closed field of characteristic zero. Let $R = K[X_1, X_2, \dots, X_N]$ and let A be the coordinate ring of a reduced hypersurface X with only isolated singularities in $\mathbf{A}_K^N$, i.e. $A = \frac{R}{(F)}$, where F is a reduced polynomial in R. By abuse of notation we call A a reduced hypersurface with only isolated singularities. We will write $\mathrm{Torsion}(\Omega_{A/K}^i)$ for the torsion submodule of $\Omega_{A/K}^i$, where $i = 0, \dots, N$, and J for the Jacobian ideal

$$(\frac{\partial F}{\partial X}1, \dots, \frac{\partial F}{\partial X}N).$$

Recall (cf. [3]) that a reduced hypersurface A has an isolated singularity at $P \in \mathrm{Spec}(A)$ if there exists an element $s \in A - P$ such that $\mathrm{Sing}(A_s) = \{P_s\}$. In [4], Theorem 1, we show:

Proposition 1.1. *Let K be an algebraically closed field of characteristic zero. Let $F \in K[X_1, X_2, \dots, X_N]$ be a reduced polynomial defining an affine hypersurface A*

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with only isolated singularities. Let

$$(J : F) = \{Q \in R : FQ \in J\}.$$

Then there exists an isomorphism η of A-modules:

$$\eta : \text{Torsion}(\Omega^{N-1}_{A/K}) \to (J : F)/J.$$

If A is defined by a reduced polynomial F has only isolated singularities, then

$$\text{ht } (F, \frac{\partial F}{\partial X_1}, \dots, \frac{\partial F}{\partial X_N}) = N$$

and $\Omega^N_{A/K}$ and (hence by [4]) $\text{Torsion}(\Omega^{N-1}_{A/K})$) are finite dimensional K vector spaces. Moreover both modules are supported only on a finite number s of maximal ideals $m_1, \dots, m_s$ in A say, or

$$Supp(\text{Torsion}(\Omega^{N-1}_{A/K})) = \{m_1, \dots, m_s\}.$$

Unlike in the case of a quasi-homogeneous hypersurface (cf. [5]) $\text{Torsion}(\Omega^{N-1}_{A/K})$ is not always cyclic even if some permutation of $\frac{\partial F}{\partial X}1, \dots, \frac{\partial F}{\partial X}N$ is an R-sequence. We have seen in [4]:

Example 1.2. Consider the plane curve with a single isolated singularity at the origin defined by $F : X^3Y^2 + Y^5 + X^7 = 0$. (Here $\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}$ is an R-sequence.) Since $(J : F)/J$ is minimally generated by $p_1 = (2X^3+5Y^3)+J$ and $p_2 = (3Y^2+7X^4)+J$, the torsion module $\text{Torsion}(\Omega^1_{A/K})$ is not cyclic.

So it is natural to ask how large the minimum number of global generators of the torsion module of differentials can get. Here we investigate the number of generators for $\text{Torsion}(\Omega^{N-1}_{A/K})$ as an A-module. The paper is organized as follows: In section 2 we prove a reduction to the local case. In section 3 we prove that for reduced curves the number of generators of the torsion module of differentials is bounded above by two. In section 4 we exhibit hypersurfaces with a single isolated singularity at the origin that require a large number of generators for the torsion module of differentials.

2. Reduction to the local case

The singular locus $Sing(X)$ of the hypersurface A is described by

$$Sing(X) := Spec(R/(F, \frac{\partial F}{\partial X}1, \dots, \frac{\partial F}{\partial X}N))$$

and it contains only finitely many maximal ideals of R.

The torsion module $\text{Torsion}(\Omega^{N-1}_{A/K})$ is supported on the (finitely many) maximal ideals in the singular locus of the hypersurface for hypersurfaces with only isolated singularities. So we can describe the torsion module by describing the finitely many localizations of it at the maximal ideals in the singular locus. We have the following isomorphism:

$$\text{Torsion}(\Omega^{N-1}_{A/K}) \simeq \prod_{m \in Support} \text{Torsion}(\Omega^{N-1}_{A/K})_m.$$

As an easy consequence of this and the Chinese Remainder Theorem, we have the following:

Theorem 2.1. *Let A be a reduced hypersurface with only isolated singularities in $\mathbf{A}^N_K$. Then:*

$$\mu(\mathrm{Torsion}(\Omega^{N-1}_{A/K})) = max\{\mu(M, \mathrm{Torsion}(\Omega^{N-1}_{A/K})) : M \in Sing(A)\},$$

where $\mu(M, \mathrm{Torsion}(\Omega^{N-1}_{A/K})) := \mu(\mathrm{Torsion}(\Omega^{N-1}_{A/K} \otimes_A A_M))$ is the number of local generators of $\mathrm{Torsion}(\Omega^{N-1}_{A/K})_M$ as an A_M-module and M denotes a maximal ideal in the singular locus.

Remark 2.2. Note for a hypersurface A with only isolated singularities we have that the number of generators of $\mathrm{Torsion}(\Omega^{N-1}_{A/K})$ is always bounded above by $\dim_K \Omega^N_{A/K} < \infty$.

Remark 2.3. By [4] we have that, for any maximal ideal M in the singular locus of the hypersurface, the local Jacobian ideal J_M is generated by an R_M sequence and $(R/J)_M$ is a local Gorenstein Artin ring.

3. Curves

We will now only consider plane singular curves: Let $R := K[X, Y]$ where K is an algebraically closed field of characteristic zero and let the reduced, singular, plane curve be defined by the polynomial F. We have the following

Theorem 3.1. *Let A be the coordinate ring of a reduced, singular, plane curve. Then the torsion module of differentials $\mathrm{Torsion}(\Omega^1_{A/K})$ can be generated by at most 2 elements.*

Proof. The singular locus $Sing(X)$ of a plane curve X is described by the ideal $(F, \frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y})$, i.e. $Sing(X) := Spec(R/(F, \frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}))$ and contains only finitely many maximal ideals of R. We will show that $\frac{(J:F)}{J}$ is generated by at most 2 elements, where J denotes the Jacobian ideal $(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y})$.

Let M be any maximal ideal in $Sing(X)$ and localize at M. The ring $(R/J)_M$ is a local Gorenstein ring (by Remark 2.3). We have that $(J_M : F_M) = (J : F) \otimes R_M := (J : F)_M$ (cf. [7]). There are two cases:

Case A. If $F_M \in J_M$, i.e. F is locally quasi-homogeneous, then $(J : F)_M = R_M$ and $\mathrm{Torsion}(\Omega^1_{A/K})$ is cyclic at M.

Case B. If $F_M \notin J_M$, then since $\frac{\partial F}{\partial X}_M, \frac{\partial F}{\partial Y}_M$ is an R_M sequence we have that $(J_M : F_M)$ has codimension 2 in R_M and $R_M/(J : F)_M$ is Gorenstein (cf. [1], ex. 21.15, p.550). By a well known result of J.P. Serre [8] it follows that $(J : F)_M$ and can be generated by two elements. Since $\mu(M, (J : F)/J) \leq \mu(M, (J : F))$, the result follows upon invoking Proposition 1.1 and Theorem 2.1. □

In [5] it was shown that for any hypersurface defined by a quasi-homogeneous polynomial the torsion module of differentials is cyclic. However the converse is false globally. We have:

Example 3.2. $F := Y^2 - X^3 - X^2$. The Jacobian ideal J is given by

$$J := (2Y, -X(3X + 2)).$$

$F \notin J$, so F is not quasi-homogeneous , but the torsion module is generated by just one element corresponding to the generator $(3X + 2) + J$ of $(J : F)/J$.

Note that the above example also shows that Case B (in the proof of Theorem 3.1) occurs.

4. Isolated hypersurface singularities with large torsion modules

In this section we show that it is possible to construct hypersurfaces with isolated singularities that require an arbitrarily large number of generators for the torsion submodule of $\Omega^d_{A/K}$, where d is the dimension of the hypersurface. More specifically, we consider the hypersurface with a single isolated singularity in $\mathbf{A}^N_K$ defined by the polynomial

$$F := X_1^{N+1} + X_2^{N+1} + \cdots + X_N^{N+1} + X_1 X_2 \cdots X_N.$$

The singular locus of this hypersurfaces is described by the ideal

$$I := (F, \frac{\partial F}{\partial X_1}, \dots, \frac{\partial F}{\partial X_N}).$$

The partial derivatives are given by:

$$F_i := \frac{\partial F}{\partial X_i} = (N+1)X_i^N + X_1 X_2 \cdots \hat{X}_i \cdots X_N.$$

First we observe the following:

Lemma 4.1. *Using the deg-lex ordering on $X_1, ..., X_N$ we have that a Groebner basis for*

$$J := (\frac{\partial F}{\partial X_1}, \dots, \frac{\partial F}{\partial X_N})$$

is J.

Proof. The initial terms form an R-sequence. □

We have that F and the Jacobian ideal are invariant under the action of the symmetric group on N letters, S_N, by permutation of variables. Moreover when using the deg-lex ordering on monomials the formation of *S-pairs* commutes with the action of S_N for any two elements in the ideal I. In order to simplify the notation, we use the Vasconcelos log-notation (cf. [9], p.10) to represent monomials. Hence $(0, 1, 5, \dots, 4)$ represents the monomial $\hat{X}_1 X_2 X_3^5 \cdots X_N^4$, where $\hat{X}_1$ means omit the variable X_1. We will need to introduce the following notation:

$$(i_1, i_2, \dots, i_N)^\Sigma$$

denotes the orbit of $X_1^{i1} X_2^{i2} \cdots X_N^{iN}$ under the action of $\Sigma := S_N$. As always S_N acts by permutation of the variables. E.g. $(0, 1, 5, \dots, 4)^\Sigma$ denotes the orbit under the action of S_N of the monomial $X_2 X_3^5 \cdots X_N^4$. We have:

Lemma 4.2. *The Groebner basis for the ideal I in R,*

$$I := (F, \frac{\partial F}{\partial X_1}, \dots, \frac{\partial F}{\partial X_N}),$$

under the deg-lex ordering is given by:

$$\{[X_2^2 \cdots X_N^2]^\Sigma, [X_1 X_3^3 \cdots X_N^3]^\Sigma,$$

$$\dots, [X_1^1 \cdots X_k^k \hat{X_{k+1}} X_{k+2}^{k+2} \cdots X_{N-1}^{k+2} X_N^{k+2}]^\Sigma, \dots,$$

$$[X_1 X_2^2 \cdots X_{N-2}^{N-2} X_{N-1}^{N-1}]^\Sigma, F_1, \dots, F_N, X_1 \cdots X_N\},$$

where $k = 0, \dots, N-2$ and we use the convention that $X^i = 1$ whenever $i \leq 0$ or $i \geq N+1$.

Proof. First note that

$$F \equiv \frac{1}{N+1} X_1 X_2 \cdots X_N \bmod J\ .$$

Hence it suffices to compute a Groebner basis for the ideal generated by

$$(F_1, \ldots, F_N, X_1 X_2 \cdots X_N).$$

The proof consists in a computation of all possible S-pairs. This daunting task becomes feasible when keeping in mind the following facts: Any S-pair between monomials is zero. An S-pair of a binomial with a monomial is again a monomial. An S-pair of binomials with co-prime leading terms is zero. We introduce the following notation for the S-pairs obtained with the F_i's and all previous S-pairs between partial derivatives and F. It is easy to see from the previous remarks that these are the only ones that could be non zero. Let the first element be:

$$g_0 = X_1 X_2 \cdots X_N.$$

Note that this is the reduction of F mod J. For $i = 1, \ldots, N$ we have:

$$g_i = S(g_0, F_i) = X_1^2 X_2^2 \cdots X_{i-1}^2 \hat{X}_i X_{i+1}^2 \cdots X_N^2.$$

In our orbit notation g_i is obtained from

$$g_1 = S(F_1, g_0) = X_2^2 X_3^2 \cdots X_N^2$$

by applying the permutation $1 \leftrightarrow i$ to the above definition to obtain the definition for g_i , where $i = 1, 2, \ldots, N$. In fact any permutation interchanging 1 and i would do.

$$g_{1,2,\ldots,k+1} := S(F_{k+1}, g_{1,\ldots,k})$$

$$= (\frac{g_{12\ldots k}}{X_{k+1}^{k+1}}) F_{k+1} - (N+1) g_{12\ldots k} X_{k+1}^{N-(k+1)}$$

$$= X_1^k X_2^{k-1} \cdots X_{k-1}^2 X_k \hat{X_{k+1}} X_{k+2}^{k+2} \cdots X_N^{k+2}.$$

Again if the i_j's are all distinct ($j = 1, \ldots, k$) the general

$$g_{i_1,\ldots,i_{k+1}}$$

is defined by applying the permutation, sending i_1 to i and keeping the remaining variables fixed to the element $g_{12\ldots k+1}$.

The proof is now complete, by the following lemma:

Lemma 4.3. *With the above notation, we have that*

$$g_{i_1\ldots i_s} = 0 \text{ mod previous S-pairs}$$

whenever the i_j's are not all distinct.

Proof. An inductive argument. The result holds when $s = 1$. In the case when $s = 2$, it is a consequence of the fact that g_i and F_i have coprime leading terms. Again it suffices to prove that

$$S(g_{12\ldots k}, F_j) \equiv 0 \text{ modulo previous S-pairs when } j \in \{1, \ldots, k\}.$$

If $j = k$, then the leading terms of $g_{12\ldots k}$ and F_j are coprime, and the S-pair is zero. Now assume $j \neq k$; then

$$S(g_{12\ldots k}, F_j) = (N+1) X^{N-j} g_{12\ldots N} - (\frac{g_{12\ldots k}}{X_j^j}) F_j = g_{k12\ldots\hat{j}\ldots k-1j}.$$

Next we observe that:

$$X_k^{N+1} = \sum_{i \neq k} \frac{1}{N+1} X_i \frac{\partial F}{\partial X_i} + \frac{2}{N+1} \frac{\partial F}{\partial X_k} - F.$$

Hence $M := (X_1, \dots, X_N)$ is the only maximal ideal in the singular locus of the hypersurface defined by F. It follows that the ring R/I is local with maximal ideal $m := M + I$. By Remark 2.3 or Theorem 2.6 in [6], the number of generators of the torsion module of differentials is equal to the socle-dimension in the ring R/I. Theorem 2.6 states:

Let A be the coordinate ring of a reduced, affine hypersurface with a single isolated singularity at the origin. Then we have that n_{Tors} , the number of generators of the torsion module of differentials $\mathrm{Torsion}(\Omega^{N-1}_{A/K})$*, is given by:*

$$n_{Tors} = dim_K soc(\Omega^N_{A/K}).$$

We have:

Theorem 4.4. *Let $N \geq 5$; then for the hypersurface defined by*

$$F := X_1^{N+1} + X_2^{N+1} + \cdots + X_N^{N+1} + X_1 X_2 \cdots X_N$$

the torsion module of differentials $\mathrm{Torsion}(\Omega^{N-1}_{A/K})$ *requires* $N!/2 + \binom{N}{2}$ *generators.*

Proof. Locally the socle is computed by $Socle(R/I) := (0 : m)$. The elements in the socle of R/I are then given by the intersection of the annihilator ideals of $X_1, \dots, X_N$ repectively. The elements in this intersection are precisely the elements in $(0 : X_1)$ say, that have exponent sequences that form complete orbits under the action of S_N. To compute $(0 : X_1)$ one subtracts one in the first coordinate of all exponent sequences in the description of the Groebner basis for I.

For $N \geq 5$ we have the following description of R/I:

The elements that reduce to zero mod I, i.e. the elements of the Groebner basis for I, can be represented by the following exponent sequences:

$$(0, 2, 2, 2, \dots, 2)^\Sigma, (1, 0, 3, \dots, 3, 3)^\Sigma, \dots, (N-4, N-3, \dots, 0, N-2, N-2)^\Sigma,$$

$$(0, 1, 2, \dots, N-3, N-2, N-1)^\Sigma.$$

The elements of $(0 : m)$ correspond to exponent sequences that reduce to 0 *mod* $GB[I]$, when one of their coordinates is increased by one. The orbits of the following monomials under the action of Σ generate $(0 : m)$:

$$(0, 1, 2, \dots, N-2, N-2)^\Sigma, (N-1, N-1, \dots, N-1, 0, 0)^\Sigma.$$

The elements in the orbit of the exponent sequence $(0, 1, 2, \dots, N-2, N-2)$ are called elements of the first type. They are obtained by subtracting one from the last coordinate of the last elements in the Groebner basis. Now consider the element $(N-1, \dots, N-1, 0, 0)$. After multiplying the corresponding element with X_1 we use the relation

$$(N, 0, 0, \dots, 0) \sim (0, 1, \dots, 1)$$

and its permutations three times, we get:

$$(N, N-1, \dots, N-1, 0, 0) \sim (0, N, N, \dots, 1, 1) \sim (1, 0, N+1, \dots, N+1, 2, 2).$$

The last exponent sequence represents an element of I and so

$$(N-1,\dots,N-1,0,0)$$

represents an element of $(0 : X_1)$. Exploiting the symmetry in the first $N-2$ entries, we see that

$$(N-1,N-1,\dots,N-1,0,0)$$

is an element of $\bigcap_{i=1}^{N-2}(0 : X_i)$. On the other hand multiplying the element by X_{N-1} or X_N yields sequences $(N-1,\dots,N-1,1,0)$ and $(N-1,\dots,N-1,0,1)$ respectively both of which represent multiples of elements in $GB[I]$. Hence in summary, if $N \geq 5$, we have that the elements in the orbit of the exponent sequence $(N-1,N-1,\dots,N-1,0,0)$ also lie in $(0 : m)$. They are called elements of the second type. These elements only arise when $N \geq 5$, because we needed to apply the relation three times on the non-zero coordinates. S_N acts N-transitively on R and there is no duplication within orbits. Hence there are $N!/2$ elements of the first type in $(0 : m)$ and

$$\binom{N}{2}$$

elements of the second type, so there are

$$\binom{N}{2} + \frac{N!}{2}$$

elements in $\mathrm{Torsion}(\Omega_{A/K}^{N-1})$. There cannot be any more, because two is the maximum number of zero entries and we have exhausted all the possibilities for $0,1,2$ zero coordinates. □

Remark 4.5. For $N = 3$ and $N = 4$ the elements of $(0 : m)$ are the orbits of the monomials with the exponent sequence: $(2,1,0)$ for $N = 3$, and the orbit of elements with exponent sequence $(3,2,1,0)$ when $N = 4$. Hence for $N = 3$ the torsion module of differentials is $\frac{3!}{2}$ dimensional and for $N = 4$ the torsion module of differentials is $\frac{4!}{2}$ dimensional.

Added after posting

Dr. Ruth Michler: March 8, 1967–November 1, 2000

Dr. Ruth Michler died tragically November 1, 2000, in a traffic accident while waiting to cross a street in Boston. She was an Associate Professor at the University of North Texas and was visiting Northeastern University for the year.

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