

A NUMERICAL RANGE CHARACTERIZATION OF UNIFORMLY SMOOTH BANACH SPACES

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ABSTRACT. We prove that a Banach space X is uniformly smooth if and only if, for every X -valued bounded function f on the unit sphere of X , the intrinsic numerical range of f is equal to the closed convex hull of the spatial numerical range of f .

1. INTRODUCTION

Let X be a Banach space over \mathbb{K} ($= \mathbb{R}$ or \mathbb{C}). We denote by S_X , B_X , and X^* the unit sphere of X , the closed unit ball of X , and the dual space of X , respectively. For u in S_X , we denote by $D(X, u)$ the set of all states of X relative to u , namely

$$D(X, u) := \{\phi \in S_{X^*} : \phi(u) = 1\},$$

and then, for x in X , we define the *numerical range* $V(X, u, x)$ of x relative to u as the nonempty, convex, and compact subset of \mathbb{K} given by the equality

$$V(X, u, x) := \{\phi(x) : \phi \in D(X, u)\}.$$

Given a mapping f from S_X into X , we can consider the so-called *spatial numerical range* $W(f)$ of f , namely

$$W(f) := \bigcup \{V(X, u, f(u)) : u \in S_X\}.$$

If the mapping f above is bounded, then it also has an *intrinsic numerical range* $V(f)$, given by the equality

$$V(f) := V(B(S_X, X), \mathbf{1}, f),$$

where $B(S_X, X)$ denotes the Banach space of all bounded functions from S_X to X , and $\mathbf{1}$ stands for the natural embedding $S_X \hookrightarrow X$. In this case we have the inclusion

$$\overline{\text{co}} W(f) \subseteq V(f),$$

where $\overline{\text{co}}$ means closed convex hull. (Indeed, for u in S_X and ϕ in $D(X, u)$, the mapping $g \rightarrow \phi(g(u))$ from $B(S_X, X)$ to \mathbb{K} is an element of $D(B(S_X, X), \mathbf{1})$.) The inclusion above is known to be an equality whenever either f is (the restriction to S_X of) a continuous linear operator on X [6], or $\mathbb{K} = \mathbb{C}$ and f is (the restriction to S_X of) a uniformly continuous function on B_X which is holomorphic on the interior of B_X [5].

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The aim of this note is to show that the equality $\overline{\text{co}}W(f) = V(f)$ can hold for every f in $B(S_X, X)$, but only if X lies in a very restrictive class of Banach spaces. Actually, we prove that the equality $\overline{\text{co}}W(f) = V(f)$ holds for every f in $B(S_X, X)$ if and only if X is uniformly smooth. Note that every uniformly smooth Banach space is super-reflexive, and that, in fact, super-reflexivity is the isomorphic side of the uniform smoothness, i.e., every super-reflexive Banach space has a uniformly smooth equivalent renorming (see for instance [2]).

2. THE RESULTS

Let X be a Banach space over \mathbb{K} ($= \mathbb{R}$ or \mathbb{C}), and u an element in S_X . For x in X , the function $\alpha \rightarrow \|u + \alpha x\|$ from \mathbb{R} to \mathbb{R} is convex, hence there exists the number

$$M^u(x) := \lim_{\alpha \rightarrow 0^+} \frac{\|u + \alpha x\| - 1}{\alpha} = \inf \left\{ \frac{\|u + \alpha x\| - 1}{\alpha} : \alpha \in \mathbb{R}^+ \right\},$$

and it is known that the equality

$$M^u(x) = \max\{\text{Re}(\lambda) : \lambda \in V(X, u, x)\}$$

holds (see for instance [3, Theorem V.9.5]). For α in \mathbb{R}^+ , we put

$$\varphi(X, u, \alpha) := \sup \left\{ \frac{\|u + \alpha x\| - 1}{\alpha} - M^u(x) : x \in B_X \right\}.$$

Theorem 1. *Let I be an infinite set, $\{X_i\}_{i \in I}$ a family of Banach spaces over \mathbb{K} , and, for each i in I , let u_i be a fixed norm-one element in X_i . Denote by Y the l_∞ -sum of the family $\{X_i\}_{i \in I}$, and by u the element of S_Y given by $u(i) := u_i$ for every i in I . Then the following assertions are equivalent:*

i) *the equality*

$$V(Y, u, y) = \overline{\text{co}} \left(\bigcup \{V(X_i, u(i), y(i)) : i \in I\} \right)$$

holds for every y in Y .

ii) $\lim_{(i, \alpha) \rightarrow (\infty, 0)} \varphi(X_i, u(i), \alpha) = 0$.

In assertion ii) above, the symbol $\lim_{(i, \alpha) \rightarrow (\infty, 0)}$ means the limit along the filter basis on $I \times \mathbb{R}^+$ consisting of all subsets of $I \times \mathbb{R}^+$ of the form $J \times (0, \delta)$, where J is a co-finite subset of I and δ is a positive number. We also note that, if the set I in the theorem fails to be infinite, then, for every y in Y , the equality

$$V(Y, u, y) = \text{co} \left(\bigcup \{V(X_i, u(i), y(i)) : i \in I\} \right)$$

is automatically true.

Proof of Theorem 1. ii) \Rightarrow i) Fix y in S_Y , and let ε be a positive number. By assumption ii), there exists $\delta > 0$, and a co-finite subset J of I , such that the inequality

$$\frac{\|u(j) + \alpha x_j\| - 1}{\alpha} - M^{u(j)}(x_j) < \varepsilon$$

holds whenever j is in J , x_j is in B_{X_j} for such a j , and $0 < \alpha < \delta$. For k in $I \setminus J$, we can choose $\delta_k > 0$ such that

$$\frac{\|u(k) + \alpha y(k)\| - 1}{\alpha} - M^{u(k)}(y(k)) < \varepsilon$$

whenever $0 < \alpha < \delta_k$. By putting $\rho := \min\{\delta, \min\{\delta_k : k \in I \setminus J\}\}$, it follows that the inequality

$$\frac{\|u(i) + \alpha y(i)\| - 1}{\alpha} < M^{u(i)}(y(i)) + \varepsilon$$

is true for every i in I and $0 < \alpha < \rho$. Therefore, for $0 < \alpha < \rho$ we have

$$\frac{\|u + \alpha y\| - 1}{\alpha} \leq \sup\{M^{u(i)}(y(i)) : i \in I\} + \varepsilon,$$

and, by letting $\alpha \rightarrow 0$, we obtain

$$M^u(y) \leq \sup\{M^{u(i)}(y(i)) : i \in I\} + \varepsilon.$$

In view of the arbitrariness of ε , we actually have

$$M^u(y) \leq \sup\{M^{u(i)}(y(i)) : i \in I\},$$

so that, replacing y by zy with z in $S_{\mathbb{K}}$, the inclusion

$$V(Y, u, y) \subseteq \overline{\text{co}}\left(\bigcup\{V(X_i, u(i), y(i)) : i \in I\}\right)$$

follows. The reverse inclusion is trivial.

i) \Rightarrow ii) Assume that ii) is not true. Then there is $\varepsilon > 0$ such that, for every co-finite subset J of I , and for every $\delta > 0$, there exists $0 < \alpha < \delta$, j in J , and x_j in B_{X_j} such that

$$\frac{\|u(j) + \alpha x_j\| - 1}{\alpha} - M^{u(j)}(x_j) \geq \varepsilon.$$

Take $0 < \alpha_1 < 1$, $i(1)$ in I , and $x_{i(1)}$ in $B_{X_{i(1)}}$ such that

$$\frac{\|u(i(1)) + \alpha_1 x_{i(1)}\| - 1}{\alpha_1} - M^{u(i(1))}(x_{i(1)}) \geq \varepsilon.$$

Assume that, for some n in \mathbb{N} , $\alpha_1, \dots, \alpha_n, i(1), \dots, i(n), x_{i(1)}, \dots, x_{i(n)}$ have been found such that $0 < \alpha_m < \frac{1}{m}$, $i(m)$ is in I , $x_{i(m)}$ belongs to $X_{i(m)}$,

$$\frac{\|u(i(m)) + \alpha_m x_{i(m)}\| - 1}{\alpha_m} - M^{u(i(m))}(x_{i(m)}) \geq \varepsilon$$

(for all m in $\{1, \dots, n\}$), and $i(m) \neq i(m')$ (for all m, m' in $\{1, \dots, n\}$ with $m \neq m'$). Then we can choose $0 < \alpha_{n+1} < \frac{1}{n+1}$, $i(n+1)$ in $I \setminus \{i(1), \dots, i(n)\}$, and $x_{i(n+1)}$ in $B_{X_{i(n+1)}}$ satisfying

$$\frac{\|u(i(n+1)) + \alpha_{n+1} x_{i(n+1)}\| - 1}{\alpha_{n+1}} - M^{u(i(n+1))}(x_{i(n+1)}) \geq \varepsilon.$$

In this way, we have constructed a sequence $\{(\alpha_n, i(n), x_{i(n)})\}_{n \in \mathbb{N}}$ with $\alpha_n \in \mathbb{R}^+$, $\{\alpha_n\} \rightarrow 0$, $i(n) \in I$, $i(n) \neq i(m)$ for $n \neq m$, $x_{i(n)} \in B_{X_{i(n)}}$, and

$$\frac{\|u(i(n)) + \alpha_n x_{i(n)}\| - 1}{\alpha_n} - M^{u(i(n))}(x_{i(n)}) \geq \varepsilon.$$

Put $r := \limsup_{n \rightarrow \infty} \{M^{u(i(n))}(x_{i(n)})\}$, so that, by discarding a finite number of terms in the sequence $\{(\alpha_n, i(n), x_{i(n)})\}$, we can assume that, in addition to the above properties, we have

$$M^{u(i(n))}(x_{i(n)}) \leq r + \frac{\varepsilon}{2}$$

for every n . Now, consider the element y in Y defined by $y(i) = x_{i(n)}$ if $i = i(n)$ for some n , and $y(i) = ru(i)$ otherwise. Then, for every n in \mathbb{N} we have

$$\begin{aligned} \frac{\|u + \alpha_n y\| - 1}{\alpha_n} &\geq \frac{\|u(i(n)) + \alpha_n y(i(n))\| - 1}{\alpha_n} \\ &= \frac{\|u(i(n)) + \alpha_n x_{i(n)}\| - 1}{\alpha_n} \geq M^{u(i(n))}(x_{i(n)}) + \varepsilon. \end{aligned}$$

By taking upper limits as $n \rightarrow \infty$, we obtain

$$M^u(y) \geq r + \varepsilon.$$

On the other hand, the definition of y yields the inequality

$$\sup\{M^{u(i)}(y(i)) : i \in I\} \leq r + \frac{\varepsilon}{2}.$$

It follows that

$$\sup\{M^{u(i)}(y(i)) : i \in I\} + \frac{\varepsilon}{2} \leq M^u(y),$$

so that the equality in assertion i) cannot be true for y . \square

Let X be a Banach space. Given u in S_X , we say that *the norm of X is strongly subdifferentiable at u* if

$$M^u(x) = \lim_{\alpha \rightarrow 0^+} \frac{\|u + \alpha x\| - 1}{\alpha} \text{ uniformly for } x \text{ in } B_X$$

(equivalently, if $\lim_{\alpha \rightarrow 0} \varphi(X, u, \alpha) = 0$). Given a nonempty set I , we denote by $B(I, X)$ the Banach space of all bounded functions from I to X . A straightforward consequence of Theorem 1 is the following.

Corollary 2 ([1]). *Let X be a Banach space, and let u be in S_X . For every nonempty set I , denote by \hat{u} the element of $B(I, X)$ defined by $\hat{u}(i) = u$ for all i in I . Then the following assertions are equivalent:*

- i) *For every nonempty set I and every f in $B(I, X)$, the equality*

$$V(B(I, X), \hat{u}, f) = \overline{\text{co}} \left(\bigcup \{V(X, u, f(i)) : i \in I\} \right)$$

holds.

- ii) *There exists an infinite set I such that*

$$V(B(I, X), \hat{u}, f) = \overline{\text{co}} \left(\bigcup \{V(X, u, f(i)) : i \in I\} \right)$$

for every f in $B(I, X)$.

- iii) *$V(B(\mathbb{N}, X), \hat{u}, f) = \overline{\text{co}}(\bigcup\{V(X, u, f(n)) : n \in \mathbb{N}\})$ for all f in $B(\mathbb{N}, X)$.*

- iv) *The norm of X is strongly subdifferentiable at u .*

Given a Banach space X , and a subset D of S_X , we say that *the norm of X is uniformly strongly subdifferentiable on D* if

$$M^u(x) = \lim_{\alpha \rightarrow 0^+} \frac{\|u + \alpha x\| - 1}{\alpha} \text{ uniformly for } (u, x) \text{ in } D \times B_X.$$

Proposition 3. *Let E be a topological space without nonempty open finite subsets (for instance, every topological T_1 -space without isolated points), X a Banach space, and $g: E \rightarrow S_X$ a continuous function. Then the following assertions are equivalent:*

i) For every f in $B(E, X)$, the equality

$$V(B(E, X), g, f) = \overline{\text{co}} \left(\bigcup \{V(X, g(t), f(t)) : t \in E\} \right)$$

holds.

ii) The norm of X is uniformly strongly subdifferentiable on the range of g .

Proof. ii) \Rightarrow i) The uniform strong subdifferentiability of the norm of X on the range of g implies $\lim_{(t,\alpha) \rightarrow (\infty,0)} \varphi(X, g(t), \alpha) = 0$. Now, apply Theorem 1.

i) \Rightarrow ii) Let $\varepsilon > 0$. By assumption i) and Theorem 1, there exist $\delta > 0$ and a co-finite subset J of E such that the inequality

$$\frac{\|g(t) + \alpha x\| - 1}{\alpha} - M^{g(t)}(x) \leq \frac{\varepsilon}{2}$$

is true whenever t is in J , x is in B_X , and $0 < \alpha < \delta$. Then, for $0 < \alpha < \delta$, $0 < \beta < \delta$, t in J , and x in B_X , we have

$$\left| \frac{\|g(t) + \alpha x\| - 1}{\alpha} - \frac{\|g(t) + \beta x\| - 1}{\beta} \right| \leq \varepsilon.$$

Since J is dense in E and g is continuous, the last inequality remains true for $0 < \alpha < \delta$, $0 < \beta < \delta$, t in E , and x in B_X . By letting $\beta \rightarrow 0$, we obtain

$$\frac{\|g(t) + \alpha x\| - 1}{\alpha} - M^{g(t)}(x) \leq \varepsilon$$

whenever t is in E , x is in B_X , and $0 < \alpha < \delta$. □

We note that Proposition 3 generalizes Corollary 2. Indeed, if I is an infinite set, and if we endow I with the trivial topology, then I becomes a topological space with no nonempty open finite subsets, and the unique continuous functions on I are the constant ones.

Let X be a Banach space. Recall that, given u in S_X , the norm of X is said to be *Fréchet differentiable at u* if there exists a continuous \mathbb{R} -linear mapping $\tau(u, \cdot) : X \rightarrow \mathbb{R}$ such that

$$\lim_{\|h\| \rightarrow 0} \frac{|\|u + h\| - 1 - \tau(u, h)|}{\|h\|} = 0.$$

It is easy to see that the norm of X is Fréchet differentiable at u if and only if X is smooth at u (i.e., $D(X, u)$ reduces to a singleton) and the norm of X is strongly subdifferentiable at u . If this is the case, then the unique element ϕ in $D(X, u)$ and the real-valued functional $\tau(u, \cdot)$ are mutually determined by means of the equality $\text{Re} \circ \phi = \tau(u, \cdot)$. If the norm of X is Fréchet differentiable at every point of S_X , and if we have

$$\lim_{\|h\| \rightarrow 0} \frac{|\|u + h\| - 1 - \tau(u, h)|}{\|h\|} = 0 \text{ uniformly for } u \text{ in } S_X,$$

then we say that X is *uniformly smooth*. The notion of uniform smoothness of X is apparently stronger than that of uniform strong subdifferentiability of the norm of X on S_X , but, as proved in [4, Proposition 4.1], both notions actually are equivalent. Noticing also that the uniform strong subdifferentiability of the norm of X on some subset D of S_X implies the uniform strong subdifferentiability of the norm of X on the closure of D (argue like in the proof of Proposition 3), we derive the next corollary.

Corollary 4. *Let E be a topological space without nonempty open finite subsets, X a Banach space, and g a continuous function from E onto a dense subset of S_X . Then the following assertions are equivalent:*

i) *For every f in $B(E, X)$, the equality*

$$V(B(E, X), g, f) = \overline{\text{co}} \left(\bigcup \{V(X, g(t), f(t)) : t \in E\} \right)$$

holds.

ii) *X is uniformly smooth.*

Taking in Corollary 4 $E = S_X$ and g equal to the inclusion mapping $S_X \hookrightarrow X$, we obtain the characterization of uniformly smooth Banach spaces announced in the introduction.

Theorem 5. *For a Banach space X , the following assertions are equivalent:*

i) *For every bounded function $f: S_X \rightarrow X$, the equality $\overline{\text{co}} W(f) = V(f)$ is true.*

ii) *X is uniformly smooth.*

To conclude the paper, let us show an elemental example of a Banach space X , together with a bounded function $f: S_X \rightarrow X$, such that the inclusion $\overline{\text{co}} W(f) \subseteq V(f)$ is strict.

Example 6. Take $X = \mathbb{R}^2$ with norm $\|(\lambda, \mu)\| := \max\{|\lambda|, |\mu|\}$. For x, y in X , put $]x, y[:= \{\alpha x + (1 - \alpha)y : 0 < \alpha < 1\}$, and define a function $f: S_X \rightarrow X$ by

$$f(u) := \begin{cases} (0, 0) & \text{if } u \in \{(1, 1), (1, -1), (-1, -1), (-1, 1)\}, \\ (0, 1) & \text{if } u \in](1, 1), (1, -1)[\cup](-1, -1), (-1, 1)[, \\ (1, 0) & \text{if } u \in](1, -1), (-1, -1)[\cup](-1, 1), (1, 1)[. \end{cases}$$

Then f is bounded with $\|f\| = 1$. Moreover, it is easily checked that the equality $\|1 + \alpha f\| = 1 + |\alpha|$ holds for every α in \mathbb{R} , and therefore $V(f)$ is equal to the closed real interval $[-1, 1]$. However, we have $V(X, u, f(u)) = \{0\}$ for every u in S_X , and hence $W(f) = \{0\}$.

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