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A STRONG UNIFORM BOUNDEDNESS PRINCIPLE IN BANACH SPACES

OLAV NYGAARD

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Abstract. We discuss the term "thick" set. With the help of this term we deduce a strong Uniform Boundedness Principle valid for all Banach spaces. As an application we give a new proof of Seever's theorem.

1. Definitions and preliminary results

In this paper X is a Banach space. In [3] the notion of a "thin" set was introduced. It turns out that if a bounded set $A \subset X$ is not thin, then for any Banach space Y, any $T \in \mathcal{L}(Y, X)$ onto A has to be onto X. Further, "not thin" is the weakest general condition such that "onto A" implies "onto X".

It is natural to ask for the weakest general condition to be put on a bounded set A to assure that pointwise boundedness of a family of operators in $\mathcal{L}(X, Y)$ implies uniform boundedness. The Banach-Steinhaus theorem tells us that "second category" is a sufficient condition, but the Nikodým boundedness theorem shows that a uniform boundedness principle is true under weaker conditions, in particular spaces at least. Our main result in this paper is that "not thin" is the condition sought for this problem as well.

Since, for our purposes at least, "not thin" seems to be more important than "thin", we prefer to use the terms "thin" and "thick" instead of "thin" and "not thin".

Definition 1.1. A bounded set $B \subset X$ is called non-norming for X^* if

$$\inf_{f \in S_{X^*}} \sup_{x \in B} |f(x)| = 0.$$

A bounded set $C \subseteq X$ is called *thin* if it is the countable union of an increasing family of sets which are non-norming for X^*. If a bounded set C is not thin, we will call it *thick*.

In [3] Kadets and Fonf showed that there is an "open mapping principle" associated to thickness:

Theorem 1.1. *Suppose $B \subset X$ is bounded. Then the following are equivalent statements:*

1) *For all Banach spaces Y and all $T \in \mathcal{L}(Y, X)$ with $TY \supset B$, we have $TY = X$.*

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2) *B is thick.*

Furthermore, if 2) *fails, an operator T contradicting* 1) *may be chosen to be* 1-1.

2. Main results

From Seever's theorem [1], [4] and the classical Nikodým boundedness theorem [1] it is natural to propose the following generalization of Nikodým's theorem:

Proposition 2.1. *Suppose Φ is a family of elements in a dual space X^* such that*

$$\sup_{x^*\in\Phi} |x^*(x)| < \infty$$

for each x belonging to a thick subset $B \subset X$. Then Φ is a bounded subset of X^*, i.e.

$$\sup_{x^*\in\Phi} ||x^*|| < \infty.$$

Proof. Write $\Phi(x) = \{x^*(x)\colon x^* \in \Phi\}$. Put

$$B_m = \{x \in B\colon |\Phi(x)| \le m\}.$$

Then (B_m) is an increasing family of sets which covers B. Since B is thick some B_q is norming. Then, using the Hahn-Banach theorem, it is easy to see that there exists a $\delta > 0$ such that

$$\overline{co}(\pm B_q) \supseteq \delta B_X.$$

But then if $x^* \in \Phi$,

$$\delta||x^*|| = \sup_{x\in\delta S_X} |x^*(x)| \le \sup_{x\in\overline{co}(\pm B_q)} |x^*(x)| \le q.$$

Thus $\sup_{x^*\in\Phi} ||x^*|| \le q/\delta$. The result follows. □

The above proof builds on ideas from [2]. Proposition 2.2 gives an analogue of the classical Banach-Steinhaus theorem for Banach spaces. It can be proved in exactly the same way as above.

Proposition 2.2. *Suppose Γ is a subset of $\mathcal{L}(X,Y)$ and B is a thick subset of S_X. If $\Gamma(x) = \{Tx\colon T \in \Gamma\}$ is a bounded subset of Y for each $x \in B$, then Γ is a bounded subset of $\mathcal{L}(X,Y)$.*

Note that the Banach-Steinhaus theorem follows as a corollary to Proposition 2.2 since every set of the second category is thick. The classical Nikodým boundedness theorem shows that there exists a Banach space for which the condition "second category" in the Banach-Steinhaus theorem can be weakened. This is, however, not the case with the condition "thick" in Proposition 2.2 as the next result shows.

Proposition 2.3. *If X and Y are Banach spaces and $B \subset S_X$ is thin, there is always a countable set $\Gamma \subset \mathcal{L}(X,Y)$ such that $\Gamma(x)$ is bounded in Y for every $x \in B$, but Γ is not bounded.*

Proof. Since B is thin we can pick a countable, increasing covering, $\bigcup B_n$ of B, consisting of non-norming sets only, and a sequence $(f_n) \subset X^*$ such that $f_n \in nS_{X^*}$ but $\sup_{B_n} |f_n(x)| < 1$. Let x be an arbitrary element of B. Then there is a natural number m such that $x \in B_m$. Thus, since (B_n) is increasing,

$$|f_k(x)| \le \sup_{1\le k<m} ||f_k||\,||x|| < m||x|| \qquad \text{if } k < m,$$

while

$$|f_k(x)| \leq 1 \qquad \text{if } k \geq m.$$

The proposition is thus proved in the case $Y =$ scalars. To extend the result to an arbitrary Banach space Y, let $y \in S_Y$ and put $T_n(x) = f_n(x)y$. □

The results obtained so far can be combined to give a more complete description of thick sets.

Theorem 2.4. *Suppose B is a bounded subset of X. The following statements are equivalent:*

1) *For all Banach spaces Y, all families $\Gamma \subset \mathcal{L}(X, Y)$ whose orbits $\Gamma(x)$ are bounded subsets of Y for every $x \in B$ are bounded subsets of $\mathcal{L}(X, Y)$.*
2) *Every countable family $\Phi \subset X^*$ which is pointwise bounded on B is uniformly bounded.*
3) *For all Banach spaces Y and all $T \in \mathcal{L}(Y, X)$ with $TY \supset B$, we have $TY = X$.*
4) *For all Banach spaces Y and all injections $T \in \mathcal{L}(Y, X)$ with $TY \supset B$, we have $TY = X$.*
5) *B is thick.*

From this theorem and the Nikodým boundedness theorem, Seever's theorem immediately follows.

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Deparment of Mathematics, Agder College, Tordenskjoldsgate 65, 4604 Kristiansand, Norway

E-mail address: Olav.Nygaard@hia.no