PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 3, Pages 881–886
S 0002-9939(00)05669-0
Article electronically published on September 20, 2000

OHKAWA'S THEOREM: THERE IS A SET OF BOUSFIELD CLASSES

WILLIAM G. DWYER AND JOHN H. PALMIERI

(Communicated by Ralph Cohen)

Abstract. We give a simple proof of Ohkawa's theorem, that there is a set of Bousfield classes. The proof leads us to consider the partially ordered set of Ohkawa classes, especially as it compares to the partially ordered set of Bousfield classes.

Introduction

Bousfield classes are a very useful tool in stable homotopy theory. Ten years ago, Ohkawa proved in [5] that there is a set of Bousfield classes, with cardinality at most $2^{2^{\aleph_0}}$. This result has some interesting consequences—see [3], for example—but it does not seem to be as well known as it should; we are trying to remedy this.

In this short note, we give a simple proof of Ohkawa's theorem, and we discuss some natural questions that arise from the proof. In particular, the proof leads us to consider "Ohkawa classes", which provide a weaker equivalence relation than Bousfield classes do. Unfortunately, Ohkawa classes seem unwieldy, and we have not found any uses for them aside from proving Ohkawa's theorem.

We define Ohkawa classes and prove Ohkawa's theorem (Theorem 1.2) in Section 1; to prove the result we show that there is a set $\mathbf{O}$ of Ohkawa classes, of cardinality at most $2^{2^{\aleph_0}}$, and that this set maps onto the collection of Bousfield classes. This map is in fact a map of partially order sets; in Section 2, we examine the partial ordering on $\mathbf{O}$. It turns out to be rather different from the ordering on the set $\mathbf{B}$ of Bousfield classes; we are able to conclude from this that the surjection $\mathbf{O} \longrightarrow \mathbf{B}$ is not one-to-one. In Section 3, we show that the cardinality of $\mathbf{O}$ is exactly $2^{2^{\aleph_0}}$, but we are unable to deduce anything useful about the cardinality of $\mathbf{B}$ from this.

Throughout, we ask a number of questions; knowing the answers would shed some light on $\mathbf{O}$ and $\mathbf{B}$ and the relationship between the two.

The authors would like to thank Dan Christensen and Neil Strickland for interesting discussions about this material.

Received by the editors May 12, 1999.
2000 *Mathematics Subject Classification.* Primary 55P42, 55P60, 55U35.
This work was partially supported by the National Science Foundation, Grant DMS98-02386.

1. Ohkawa's theorem

The following definition is due to Bousfield [1, 2].

Definition 1.1. The *Bousfield class* $\langle E\rangle$ of a spectrum E is defined to be the collection of E-acyclic spectra:

$$\langle E\rangle = \{X : \ E \wedge X = 0\}.$$

Two spectra E and F are *Bousfield equivalent* if $\langle E\rangle = \langle F\rangle$. Let $\mathbf{B}$ be the collection of Bousfield classes.

Our goal for this paper is to provide a simple proof of the following result, and to discuss some related results. We write $\aleph_0$ for the cardinality of the integers.

Theorem 1.2 (Ohkawa [5]). $\mathbf{B}$ *is a set, with cardinality at most* $2^{2^{\aleph_0}}$.

Our proof is a reformulation of Neil Strickland's reformulation (private communication) of Ohkawa's original proof. As we hope will be clear, this result holds in a fair amount of generality; Lemma 1.4 contains the main facts we need.

Definition 1.3. Let $\mathcal{F}$ be the homotopy category of finite spectra, and let $\overline{\mathcal{F}}$ be the set of isomorphism classes of objects of $\mathcal{F}$.

We need two facts about finite spectra.

Lemma 1.4. (a) $\overline{\mathcal{F}}$ *has cardinality* $\aleph_0$*, and for any two finite spectra* A *and* B*, the set* $[A, B]$ *of homotopy classes of maps from* A *to* B *is at most countable.*
(b) *Every spectrum* X *can be written as a colimit of finite spectra.*

Proof. This is standard. Note that part (b) uses Brown representability of homology functors; see [4, Theorem 4.2.4] for a general statement. □

These are the main requirements for the proof of Theorem 1.2, so that theorem will hold essentially whenever these conditions are satisfied. For example, in any stable homotopy category (as defined in [4]) which satisfies Brown representability for homology functors, the collection $\mathbf{B}$ of Bousfield classes is a set.

Definition 1.5. A *left ideal* I in the category $\mathcal{F}$ is a set of maps between finite spectra which is closed under left composition: if $f\colon A \to B$ is in I, then so is $g \circ f\colon A \to C$ for any map $g\colon B \to C$. If every map in I has the same domain A, we say that I is *based* at A.

Let E be a spectrum. Given a finite spectrum A and a homology class $x \in E_*A$, define the *annihilator ideal* of x to be

$$\operatorname{ann}_A(x) = \{f \in [A, B] : \ [B] \in \overline{\mathcal{F}},\ (E_*f)(x) = 0\}.$$

This is a left ideal based at A. Define the *Ohkawa class* of E, written $\langle\langle E\rangle\rangle$, to be

$$\langle\langle E\rangle\rangle = \{\operatorname{ann}_A(x) : \ [A] \in \overline{\mathcal{F}},\ x \in E_*A\}.$$

Write $\mathbf{O}$ for the collection of all Ohkawa classes.

Lemma 1.6. $\mathbf{O}$ *is a set, with cardinality at most* $2^{2^{\aleph_0}}$.

(In Theorem 3.1, we show that $\mathbf{O}$ has cardinality exactly $2^{2^{\aleph_0}}$.)

Proof. By Lemma 1.4(a), the set of left ideals has cardinality at most $2^{\aleph_0}$. Since Ohkawa classes are sets of annihilator ideals, then the collection $\mathbf{O}$ of all of them is a set, with cardinality at most $2^{2^{\aleph_0}}$. □

In order to show that the Bousfield classes form a set, we will show that the function sending $\langle\langle E\rangle\rangle$ to $\langle E\rangle$ defines a surjection $\mathbf{O} \twoheadrightarrow \mathbf{B}$. In fact, this is not just a map of sets: both $\mathbf{O}$ and $\mathbf{B}$ have partial orderings, and we will show that this map is a map of posets.

The partial ordering on $\mathbf{B}$ is defined as follows: we say that $\langle E\rangle \geq \langle F\rangle$ if $E\wedge X = 0 \Longrightarrow F \wedge X = 0$. Equivalently, viewing Bousfield classes as collections of acyclics, as in Definition 1.1, the ordering is by reverse inclusion. The partial ordering on $\mathbf{O}$ is by inclusion: $\langle\langle E\rangle\rangle \geq \langle\langle F\rangle\rangle$ if for all annihilator ideals $\operatorname{ann}_A(x) \in \langle\langle F\rangle\rangle$, then $\operatorname{ann}_A(x) = \operatorname{ann}_A(y)$ for some $y \in E_*A$.

Lemma 1.7. *If $\langle\langle E\rangle\rangle \geq \langle\langle F\rangle\rangle$, then $\langle E\rangle \geq \langle F\rangle$.*

Proof. Suppose that $\langle\langle E\rangle\rangle \geq \langle\langle F\rangle\rangle$ and that $E\wedge X = 0$; we want to show that $F\wedge X = 0$. Write X as a colimit of finite spectra: $X = \operatorname{colim} X_\alpha$. Since homology commutes with direct limits, it suffices to show that for all α and all $x \in F_*(X_\alpha)$, then $x \mapsto 0$ in F_*X. Given such an x, then because $\langle\langle E\rangle\rangle \geq \langle\langle F\rangle\rangle$, we have $\operatorname{ann}_{X_\alpha}(x) = \operatorname{ann}_{X_\alpha}(y)$ for some $y \in E_*(X_\alpha)$. Since $E \wedge X = 0$, then $i_{\alpha\beta}\colon X_\alpha \to X_\beta$ is in $\operatorname{ann}_{X_\alpha}(y)$ for all large β; hence it is also in $\operatorname{ann}_{X_\alpha}(x)$. Thus x goes to zero in $F_*X = \operatorname{colim} F_*X_\alpha$, and thus $F \wedge X = 0$. ∎

Corollary 1.8. *The map $f\colon \mathbf{O} \to \mathbf{B}$ defined by $f\langle\langle E\rangle\rangle = \langle E\rangle$ is well-defined, surjective, and order-preserving.*

Theorem 1.2 follows immediately. By the way, we point out in Corollary 2.7 that the map $\mathbf{O} \longrightarrow \mathbf{B}$ is not one-to-one.

2. The partial ordering on O

In this section, we examine the partial ordering on $\mathbf{O}$. There is a minimal element and a maximal element, and we also find that the partial orderings on $\mathbf{O}$ and $\mathbf{B}$ are rather different: many pairs of Ohkawa classes are incomparable, while the corresponding Bousfield classes are not. We use this observation to deduce that the map $\mathbf{O} \longrightarrow \mathbf{B}$ is not one-to-one.

Given a left ideal I in $\mathcal{F}$ based at a finite spectrum A, we write $\operatorname{dom}(I)$ for A, the domain of the maps in I. For any finite A, we let $(1)_A$ denote the left ideal consisting of every map with domain A.

Lemma 2.1. $\langle\langle E \vee F\rangle\rangle = \{I \cap J : I \in \langle\langle E\rangle\rangle,\ J \in \langle\langle F\rangle\rangle,\ \operatorname{dom}(I) = \operatorname{dom}(J)\}$. *More generally, for any set of spectra $\{E_\alpha\}$, the Ohkawa class of $\bigvee_\alpha E_\alpha$ consists of all finite intersections of elements of the Ohkawa classes of the E_α:*

$$\langle\langle \bigvee_\alpha E_\alpha\rangle\rangle = \{I_{\alpha_1} \cap \cdots \cap I_{\alpha_n} : I_{\alpha_j} \in \langle\langle E_{\alpha_j}\rangle\rangle,\ \operatorname{dom}(I_{\alpha_j}) = \operatorname{dom}(I_{\alpha_k})\}.$$

Proof. This is clear. ∎

Corollary 2.2. *$\langle\langle E\rangle\rangle \vee \langle\langle F\rangle\rangle = \langle\langle E \vee F\rangle\rangle$ is well-defined, and is at least as large as both $\langle\langle E\rangle\rangle$ and $\langle\langle F\rangle\rangle$; the analogous statement holds for $\bigvee_\alpha \langle\langle E_\alpha\rangle\rangle$.*

Proof. To see that $\langle\langle E\rangle\rangle \vee \langle\langle F\rangle\rangle \geq \langle\langle E\rangle\rangle$, given an ideal $\operatorname{ann}_A(x) \in \langle\langle E\rangle\rangle$, consider the class $x \oplus 0 \in E_*A \oplus F_*A$. This class has annihilator ideal equal to $\operatorname{ann}_A(x) \cap (1)_A = \operatorname{ann}_A(x)$. ∎

Question 2.3. Is $\langle\langle E \vee F\rangle\rangle$ the least upper bound of $\langle\langle E\rangle\rangle$ and $\langle\langle F\rangle\rangle$?

The answer is probably "no"; in fact, $\langle\!\langle E\rangle\!\rangle \vee \langle\!\langle E\rangle\!\rangle$ will be strictly larger than $\langle\!\langle E\rangle\!\rangle$ whenever $\langle\!\langle E\rangle\!\rangle$ is not closed under intersections of ideals.

Lemma 2.4. *Write* 0 *for the trivial spectrum. Then* $\langle\!\langle 0\rangle\!\rangle$ *is the minimal element in the poset* $\mathbf{O}$.

Proof. $\langle\!\langle 0\rangle\!\rangle$ contains only the ideals $(1)_A$ for each finite A, and for any spectrum E, $(1)_A$ is the annihilator ideal of $0 \in E_*A$. Hence $\langle\!\langle 0\rangle\!\rangle \leq \langle\!\langle E\rangle\!\rangle$ for all E. □

This is somewhat heartening, because $\langle 0\rangle$ is the minimal element in $\mathbf{B}$. Things rapidly turn sour, though.

Lemma 2.5. $\langle\!\langle S^0\rangle\!\rangle \not\geq \langle\!\langle H\mathbf{Q}\rangle\!\rangle$.

In contrast, in the Bousfield lattice $\mathbf{B}$, $\langle S^0\rangle$ is the unique maximal element; hence it is at least as large as any Bousfield class, while $\langle H\mathbf{Q}\rangle$ is a minimal nonzero element.

Proof. Consider the unit $\eta\colon S^0 \to H\mathbf{Q}$, viewed as an element in $H\mathbf{Q}_*S^0$. Then the annihilator ideal of η contains every map from S^0 to every finite torsion spectrum. On the other hand, given any nonzero element $y \in \pi_*S^0$, for some prime p and for every $n \gg 0$, y is not p^n-divisible, so the composite $j_n \circ y$ is non-trivial, where $j_n\colon S^0 \to M(p^n)$ is the inclusion of the bottom cell. Hence $\mathrm{ann}_{S^0}(\eta)$ is not an element of $\langle\!\langle S^0\rangle\!\rangle$. □

A similar argument should show that $\langle\!\langle S^0_{(p)}\rangle\!\rangle \not\geq \langle\!\langle K(n)\rangle\!\rangle$ for all $n \geq 0$, where $S^0_{(p)}$ denotes the p-local sphere. (And as with $\langle H\mathbf{Q}\rangle$, each $\langle K(n)\rangle$ is a minimal nonzero Bousfield class.) We conclude that the partial ordering on $\mathbf{O}$ does not bear much relation to the partial ordering on $\mathbf{B}$.

Corollary 2.6. $\langle\!\langle S^0\rangle\!\rangle < \langle\!\langle S^0 \vee H\mathbf{Q}\rangle\!\rangle$.

Since S^0 and $S^0 \vee H\mathbf{Q}$ have the same Bousfield class, we have the following.

Corollary 2.7. *The surjection* $f\colon \mathbf{O} \to \mathbf{B}$ *is not one-to-one.*

Corollary 2.8. $\langle\!\langle S^0\rangle\!\rangle$ *is not the largest element in* $\mathbf{O}$.

Note that $\mathbf{O}$ does have a largest element: $\displaystyle\bigvee_{\langle\!\langle E\rangle\!\rangle \in \mathbf{O}} \langle\!\langle E\rangle\!\rangle$.

Question 2.9. Is this the Ohkawa class of some familiar spectrum?

Remark 2.10. Dan Christensen has suggested several modifications to the definition of Ohkawa class, in order to make the partial ordering better behaved. If one defines the *modified Ohkawa class* of E to be $[\![E]\!] = \langle\!\langle \bigvee_{n=1}^{\infty} E\rangle\!\rangle$, then this has the effect of closing $\langle\!\langle E\rangle\!\rangle$ under finite intersections, and hence making $\vee$ the least upper bound. Another more involved modification changes the partial ordering so that, for example, the class of the sphere is larger than that of $H\mathbf{Q}$.

3. The cardinality of O

In this section, we determine the cardinality of the set $\mathbf{O}$ of Ohkawa classes.

Theorem 3.1. *There are at least* $2^{2^{\aleph_0}}$ *Ohkawa classes.*

Hence by Lemma 1.6, $\mathbf{O}$ has cardinality $2^{2^{\aleph_0}}$. In order to prove the theorem, we need some notation and some definitions.

Definition 3.2. Given a spectrum E and a finite spectrum A, we write $\langle\langle E\rangle\rangle_A$ for the annihilator ideals in $\langle\langle E\rangle\rangle$ based at A. Let $\mathcal{P} = \{2, 3, 5, \dots\}$ be the set of positive prime numbers. If I is an ideal based at S^0, let $\mathcal{P}(I)$ be this set of prime numbers:

$$\mathcal{P}(I) = \{p \in \mathcal{P} : S^0 \to S^0 \cup_p e^1 \in I\}.$$

Given two sets of primes S and T, we say that S and T are *commensurable* if $(S - T) \cup (T - S)$ is finite. We write $\overline{S}$ for the commensurability class of S.

Proof of Theorem 3.1. Given an infinite set $S = \{p_1, p_2, \dots\}$ of prime numbers, define a spectrum $X(S)$ by

$$X(S) = \operatorname{colim}(S_0 \xrightarrow{p_1} S^0 \xrightarrow{p_2} S^0 \xrightarrow{p_3} \dots).$$

We want to examine the Ohkawa class of $X(S)$ in terms of sets of primes. First, we note that

$$\pi_0 X(S) = \{\frac{a}{b} \in \mathbf{Q} : b = p_{n_1} \cdots p_{n_k},\ p_{n_i} \in S,\ n_1 < \cdots < n_k\},$$

while $\pi_i X(S) = \pi_i S^0$ if $i \neq 0$. For each ideal I based at S^0 in the Ohkawa class of $X(S)$, we can then determine the possible commensurability classes of $\mathcal{P}(I)$:

$$\{\overline{\mathcal{P}(I)} : I \in \langle\langle X(S)\rangle\rangle_{S^0}\} = \{\overline{\emptyset},\ \overline{S}, \overline{\mathcal{P}}\}.$$

For instance, given $x = \dfrac{a}{b} \in \pi_0 X(S)$ with $b = p_{n_1} \dots p_{n_k}$, then $\mathcal{P}(\operatorname{ann}_{S^0}(x))$ will contain all primes in S except the n_i; hence $\mathcal{P}(\operatorname{ann}_{S^0}(x))$ and S are commensurable.

Now consider a set of subsets of $\mathcal{P}$, indexed by some indexing set J: $\{S_\alpha : \alpha \in J\}$. Then by Lemma 2.1,

$$\{\overline{\mathcal{P}(I)} : I \in \langle\langle \bigvee_{\alpha \in J} X(S_\alpha)\rangle\rangle_{S^0}\}$$

consists of $\overline{\emptyset}$, $\overline{\mathcal{P}}$, $\overline{S_\alpha}$ for each α, and the commensurability classes of finite intersections of the S_α. Lemma 3.3 below shows that there is a set $\mathbf{T}$ of subsets of $\mathcal{P}$, with cardinality $2^{\aleph_0}$, so that no member of $\mathbf{T}$ is commensurable with any subset of any other member. Given any set $\{S_\alpha\}$ of elements of $\mathbf{T}$, one can recover the commensurability classes of the S_α from the set of all commensurability classes of finite intersections of them—the $\overline{S_\alpha}$ are the maximal elements. Hence the spectra $\bigvee_J X(S_\alpha)$, with the S_α chosen from $\mathbf{T}$, provide $2^{2^{\aleph_0}}$ different Ohkawa classes. □

Lemma 3.3. *There is a set $\mathbf{T}$ of subsets of the set $\mathcal{P}$ of prime numbers, so that $\mathbf{T}$ has cardinality $2^{\aleph_0}$, and so that no element of $\mathbf{T}$ is commensurable with a subset of any other element of $\mathbf{T}$.*

Proof. First, note that any countably infinite set $\mathcal{N}$ has $2^{\aleph_0}$ subsets, none of which is a subset of any other. For example, partition $\mathcal{N}$ into countably many sets N_1, N_2, N_3, … , where each N_i has cardinality at least 2. Then the subsets of $\mathcal{N}$ of the form

$$\{x_i : i = 1, 2, \dots,\ x_i \in N_i\}$$

do the job.

Now, partition $\mathcal{P}$ into infinitely many infinite sets S_1, S_2, S_3, … . The set $\mathcal{N} = \{1, 2, 3, \dots\}$ has $2^{\aleph_0}$ subsets, none of which is a subset of any other; for each such subset $T_\alpha \subseteq \mathcal{N}$, let

$$S_\alpha = \bigcup_{n \in T_\alpha} S_n.$$

Since no T_α is a subset of any other, then this gives $2^{\aleph_0}$ subsets S_α of $\mathcal{P}$, none of which is commensurable with a subset of any other. $\square$

Note that each $X(S)$ constructed in the proof of Theorem 3.1 is Bousfield equivalent to the sphere, so this result gives little insight into the cardinality of the set of Bousfield classes. This also gives a dramatic example of the failure of $\mathbf{O} \to \mathbf{B}$ to be one-to-one. By way of comparison, we have this result, which was pointed out by Neil Strickland.

Lemma 3.4. *There are at least $2^{\aleph_0}$ Bousfield classes.*

Proof. For each subset S of the set $\{0, 1, 2, \dots\} \cup \{\infty\}$, we have a distinct Bousfield class: $\langle \bigvee_{n \in S} K(n) \rangle$. $\square$

Question 3.5. What is the cardinality of $\mathbf{B}$?

References

[1] A. K. Bousfield, *The Boolean algebra of spectra*, Comment. Math. Helv. **54** (1979), no. 3, 368–377. MR **81a:**55015

[2] ———, *The localization of spectra with respect to homology*, Topology **18** (1979), no. 4, 257–281. MR **80m:**55006

[3] M. Hovey and J. H. Palmieri, *The structure of the Bousfield lattice*, Homotopy invariant algebraic structures (J.-P. Meyer, J. Morava, and W. S. Wilson, eds.), Contemp. Math., vol. 239, Amer. Math. Soc., Providence, RI, 1999.

[4] M. Hovey, J. H. Palmieri, and N. P. Strickland, *Axiomatic stable homotopy theory*, Mem. Amer. Math. Soc. **128** (1997), no. 610, x+114. MR **98a:**55017

[5] T. Ohkawa, *The injective hull of homotopy types with respect to generalized homology functors*, Hiroshima Math. J. **19** (1989), no. 3, 631–639. MR **90j:**55013

Department of Mathematics, University of Notre Dame, Notre Dame, Indiana 46556
E-mail address: dwyer.1@nd.edu

Department of Mathematics, University of Notre Dame, Notre Dame, Indiana 46556
Current address: Department of Mathematics, University of Washington, Box 354350, Seattle, Washington 98195-4350
E-mail address: palmieri@member.ams.org