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THE COHOMOLOGY RINGS OF THE ORBIT SPACES OF FREE TRANSFORMATION GROUPS OF THE PRODUCT OF TWO SPHERES

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Abstract. Let $G = Z_p$, p a prime (resp. S^1), act freely on a finitistic space X with mod p (resp. rational) cohomology ring isomorphic to that of $S^m \times S^n$. In this paper we determine the possible cohomology algebra of the orbit space X/G.

1. Introduction

Let $G = Z_p$, p a prime (resp. S^1, the circle group), act on a space X with mod p (resp. rational) cohomology ring isomorphic to that of $S^m \times S^n$; we abbreviate this as $X \sim_p S^m \times S^n$ (resp. $X \sim_Q S^m \times S^n$). There are two spaces associated with the transformation group (G, X); viz. the fixed point set X^G and the orbit space X/G. The homological nature of X^G has been studied in detail by Adem [1], Bredon [3], Hsiang [4], Su [6] and Tomter [7]. However, to our knowledge, no one has investigated the homological structure of the space X/G. We find here the possibilities for the cohomology algebra $H^*(X/G)$ when the action is free. Throughout this paper, we use Čech cohomology with coefficients in the field F_p of p elements or Q of rational numbers, unless otherwise indicated. The mod p Bockstein cohomology operation associated with the coefficient sequence $0 \to Z_p \to Z_{p^2} \to Z_p \to 0$ will be denoted by β. We prove the following:

Theorem 1. *Let $G = Z_p$, p an odd prime, act freely on a finitistic space $X \sim_p S^m \times S^n$, $0 < m \leq n$, and assume that $H^*(X; Z)$ is of finite type. Then $H^*(X/G; Z_p)$ is isomorphic to $Z_p[x, y, z]/\phi(x, y, z)$ as a graded commutative algebra, where $\phi(x, y, z)$ is one of the following graded ideals*:

(i) $(x^2, y^{(m+1)/2}, z^2)$, *m odd,* $\deg x = 1, y = \beta(x), \deg z = n$;

(ii) $(x^2, y^{(m+n+1)/2}, y^{(n-m+1)/2}z - ay^{(n+1)/2}, z^2 - by^m)$, *$m$ even, n odd,* $\deg x = 1, y = \beta(x), \deg z = m, a, b \in Z_p$ *and $a = 0$ necessarily when $n < 2m$*;

(iii) $(x^2, y^{(n+1)/2}, z^2 - by^m)$, *$n$ odd,* $\deg x = 1, y = \beta(x), \deg z = m, b \in Z_p, b \neq 0$ *only when m is even and $2m < n$.*

Theorem 2. *Let $G = Z_2$ act freely on a finitistic space $X \sim_2 S^m \times S^n$, $0 < m \leq n$. Then $H^*(X/G, Z_2)$ is isomorphic to $Z_2[y, z]/\psi(y, z)$ as a graded algebra, where $\psi(y, z)$ is one of the following graded ideals*:

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(i) (y^{m+2}, z^2), $\deg y = 1, \deg z = n$;

(ii) $(y^{m+n+1}, y^{n-m+1}, z, z^2 - ay^m z - by^{2m})$, $\deg y = 1, \deg z = m, a, b \in Z_2$ *and* $a = 0$ *necessarily when* $n < 2m$;

(iii) $(y^{n+1}, z^2 - ay^m z - by^{2m})$, $\deg y = 1, \deg z = m, a, b \in Z_2$ *and* $b = 0$ *necessarily when* $m = n$ *or* $n < 2m$.

Theorem 3. *Let* $G = S^1$ *act freely on a finitistic space* $X \sim_Q S^m \times S^n$, $0 < m \leq n$. *Then* $H^*(X/G; Q)$ *is isomorphic to* $Q[y, z]/\psi(y, z)$ *as a graded algebra, where* $\psi(y, z)$ *is one of the following graded ideals*:

(i) $(y^{(m+1)/2}, z^2)$, m *odd,* $\deg y = 2, \deg z = n$.

(ii) $(y^{(m+n+1)/2}, zy^{(n-m+1)/2} - ay^{(n+1)/2}, z^2 - by^m)$, m *even,* n *odd,* $\deg y = 2, \deg z = m$ *and* $a = 0$ *necessarily when* $n < 2m$.

(iii) $(y^{(n+1)/2}, z^2 - by^m)$, n *odd,* $\deg y = 2$, $\deg z = m, b \neq 0$ *only when* m *is even and* $2m < n$.

The main gadget employed in our proofs is the Leray-Serre spectral sequence of the fibration, $X \xrightarrow{\iota} X_G \xrightarrow{\pi} B_G$, which has $E_2^{k,l} = H^k(B_G; H^l(X))$ as its E_2-term and converges to $H^{k+l}(X_G)$, in the sense of Bredon [2], where the coefficients $H^*(X)$ are twisted by the action of $\pi_1(B_G)$ and $X_G = (E_G \times X)/G$ is the Borel construction of X associated to a universal G-bundle $E_G \to B_G$. It can easily be seen that X_G is paracompact, when X is so.

2. Some known results

Suppose $G = Z_p$, p a prime, acts on a finitistic space $X \sim_p S^m \times S^n$. The following facts can be easily deduced.

Proposition 1. *If* G *acts trivially on* $H^*(X)$ *and the spectral sequence of the map* $\pi : X_G \to B_G$ *degenerates, then* $\sum_k \operatorname{rk} H^k(X^G) = 4$ [2, VII, 1.6].

Proposition 2. *If* m *and* n *are even and* $p > 2$, *then* $X^G \neq \Phi$ [2, III, 7.10].

Proposition 3. *If* $H^*(X; Z)$ *is of finite type,* $p > 2$ *and* G *acts nontrivially on* $H^*(X)$, *then* $p = 3$ *and* $X^G \neq \Phi$ [6].

Proposition 4. *If* $p = 2$ *and* G *acts nontrivially on* $H^*(X)$, *then* $X^G \neq \Phi$ *and* $m = n$ [2, VII, 7.5].

We recall that for $G = Z_p$,

$$H^*(B_G; Z_p) = \begin{cases} Z_p[t], & \deg t = 1, p = 2, \\ \bigwedge(s) \otimes Z_p[t], & \deg s = 1, t = \beta(s), p > 2, \end{cases}$$

and for $G = S^1$,

$$H^*(B_G; Q) = Q[t], \qquad \deg t = 2.$$

3. Proofs

Proof of Theorem 1. Since there are no fixed points, it follows from Propositions 1, 2, and 3 that m and n cannot both be even and that Z_p acts trivially on $H^*(X)$. Hence the Leray-Serre spectral sequence of the map $\pi : X_G \to B_G$ does not collapse at the E_2-term and $E_2^{k,l} = H^k(B_G) \otimes H^l(X)$. Let $r \geq 2$ be the smallest integer such that $d_r \neq 0$. By the multiplicative properties of the spectral sequence, we have $d_r(1 \otimes v_1) \neq 0$ or $d_r(1 \otimes v_2) \neq 0$. Suppose, first, that $d_r(1 \otimes v_1) \neq 0$. Then $r = m+1$

and m must be odd. So we can write $d_{m+1}(1 \otimes v_1) = t^{(m+1)/2} \otimes 1$. Now, we either have $d_{m+1}(1 \otimes v_2) = 0$ or $n = m$ and $d_{m+1}(1 \otimes v_2) = at^{m+1} \otimes 1$, $0 \neq a \in Z_p$. For $n \neq m$, obviously, $d_{m+1}(1 \otimes v_2) = 0$ and $d_{m+1}(1 \otimes v_3) = t^{(m+1)/2} \otimes v_2$. Thus the differentials

$$d_{m+1} : E_{m+1}^{k,m} \to E_{m+1}^{k+m+1,0}, \text{ and}$$
$$d_{m+1} : E_{m+1}^{k,m+n} \to E_{m+1}^{k+m+1,n}$$

are isomorphisms and we have $E_\infty = E_{m+2}$. Consequently, the only nonzero vector spaces in the E_∞-term are $E_\infty^{k,n} = Z_p = E_\infty^{k,0}$, $0 \leq k \leq m$. Thus,

$$H^k(X_G) = \begin{cases} Z_p, & 0 \leq k \leq m \text{ and } n \leq k \leq m+n; \\ 0, & \text{otherwise.} \end{cases}$$

If $n = m$ and $d_{m+1}(1 \otimes v_2) = at^{(m+1)/2} \otimes 1$, $a \in Z_p$, then $d_{m+1}(1 \otimes v_3) = t^{(m+1)/2} \otimes (v_2 - av_1)$. So the differential

$$d_{m+1} : E_{m+1}^{k,m} \to E_{m+1}^{k+m+1,0}$$

is surjective with $\ker(d_{m+1})$ generated by $\zeta_k \otimes (v_2 - av_1)$, ζ_k is the generator of $H^k(B_G)$ and the differential

$$d_{m+1} : E_{m+1}^{k,2m} \to E_{m+1}^{k+m+1,m}$$

is injective with $im(d_{m+1})$ generated by $\zeta_{k+m+1} \otimes (v_2 - av_1)$. Consequently, $E_\infty = E_{m+2}$ and the only nonzero entries in the E_∞-term are $E_\infty^{k,m} = Z_p = E_\infty^{k,0}$, for $0 \leq k \leq m$. We obtain

$$H^k(X_G) = \begin{cases} Z_p, & 0 \leq k \leq 2m \text{ and } k \neq m; \\ Z_p \oplus Z_p, & k = m; \\ 0, & 2m < k. \end{cases}$$

The multiplication by $t \in H^2(B_G)$:

$$t \cup (\cdot) : E_\infty^{k,l} \to E_\infty^{k+2,l},$$

regarded as a spectral sequence endomorphism, is an isomorphism for $0 \leq k \leq m-2$, $m > 1$ and $l = 0$ or n. If $m \neq n$ (resp. $m = n$) the element $1 \otimes v_2$ (respectively $1 \otimes (v_2 - av_1)$) of $E_2^{0,n}$ is a permanent cocycle and gives a nonzero element $w \in E^{0,n}$. There are elements $x \in H^1(X_G)$ and $y \in H^2(X_G)$ with $x = \pi^*(s)$ and $y = \pi^*(t)$. It is easily checked that the total complex

$$\operatorname{Tot} E_\infty^{*,*} \cong Z_p[x, y, w]/(x^2, y^{(m+1)/2}, w^2)$$

as a graded commutative algebra. We choose an element $z \in H^n(X_G)$ such that $\iota^*(z) = v_2$ (resp. $v_2 - av_1$) if $m \neq n$ (resp. $m = n$). Then $xz \neq 0 \neq yz$ and $z^2 = 0$. The multiplication

$$y \cup (\cdot) : H^k(X_G) \to H^{k+2}(X_G)$$

is an isomorphism in degrees k such that $0 \leq k \leq m-2$ and $n < k \leq m+n-2$. So

$$H^*(X_G) \cong Z_p[x, y, z]/(x^2, y^{(m+1)/2}, z^2).$$

For $m = 1$, we have

$$H^*(X_G) \cong Z_p[x, z]/(x^2, z^2).$$

Since G acts freely on X, $H^*(X_G)$ is isomorphic to $H^*(X/G)$ as a ring and we are in case (i).

Suppose, now, that $d_r(1 \otimes v_1) = 0$ and $d_r(1 \otimes v_2) \neq 0$. We then have either $r = n - m + 1$ and $d_r(1 \otimes v_2) = A \otimes v_1$ or $r = n + 1$ and $d_r(1 \otimes v_2) = A \otimes 1$, $0 \neq A \in H^*(B_G)$. In the former case if n were even, then we would have

$$0 = d_r(1 \otimes v_2^2) = A \otimes v_1 v_2 + (-1)^{n(1+r)} A \otimes v_2 v_1 = 2A \otimes v_3 \neq 0.$$

Hence n is odd. We then observe that m must be even. Assume the contrary and consider the spectral sequence of the map π with coefficients in $\mathbb{Z}$, the ring of integers. Since $H^*(X;\mathbb{Z})$ is finitely generated, it has no p-torsion elements; consequently, we have $\widetilde{E}_2^{k,l} = H^k(B_G; H^l(X;\mathbb{Z})) = 0$, for all k odd. Thus $\widetilde{E}_r^{k,l} = 0$, for all k odd and $r \geq 2$. The coefficients homomorphism $q : \mathbb{Z} \to \mathbb{Z}_p$ gives the commutative diagram:

$$\begin{array}{ccc} \widetilde{E}_{n-m+1}^{0,n} & \xrightarrow{d_{n-m+1}} & \widetilde{E}_{n-m+1}^{n-m+1,m} \\ \big\downarrow q^* & & \big\downarrow q^* \\ E_{n-m+1}^{0,n} & \xrightarrow{d_{n-m+1}} & E_{n-m+1}^{n-m+1,m} \end{array}$$

The composition $d_{n-m+1} \circ q^*$ is the trivial homomorphism, for $n - m + 1$ is odd. Since q^* in the left is surjective, the bottom d_{n-m+1} is trivial. But this is not the case; hence our assertion. Thus we must have m even and n odd so that we can write $d_{n-m+1}(1 \otimes v_2) = t^{(n-m+1)/2} \otimes v_1$. It follows that the differential

$$d_{n-m+1} : E_{n-m+1}^{*,n} \to E_{n-m+1}^{*,m}$$

is an isomorphism and $d_{n-m+1}(E_{n-m+1}^{*,m}) = 0 = d_{n-m+1}(E_{n-m+1}^{*,m+n})$. So we have $E_r^{k,n} = 0 = E_r^{k+n-m+1,m}, E_r^{k,m+n} = E_2^{k,m+n}$ and $E_r^{k,0} = E_2^{k,0}$ for all $k \geq 0$ and $r = n - m + 2$. It is easily seen that the differential

$$d_{m+1} : E_{m+1}^{k,m} \to E_{m+1}^{k+m+1,0}$$

is trivial for $0 \leq k \leq n - m$, since $E_{m+1}^{k,m} = E_2^{k,m}$. Because there are no fixed points, the differential

$$d_{n+m+1} : E_{n+m+1}^{0,m+n} \to E_{n+m+1}^{n+m+1,0}$$

must be nontrivial so that we can assume $d_{n+m+1}(1 \otimes v_3) = t^{(n+m+1)/2} \otimes 1$. Then, the differential

$$d_{n+m+1} : E_{n+m+1}^{*,m+n} \to E_{n+m+1}^{*,0}$$

is an isomorphism. Consequently, $E_\infty = E_{m+n+2}$ and the only nonzero vector spaces in the E_∞-term are $E_\infty^{k,m} = Z_p$ for $0 \leq k \leq n - m$ and $E_\infty^{k,0} = Z_p$ for $0 \leq k \leq m + n$. It follows that

$$H^k(X_G) = \begin{cases} Z_p, & 0 \leq k < m \text{ and } n < k \leq m + n; \\ Z_p \oplus Z_p, & m \leq k \leq n; \\ 0, & m + n < k. \end{cases}$$

We note that $1 \otimes v_1 \in E_2^{0,m}$ is a permanent cocycle and determines an element $w \in E_\infty^{0,m}$. We have

$$\operatorname{Tot} E_\infty^{*,*} \cong Z_p[x, y, w]/(x^2, w^2, y^{(m+n+1)/2}, y^{(n-m+1)/2} w)$$

as graded commutative algebras, where x and y satisfy $\pi^*(s) = x$ and $\pi^*(t) = y$. The multiplication

$$y \cup (\cdot) : H^k(X_G) \to H^{k+2}(X_G)$$

is an isomorphism in degrees k for $0 \leq k \leq n-2$ and $n < k \leq m+n-2$. We choose an element $z \in H^m(X/G)$ such that $\iota^*(z) = v_1$. Then $y^r z$ and $y^{(m+2r)/2}$ are linearly independent over Z_p for $r \leq (n-m-1)/2$. It is possible to change z suitably so that $y^{(n-m+1)/2}z = 0$ and $z^2 = by^m$, $b \in Z_p$, when $n < 2m$ and $y^{(n-m+1)/2}z = ay^{(n+1)/2}$ and $z^2 = by^m$, $a, b \in Z_p$, when $2m < n$. Therefore,

$$H^*(X_G) \cong Z_p[x, y, z]/(x^2, y^{(m+n+1)/2}, y^{(n-m+1)/2}z - ay^{(n+1)/2}, z^2 - by^m)$$

and we are in case (ii).

Finally, consider the possibility $r = n+1$, $d_r(1 \otimes v_1) = 0$ and $d_r(1 \otimes v_2) = A \otimes 1$, $0 \neq A \in H^*(B_G)$. Then n must be odd and we can set $d_{n+1}(1 \otimes v_2) = t^{(n+1)/2} \otimes 1$. So $d_{n+1}(1 \otimes v_3) = \pm t^{(n+1)/2} \otimes v_1$; consequently the differentials

$$d_{n+1} : E_{n+1}^{k,n} \to E_{n+1}^{k+n+1,0},$$
$$d_{n+1} : E_{n+1}^{k,m+n} \to E_{n+1}^{k+n+1,m}$$

are isomorphisms. We obtain $E_\infty = E_{n+2}$ and the only nonzero vector spaces in the E_∞-term are $E_\infty^{k,m} = Z_p = E_\infty^{k,0}$ for $0 \leq k \leq n$. Thus

$$H^k(X_G) = \begin{cases} Z_p, & 0 \leq k < m \text{ and } n < k \leq m+n; \\ Z_p \oplus Z_p, & m \leq k \leq n; \\ 0, & m+n < k. \end{cases}$$

We note that $1 \otimes v_1 \in E_2^{0,m}$ is, again, a permanent cocycle and gives an element $w \in E_\infty^{0,m}$. Choosing $x \in H^1(X_G)$ and $y \in H^2(X_G)$ such that $\pi^*(s) = x$ and $\pi^*(t) = y$, we obtain

$$\operatorname{Tot} E_\infty^{*,*} \cong Z_p[x, y, w]/(x^2, y^{(n+1)/2}, w^2)$$

as graded commutative algebras. Now, we choose $z \in H^m(X_G)$ such that $\iota^*(z) = v_1$. Then z represents w and the multiplication

$$y \cup (\cdot) : H^k(X_G) \to H^{k+2}(X_G)$$

is an isomorphism for $0 \leq k \leq n-2$ and $n < k \leq m+n-2$, so that $y^2 z \neq 0$ for $r \leq (n-1)/2$. If m is even and $n < 2m$, then $z^2 = by^{m/2}z, b \in Z_p$. We replace z by $z - (b/2)y^{m/2}$ and have $z^2 = 0$. If m is even and $2m < n$, then $z^2 = by^{m/2}z + cy^m$, $b, c \in Z_p$. Replacing z by $z - (b/2)y^{m/2}$ we obtain the relation $z^2 = b'y^m$. Thus

$$H^*(X_G) \cong Z_0[x, y, z]/(x^2, y^{(n+1)/2}, z^2 - by^m)$$

as a graded commutative algebra and we obtain case (iii). This completes the proof of the theorem. □

Proof of Theorem 2. The proof is analogous to that of Theorem 1; we describe it rather briefly. The spectral sequence of the map π is nontrivial. Let $r \geq 2$ be the least integer such that $d_r \neq 0$. Then, we must have $d_r(1 \otimes v_1) \neq 0$ or

$d_r(1\otimes v_2)\neq 0$. Suppose that $r=m+1, d_{m+1}(1\otimes v_1)=t^{m+1}\otimes 1$ and $m\neq n$. Then $d_{m+1}(1\otimes v_2)=0, d_{m+1}(1\otimes v_3)=t^{m+1}\otimes v_2$ so that $E_\infty=E_{m+2}$ and we obtain

$$H^k(X_G)=\begin{cases} Z_2, & 0\leq k\leq m \text{ and } n\leq k\leq m+n;\\ 0, & \text{otherwise.}\end{cases}$$

The element $1\otimes v_2\in E_2^{0,n}$, being a permanent cocycle, yields an element $w\in E_\infty^{0,n}$. So, the total complex $\operatorname{Tot} E_\infty^{*,*}$ is the graded algebra $Z_2[y,w]/(y^1,w^2)$, where $y=\pi^*(t)$, $t\in H^1(B_G)$. Choose $z\in H^n(X_G)$ such that $\iota^*(z)=v_2$. Then z determines w and satisfies $z^2=0$. Since the multiplication

$$y\cup(\cdot): H^k(X_G)\to H^{k+1}(X_G)$$

is an isomorphism for $0\leq k<m$ and $n\leq k<m+n$, we have $y^rz\neq 0$, $1\leq r\leq m$. Therefore

$$H^*(X_G)\cong Z_2[y,z]/(y^{m+1},z^2).$$

As the action of G is free, $H^*(X_G)\cong H^*(X/G)$ and we have the case (i). If $m=n$, $d_{m+1}(1\otimes v_1)=t^{m+1}\otimes 1$ and $d_{m+1}(1\otimes v_2)=ct^{m+1}\otimes 1$, $c\in Z_2$, then $d_{m+1}(1\otimes v_3)=t^{m+1}\otimes(v_2+cv_1)$. We obtain $E_\infty=E_{m+2}$ and

$$H^k(X_G)=\begin{cases} Z_2, & 0\leq k\leq 2m \text{ and } k\neq m;\\ Z_2\oplus Z_2, & k=m;\\ 0, & 2m<k.\end{cases}$$

In this case, the element $1\otimes(v_2+cv_1)\in E_2^{0,m}$ is a permanent cocycle and determines an element $w\in E_\infty^{0,m}$. Let $y\in H^1(X_G)$ and $z\in H^m(X_G)$ be such that $\pi^*(t)=y$ and $\iota^*(z)=v_2+cv_1$. Then z represent w in $E_\infty^{0,m}$ and satisfies $z^2=ay^mz$ for some $a\in Z_2$. The element yz represents $0\neq tw\in E_\infty^{1,m}$ and the multiplication

$$y\cup(\cdot): H^k(X_G)\to H^{k+1}(X_G)$$

is an isomorphism for $m<k<2m$ so that $y^rz\neq 0$, $1\leq r\leq m$. Thus

$$H^*(X_G)\cong Z_2[y,z]/(y^{m+1},z^2-ay^mz)$$

and we are in case (iii) with $b=0$.

Next, when $d_r(1\otimes v_1)=0$, $r=n-m+1$ and $d_r(1\otimes v_2)=t^r\otimes v_1$, then $d_r(1\otimes v)=Q$. So, we have $E_{r+1}^{k,m+n}=E_2^{k,m+n}, E_{r+1}^{k,0}=E_2^{k,0}$, for $k\geq 0$, and $E_{r+1}^{k,m}$ is trivial for $k>n-m$ and $E_2^{k,m}$ for $k\leq n-m$. It is easily seen that the differential

$$d_{m+1}: E_{m+1}^{k,m}\to E_{m+1}^{k+m+1,0}$$

is also trivial for $0\leq k\leq n-m$. Then the differential

$$d_{n+m+1}: E_{m+n+1}^{k,m}\to E_{m+n+1}^{k+m+n+1,0}$$

must be an isomorphism for all k, because the action of G on X is free. Thus, the only nonzero vector spaces in the E_∞-term are $E_\infty^{k,m}=Z_2$ for $0\leq k\leq n-m$ and $E_\infty^{k,0}=Z_2$, for $0\leq k\leq m+n$. Consequently

$$H^k(X_G)=\begin{cases} Z_2, & 0\leq k<m \text{ and } n<k\leq m+n;\\ Z_2\oplus Z_2, & m\leq k\leq n;\\ 0, & m+n<k.\end{cases}$$

The element $1 \otimes v_1 \in E_2^{0,m}$ is a permanent cocycle and gives an element $w \in E_\infty^{0,m}$. We obtain

$$\operatorname{Tot} E_\infty^{*,*} \cong Z_2[y,w]/(y^{m+n+1}, w^2, y^{n-m+1}w)$$

as graded algebras. The multiplication by $\pi^*(t) = y$,

$$y \cup (\cdot) : H^k(X_G) \to H^{k+1}(X_G),$$

is an isomorphism for $0 \leq k \leq n-1$ and $n < k < m+n$. Since the composition $\iota^*\pi^*$ is trivial in positive degrees, it is possible to choose $z \in H^m(X_G)$ such that $\iota^*(z) = v_1, y^{n-m+1}z = 0$ and $z^2 = ay^m z + by^{2m}$, where $a = 0$ when $2m > n$. Thus we have

$$H^*(X_G) \cong Z_2[y,z]/(y^{m+n+1}, y^{n-m+1}z, z^2 - ay^m z - by^{2m})$$

as graded algebras and we are in case (ii).

Finally, consider the case $d_r(1 \otimes v_1) = 0$, $r = n+1$ and $d_{n+1}(1 \otimes v_2) = t^{n+1}$. Then $d_{n+1}(1 \otimes v_3) = t^{n+1} \otimes v_1$. So the differentials

$$d_{n+1} : E_{n+1}^{k,n} \to E_{n+1}^{k+n+1,0}, \text{ and}$$
$$d_{n+1} : E_{n+1}^{k,m+n} \to E_{n+1}^{k+n+1,m}$$

are isomorphisms. We have $E_\infty = E_{n+2}$ and the only nonzero vector spaces in the E_∞-term are $E_\infty^{k,m} = Z_2 = E_\infty^{k,0}$, for $0 \leq k \leq n$. Therefore

$$H^k(X_G) = \begin{cases} Z_2, & 0 \leq k < m \text{ and } n < k \leq m+n; \\ Z_2 \oplus Z_2, & m \leq k \leq n; \\ 0, & m+n < k. \end{cases}$$

Taking $y \in H^1(X_G)$ and $z \in H^m(X_G)$ with $\pi^*(t) = y$ and $\iota^*(z) = v_1$, it can be easily seen that

$$H^*(X_G) \cong Z_2[y,z]/(y^{n+1}, z^2 - ay^m z - by^{2m}),$$

as graded algebras, where $b = 0$, if $n < 2m$. This completes the proof. □

Proof of Theorem 3. Though arguments given herein are different, the technique of the proof remains the same. Since there are no fixed points, the spectral sequence of the map $\pi : X_G \to B_G$ does not collapse at the E_2-term and $E_2^{k,l} = H^k(B_G) \otimes H^l(X)$, for the action on $H^*(X)$ is trivial. Let $r \geq 2$ be the integer such that $d_r \neq 0$. By the multiplicative properties of the spectral sequence, we have $d_r(1 \otimes v_1) \neq 0$ or $d_r(1 \otimes v_2) \neq 0$.

First, suppose that $d_r(1 \otimes v_1) \neq 0$. Then $r = m+1$ where m is odd. We can write $d_r(1 \otimes v_1) = at^{(m+1)/2} \otimes 1$, $0 \neq a \in Q$. If $n = 2m$, $d_{m+1}(1 \otimes v_2) = bt^{(m+1)/2} \otimes v_1$, then

$$0 = d_{m+1}(1 \otimes v_2^2) = 2bt^{(m+1)/2} \otimes v_1 v_2 \neq 0,$$

a contradiction. Therefore, either $d_{m+1}(1 \otimes v_2) = 0$ or $m = n$, $d_{m+1}(1 \otimes v_2) = bt^{m+1} \otimes 1$, $a \neq b \in Q$. In the first case, $d_{m+1}(1 \otimes v_1 v_2) = at^{(m+1)/2} \otimes v_2$; so the differentials

$$d_{m+1} : E_{m+1}^{k,m} \to E_{m+1}^{k+m+1,0}, \text{ and}$$
$$d_{m+1} : E_{m+1}^{k,m+n} \to E_{m+1}^{k+m+1,n}$$

are isomorphisms and we have $E_\infty = E_{m+2}$. Thus, the only nonzero vector spaces in the E_∞-term are $E_\infty^{k,n} = Q = E_\infty^{k,0}$, k even, $0 \le k \le m-1$. We have

$$H^k(X_G) = \begin{cases} Q, & 0 \le k \le m-1,\ k \text{ even};\ n \le k \le n+m-1,\ k-n \text{ even}; \\ 0, & \text{otherwise.} \end{cases}$$

In this case, $d_{m+1}(1 \otimes v_2) = bt^{(m+1)/2} \otimes 1,\ b \neq 0, d_{m+1}(1 \otimes v_1 v_2) = t^{(m+1)/2} \otimes (av_2 - bv_1)$. So the differential

$$d_{m+1} : E_{m+1}^{k,m} \to E_{m+1}^{k+m+1,0}$$

is surjective, with $\ker(d_{m+1})$ generated by $\xi_k \otimes (av_2 - bv_1)$, ξ_k is the generator of $H^k(B_G)$ and the differential

$$d_{m+1} : E_{m+1}^{k,2m} \to E_{m+1}^{k+m+1,m}$$

is injective with $im(d_{m+1})$ generated by $\xi_{k+m+1} \otimes (av_2 - bv_1)$. Consequently, $E_\infty = E_{m+2}$ and the nonzero E_∞-terms are $E_\infty^{k,m} = Q = E_\infty^{k,0}$, $0 \le k \le m-1$ and k even. We have

$$H^k(X_G) = \begin{cases} Q, & 0 \le k \le m-1,\ k \text{ even};\ m \le k \le 2m-1, k \text{ odd}; \\ 0, & 2m \le k. \end{cases}$$

If $m = 1$, then $H^*(X_G) = Q[z]/(z^2)$. So, we can assume that $m > 1$. Then multiplication by $t \in H^2(B_G)$,

$$t \cup (\cdot) : E_\infty^{k,l} \to E_\infty^{k+2,l},$$

regarded as a spectral sequence endomorphism, is an isomorphism, for $0 \le k \le m-2$ and $l = 0, n$. If $m \neq n$ (resp. $m = n$) the element $1 \otimes v_2$ (resp. $1 \otimes (av_2 - bv_1)$) of $E_2^{0,n}$ is a permanent cocycle and gives a nonzero element $w \in E_\infty^{0,n}$. Let $y = \pi^*(t) \in H^2(X_G)$. Then the total complex

$$\operatorname{Tot} E_\infty^{*,*} \cong Q[y,w]/(y^{(m+1)/2}, w^2)$$

as a graded commutative algebra. We choose an element $z \in H^n(X_G)$ such that $\iota^*(z) = v_2$ (resp. $av_2 - bv_1$) if $m \neq n$ (resp. $m = n$). Then $yz \neq 0$ and $z^2 = 0$. The multiplication

$$y \cup (\cdot) : H^k(X_G) \to H^{k+2}(X_G)$$

is an isomorphism in degrees k such that $0 \le k \le m-2$ and $n \le k \le m+n-3$. So

$$H^*(X_G) \cong Q[y,z]/(y^{(m+1)/2}, z^2),$$

as mentioned in case (i).

Next, suppose that $d_r(1 \otimes v_1) = 0$ and $d_r(1 \otimes v_2) \neq 0$, where $r = n - m + 1$. If n is even, then m must be odd. Consequently, $0 = d_{n-m+1}(1 \otimes v_2^2) = 2at^q \otimes v_1 v_2 \neq 0$, where $d_r(1 \otimes v_2) = at^q \otimes v_1$, $2q = n - m + 1$. Therefore n is odd and m is even. It follows that $d_r(1 \otimes v_1 v_2) = 0$ and the differential

$$d_r : E_r^{*,n} \to E_r^{*,m}$$

is an isomorphism. Thus

$$E_{n-m+2}^{k,n} = 0 = E_{n-m+2}^{k+n-m+1,m}, \quad E_{n-m+2}^{*,m+n} = E_2^{*,m+n}, \quad E_{n-m+2}^{*,0} = E_2^{*,0}.$$

Since m is even,

$$d_{m+1} : E_{m+1}^{k,m} \to E_{m+1}^{k+m+1,0}$$

is trivial, for $0 \leq k \leq n-m$. Since there are no fixed points, the differential

$$d_{n+m+1} : E_{n+m+1}^{0,m+n} \to E_{n+m+1}^{n+m+1,0}$$

must be nontrivial so that we can assume $d_{n+m+1}(1 \otimes v_1 v_2) = at^{(n+m+1)/2} \otimes 1$, $a \neq 0$. Then the differential

$$d_{n+m+1} : E_{n+m+1}^{*,m+n} \to E_{n+m+1}^{*,0}$$

is an isomorphism. So $E_\infty = E_{n+m+2}$ and the nonzero vector spaces in the E_∞-term are $E_\infty^{k,m} = Q$, $0 \leq k \leq n-m$ and $E_\infty^{k,0} = Q$, $0 \leq k \leq m+n$; k even. We have

$$H^k(X_G) = \begin{cases} Q, & 0 \leq k \leq m-1,\ n+1 \leq k \leq n+m-1,\ k \text{ even}; \\ Q \oplus Q, & m \leq k \leq n-1,\ k \text{ even}; \\ 0, & \text{otherwise}. \end{cases}$$

The element $1 \otimes v_1 \in E_2^{0,m}$ is a permanent cocycle and determines an element $w \in E_\infty^{0,m}$. We see that

$$\operatorname{Tot} E_\infty^{*,*} \cong Q[y,w]/(w^2, y^{(n+m+1)/2}, wy^{(n-m+1)/2}),$$

as graded commutative algebras, where $y = \pi^*(t)$, as above. The multiplication

$$y \cup (\cdot) : H^k(X_G) \to H^{k+2}(X_G)$$

is an isomorphism in degrees k, for $0 \leq k \leq n-2$ and $n+1 \leq k \leq n+m-3$. We find an element $z \in H^m(X_G)$ such that $\iota^*(z) = v_1$. Then $y^r z$ and $y^{(m+2r)/2}$ are linearly independent over Q, for $r \leq (n-m-1)/2$. We can change z suitably so that $zy^{(n-m+1)/2} = 0$ and $z^2 = by^m$, $b \in Q$, when $n < 2m$ and $zy^{(n-m+1)/2} = ay^{(n+1)/2}$ and $z^2 = by^m$, $a, b \in Q$, when $2m < n$. Therefore,

$$H^*(X_G) \cong Q[y,z]/(y^{(n+m+1)/2}, zy^{(n-m+1)/2} - ay^{(n+1)/2}, z^2 - by^m)$$

as mentioned in case (ii).

Finally, let us suppose that $d_r(1 \otimes v_1) = 0$, $r = n+1$ and $d_r(1 \otimes v_2) = at^{(n+1)/2} \otimes 1$, $0 \neq a \in Q$. Then, n must be odd. We have $d_{n+1}(1 \otimes v_1 v_2) = \pm at^{(n+1)/2} \otimes v_1$; consequently the differentials

$$d_{n+1} : E_{n+1}^{k,n} \to E_{n+1}^{k+n+1}, \text{ and}$$
$$d_{n+1} : E_{n+1}^{k,m+n} \to E_{n+1}^{k+n+1,m}$$

are isomorphisms. We obtain $E_\infty = E_{n+2}$ and the only nonzero vector spaces in the E_∞-term are $E_\infty^{k,m} = Q = e_\infty^{k,0}$, $0 \leq k \leq n$, k even. Thus for m even

$$H^k(X_G) = \begin{cases} Q, & 0 \leq k \leq m-1,\ n+1 \leq k \leq n+m-1,\ k \text{ even}; \\ Q \oplus Q, & m \leq k \leq n,\ k \text{ even}; \\ 0, & \text{otherwise}, \end{cases}$$

and for m odd,

$$H^k(X_G) = \begin{cases} Q, & 0 \leq k \leq m=1,\ k \text{ even};\ m \leq k \leq m+n-1,\ k \text{ odd}; \\ 0, & \text{otherwise}. \end{cases}$$

The element $1 \otimes v_1 \in E_2^{0,m}$ is a permanent cocycle and yields an element $w \in E_\infty^{0,m}$. We have

$$\operatorname{Tot} E_\infty^{*,*} \cong Q[y,w]/(y^{(n+1)/2}, w^2)$$

as graded commutative algebras, where $y = \pi^*(t) \in H^2(X_G)$. Choose $z \in H^m(X_G)$ such that $\iota^*(z) = v_1$. Then z represents w and the multiplication

$$y \cup (\cdot) : H^k(X_G) \to H^{k+2}(X_G)$$

is an isomorphism, for $0 \leq k \leq n-2$ and $n < k \leq m+n-2$. Accordingly, $y^i z \neq 0$ for $i \leq (n-1)/2$. When m is even, z can be chosen so that $z^2 = by^{m/2}z$, $b \in Q$, is zero for $n < 2m$. So

$$H^*(X_G) \cong Q[y,z]/(y^{(n+1)/2}, z^2 - by^m),$$

as in case (iii).

Since G acts freely on X, $H^*(X_G) \cong H^*(X/G)$ and this completes the proof. □

Examples. An example of case (i) in Theorem 1 is obtained by considering the diagonal action of G on $S^m \times S^n$ where G acts freely on S^m and trivially on S^n. In fact, this possibility can be put more succinctly as $X/G \sim_p L^m \times S^n$. A similar consideration gives examples of case (3), except when $2m < n$ and m is even. By the same method one obtains an example of case (i) in Theorem 2, which can be described as $X/G \sim_2 \mathbb{R}P^m \times S^n$. The space $\mathbb{C}P^{(m-1)/2} \times S^n$ has the cohomology of type (i) in Theorem 3 and is obtained by taking the diagonal action of S^1 on $S^m \times S^n$, where S^1 acts freely on S^m and trivially on S^n. Similarly, case (iii) can also be illustrated.

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