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DECAY OF GLOBAL SOLUTIONS, STABILITY AND BLOWUP FOR A REACTION-DIFFUSION PROBLEM WITH FREE BOUNDARY

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Abstract. We consider a one-phase Stefan problem for the heat equation with a nonlinear reaction term. We first exhibit an energy condition, involving the initial data, under which the solution blows up in finite time in L^∞ norm. We next prove that all global solutions are bounded and decay uniformly to 0, and that either: (i) the free boundary converges to a finite limit and the solution decays at an exponential rate, or (ii) the free boundary grows up to infinity and the decay rate is at most polynomial. Finally, we show that small data solutions behave like (i).

1. Introduction

Consider a substance which is heat-diffusive and chemically reactive in its liquid phase, and neutral in its solid phase. Assume that the (one-dimensional) liquid is surrounded by the solid at melting temperature 0 at one end, and is isolated at the other end. Assuming a power-type reaction term, one is then led to the following one-phase Stefan problem:

$$\text{(SP)} \qquad \begin{cases} u_t - u_{xx} = u^p, \quad 0 < t < T, \quad 0 < x < s(t), \\ u(0,x) = u_0(x) \geq 0, \quad 0 < x < s_0, \quad s(0) = s_0 > 0, \\ u(t,s(t)) = u_x(t,0) = 0, \quad 0 < t < T, \\ s'(t) = -u_x(t,s(t)), \quad 0 < t < T, \end{cases}$$

where we suppose $p > 1$. In the present paper, we will address the following questions.

A. What conditions on the initial data imply that thermal runaway, that is, finite time blowup of u, will occur?
B. Is the 0 solution stable, in the sense that the solution of (SP) is global and bounded for suitably small initial data?
C. Can one classify all possible asymptotic behaviors of the global solutions of (SP)? In particular, can one rule out the existence of unbounded global solutions?

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All these questions have been the object of extensive investigation for the corresponding problem on a fixed domain, that is,

$$\begin{cases} u_t - u_{xx} = u^p, \quad 0 < t < T, \quad 0 < x < L, \\ u(t, L) = u_x(t, 0) = 0, \quad 0 < t < T \end{cases} \tag{1.1}$$

(also in higher dimensions). For the further question of asymptotic behavior of blowing up solutions of (1.1) and its higher-dimensional analogue, many recent references can be found for instance in [M]. In contrast, the questions of blowup and global existence for free-boundary problems with superlinear source terms like (SP) seem to have been almost unexplored so far. We are only aware of the work [A] on the problem (SP), where interesting results on blowup profiles were obtained for special classes of initial data, and of the numerical study in [IK].

Concerning question A, taking into account the known blowup results for equation (1.1), it is expected that finite time blowup of u should occur if u_0 is suitably large. A typical condition for blowup in problem (1.1) involves the natural energy, defined as:

$$E(u_0) = \frac{|u_{0,x}|_2^2}{2} - \frac{|u_0|_{p+1}^{p+1}}{p+1} = \int_0^L \left(\frac{(u_{0,x})^2}{2} - \frac{u_0^{p+1}}{p+1} \right)(x)\, dx.$$

In the case of a fixed boundary, it is well-known [Le, B] that negative energy $E(u_0) < 0$ leads to finite time blowup in L^∞-norm. On the other hand, a simple maximum principle argument shows that the solution of the Stefan problem (SP) dominates the solution of (1.1) with $L = s_0$ and same initial data $u_0 \geq 0$. As a consequence, negative initial energy also implies finite time blowup for the Stefan problem. Going further, we will establish a weaker energy condition for blowup, which demonstrates that the Stefan problem is in some sense less stable than the problem with fixed boundary. Namely, we will prove blowup under the condition:

$$E(u_0) < \frac{C|u_0|_1^3}{(s_0 + |u_0|_1)^4},$$

where $C > 0$ is some (explicitly determined) constant.

We next consider the question B of stability. Extending the above remark, it follows from the maximum principle that the solution of the Stefan problem (SP) is dominated by the solution of the Cauchy problem:

$$\begin{cases} u_t - u_{xx} = u^p, \quad 0 < t < T, \quad -\infty < x < -\infty, \\ u(0, x) = u_0(|x|), \quad -s_0 < x < s_0, \quad u(0, x) = 0, \quad x \in \mathbb{R} \backslash (-s_0, s_0). \end{cases} \tag{1.2}$$

We know that the solution of (1.1) exists globally if the initial data is sufficiently small in L^∞ norm, while for the Cauchy problem (1.2), the existence of nontrivial nonnegative global solutions may depend on the value of p: none exists if $p \leq 3$, and both nonglobal and small global solutions exist if $p > 3$. (See [Fu, H, W]. See also [S2] for related stability/instability results in general unbounded domains of $\mathbb{R}^N$.) Since the moving boundary problem can be thought of as a sort of intermediate between the cases of bounded and unbounded intervals, it is not clear whether the solution of the Stefan problem should exist globally for small initial data whatever the value of p. However, we will show that this is indeed the case.

As for question C, the first natural question is whether all global solutions are bounded or not. This question has been studied in detail in the case of problems in fixed domains (see, e.g., [NST, CL, G, Fi, Q] for bounded domains, and [K, S1]

for the Cauchy problem). In these works, it is proved that in many cases, the answer is positive (at least for nonnegative solutions and subcritical p; hence, in particular in one-dimensional problems). We will show that for the Stefan problem (SP) also, all global solutions are uniformly bounded. Actually, we will prove some more precise facts concerning the global solutions of (SP). Indeed, it will be shown that all of them decay uniformly to 0 as t goes to infinity. Furthermore, we will prove that there are only two possible behaviors for global solutions. In the first one, the free boundary converges to a finite limit and the solution decays with an exponential rate. In the second one, the free boundary grows up to infinity and the decay rate is at most polynomial. Moreover, the solution of (SP) is always global and exponentially decaying if the initial data is sufficiently small in L^∞ norm (depending on s_0).

The outline of the article is as follows. Blowup is treated in Section 2, and Section 3 is devoted to the study of global solutions.

2. Finite time blowup

In what follows, we assume $p > 1$, $s_0 > 0$, $u_0 \in C^1([0, s_0])$, $u_0 \geq 0$, with $u(s_0) = (u_0)_x(0) = 0$. Under these assumptions, there exists a unique, maximal in time, classical solution (u, s) of (SP), which satisfies $u \geq 0$ and $s' \geq 0$ (see [FP, Theorem 1 and Remark 1], or [A, Proposition 1.1]). We denote by $T^* = T^*(u_0) \in (0, \infty]$ its maximal existence time. Moreover, if $T^* < \infty$, we then have

$$\limsup_{t \to T^*} |u(t)|_\infty = \infty, \tag{2.1}$$

and we say that u blows up in finite time (see [A, Proposition 3.1]).

To state our blowup result, we introduce the energy

$$E(u_0) = \int_0^{s_0} \left(\frac{(u_{0,x})^2}{2} - \frac{u_0^{p+1}}{p+1} \right)(x)\, dx.$$

Also, we set $|u_0|_1 = \int_0^{s_0} u_0(x)\, dx$.

Theorem 2.1. *Let u be the solution of the problem* (SP) *and set $C = \frac{\pi^2}{256}$. Then we have $T^* < \infty$ whenever*

$$E(u_0) < H(u_0) \equiv C \frac{|u_0|_1^3}{(s_0 + |u_0|_1)^4}. \tag{2.2}$$

One of the main ingredients of the proof of Theorem 2.1 is the classical concavity argument of Levine [Le]. However, some extra work is needed to exhibit the special destabilizing effect of the free boundary condition. We begin with two lemmas.

Lemma 2.2 (Energy identities)**.** *Let u be the solution of the problem* (SP)*, and define the energy of the solution at time t by*

$$\widetilde{E}(t) = \int_0^{s(t)} \left(\frac{(u_x)^2}{2} - \frac{u^{p+1}}{p+1} \right)(t, x)\, dx,$$

and its L^1-norm by $|u(t)|_1 = \int_0^{s(t)} u(t, x)\, dx$. Then we have the relations

$$\frac{d\widetilde{E}}{dt} = -\int_0^{s(t)} u_t^2(t, x)\, dx - \frac{s'^3}{2} \tag{2.3}$$

and

$$|u(t)|_1 - |u_0|_1 = s_0 - s(t) + \int_0^t \int_0^{s(t)} u^p(\tau, x)\, dx\, d\tau. \tag{2.4}$$

Proof. We compute

$$\frac{d\widetilde{E}}{dt} = \int_0^{s(t)} (u_x u_{xt} - u^p u_t)(t,x)\, dx + s'(t) \left(\frac{(u_x)^2}{2}(t, s(t)) - \frac{u^{p+1}}{p+1}(t, s(t)) \right).$$

Integrating by parts and using $u_x(t,0) = 0$, we get $\int_0^{s(t)} u_x u_{xt} dx = -\int_0^{s(t)} u_{xx} u_t dx + u_x u_t(t, s(t))$. Noticing that $0 = \frac{d}{dt}(u(t, s(t))) = u_t(t, s(t)) + s'(t) u_x(t, s(t))$, it follows that $u_x u_t(t, s(t)) = -s'(t) u_x^2(t, s(t)) = -s'^3(t)$. By substitution, we then obtain

$$\frac{d\widetilde{E}}{dt} = -\int_0^{s(t)} (u^p u_t + u_{xx} u_t)(t,x) dx - s'^3(t) + \frac{s'^3(t)}{2} = -\int_0^{s(t)} u_t^2(t,x)\, dx - \frac{s'^3(t)}{2},$$

that is, (2.3). Finally, from (SP), we see that

$$\begin{aligned} \frac{d}{dt} \int_0^{s(t)} u(t,x)\, dx &= \int_0^{s(t)} u_t(t,x)\, dx + s'(t) u(t, s(t)) \\ &= \int_0^{s(t)} u_{xx}(t,x)\, dx + \int_0^{s(t)} u^p(t,x)\, dx = -s'(t) + \int_0^{s(t)} u^p(t,x)\, dx, \end{aligned}$$

and (2.4) follows by integrating between 0 and t.

Lemma 2.3. *Assume $T^* = \infty$, and let $A = \int_0^\infty s'^3(t)\, dt$. Then we have $A \geq \frac{\pi^2}{128} H(u_0)$.*

Proof. Let v be the solution of the following auxiliary free-boundary problem:

$$\begin{cases} v_t - v_{xx} = 0, \quad 0 < t < \infty, \quad 0 < x < \sigma(t), \\ v(0,x) = u_0(x), \quad 0 < x < s_0, \quad \sigma(0) = s_0, \\ v(t, \sigma(t)) = v_x(t, 0) = 0, \quad 0 < t < \infty, \\ \sigma'(t) = -v_x(t, \sigma(t)), \quad 0 < t < \infty. \end{cases}$$

It is well-known that v exists for all $t > 0$ (see, e.g., [Fr, Chapter 8]) and one can deduce from the maximum principle that $u \geq v \geq 0$ and $s(t) \geq \sigma(t) \geq s_0$ on $(0, T^*)$. By the same arguments as in Lemma 2.2, denoting $|v(t)|_1 = \int_0^{\sigma(t)} v(t,x)\, dx$, one easily obtains

$$\sigma(t) - s_0 = |u_0|_1 - |v(t)|_1. \tag{2.5}$$

Using Hölder's inequality and $s(t) \geq \sigma(t) \geq s_0$, it follows that for all $t \geq 0$,

$$A \geq t^{-2} \left(\int_0^t s' \right)^3 = t^{-2} (s(t) - s_0)^3 \geq t^{-2} (|u_0|_1 - |v(t)|_1)^3. \tag{2.6}$$

On the other hand, by the maximum principle, we have $v \leq w$, where w is the solution of the Cauchy problem

$$\begin{cases} w_t - w_{xx} = 0, \quad t > 0, \quad -\infty < x < \infty, \\ w(0,x) = \overline{u}_0(x) = \begin{cases} u_0(|x|), & -s_0 \leq x \leq s_0, \\ 0, & x \in \mathbb{R}/[-s_0, s_0]. \end{cases} \end{cases}$$

By the $L^1 - L^\infty$ estimate for the heat equation, we have

$$|v(t)|_\infty \le |w(t)|_\infty \le (4\pi t)^{-1/2}|w(0)|_1 = (\pi t)^{-1/2}|u_0|_1,$$

hence, by (2.5),

$$|v(t)|_1 \le \sigma(t)(\pi t)^{-1/2}|u_0|_1 \le (s_0 + |u_0|_1)(\pi t)^{-1/2}|u_0|_1.$$

Therefore, we have $|v(t_0)|_1 \le |u_0|_1/2$ for $t_0 = 4\pi^{-1}(s_0 + |u_0|_1)^2$. The desired estimate is then obtained by plugging the value $t = t_0$ into inequality (2.6). □

Proof of Theorem 2.1. Define the function

$$F(t) = \int_0^t \int_0^{s(\tau)} u^2(\tau, x)\, dx\, d\tau.$$

We compute $F'(t) = \int_0^{s(t)} u^2(t,x)\, dx$ and

$$\begin{aligned} F''(t) &= \int_0^{s(t)} 2uu_t(t,x)\, dx + s'(t)u^2(t,s(t)) = \int_0^{s(t)} 2uu_t(t,x)\, dx \\ &= 2\int_0^{s(t)} (u^{p+1} + uu_{xx})(t,x)dx = 2\int_0^{s(t)} (u^{p+1} - u_x^2)(t,x)\, dx + 2[uu_x(t,x)]_0^{s(t)} \\ &= -2(p+1)\widetilde{E}(t) + (p-1)\int_0^{s(t)} u_x^2(t,x)\, dx. \end{aligned}$$

Using identity (2.3), we get

$$\begin{aligned} F''(t) &= 2(p+1)\int_0^t \int_0^{s(\tau)} u_t^2(\tau,x)\, dx\, d\tau \\ &\quad + 2(p+1)\left[\int_0^t \frac{s'^3(\tau)}{2} d\tau - \widetilde{E}(0)\right] + (p-1)\int_0^{s(t)} u_x^2(t,x)\, dx. \end{aligned} \tag{2.7}$$

Now assume $T^* = \infty$, for contradiction. The assumption (2.2), together with Lemma 2.3, implies that $\widetilde{E}(0) < \frac{1}{2}\int_0^t s'^3(\tau)\, d\tau$ for all $t \ge t_0$ sufficiently large, so that

$$F''(t) > 2(p+1)\int_0^t \int_0^{s(\tau)} u_t^2(\tau,x)\, dx\, d\tau, \qquad t \ge t_0. \tag{2.8}$$

The end of the proof then consists in the classical concavity argument of Levine [Le], which we recall for the convenience of the reader. By applying the Cauchy-Schwarz inequality, we get:

$$\begin{aligned} FF''(t) &\ge 2(p+1)\int_0^t \int_0^{s(\tau)} u_t^2\, dx\, d\tau \int_0^t \int_0^{s(\tau)} u^2\, dx\, d\tau \\ &\ge 2(p+1)\left(\int_0^t \int_0^{s(\tau)} uu_t\, dx\, d\tau\right)^2 = \frac{p+1}{2}(F'(t) - F'(0))^2. \end{aligned}$$

On the other hand, (2.8) implies that

$$F'(t) \ge F'(t_0+1) = \int_0^{s(t_0+1)} u^2(t_0+1, x)\, dx > 0, \qquad t \ge t_0 + 1,$$

so that $\lim_{t\to\infty} F(t) = \infty$. We then obtain

$$FF''(t) \ge \frac{p+3}{4} F'^2, \qquad t \ge t_1,$$

for some large $t_1 \geq t_0 + 1$ (since $p > 1$). Defining $G(t) = F^{-\alpha}(t)$ for $t \geq t_1$, with $\alpha = (p-1)/4$, it follows that

$$G'(t) = -\alpha F'(t) F^{-(\alpha+1)}(t) < 0, \qquad t \geq t_1,$$

and that

$$G''(t) = -\alpha F^{-(\alpha+2)}(t) \left(FF'' - \frac{p+3}{4} F'^2 \right)(t) \leq 0, \qquad t \geq t_1.$$

This implies that G is concave, decreasing, positive, for $t \geq t_1$, which is impossible. This contradiction shows that $T^* < \infty$, which completes the proof of Theorem 2.1. □

3. Long time behavior of global solutions

We keep the assumptions stated at the beginning of Section 2. The following result shows that all global solutions are bounded and decay uniformly to 0. Moreover, all the possible asymptotic behaviors are described.

Theorem 3.1. *Let u be the solution of the problem* (SP)*, and assume $T^* = \infty$. Let $s_\infty = \lim_{t\to\infty} s(t) \leq \infty$ (recall that $s(t)$ is nondecreasing). Then one of the following two possibilities occurs*:

(i) *$s_\infty < \infty$ and there exist some real numbers C, $\alpha > 0$ (depending on u) such that*

$$|u(t)|_\infty \leq Ce^{-\alpha t}, \qquad t \geq 0; \tag{3.1}$$

(ii) *$s_\infty = \infty$ and $\lim_{t\to\infty} |u(t)|_\infty = 0$. Moreover, in this case, one has the estimates*

$$s(t) = O(t^{2/3}), \qquad t \to \infty, \tag{3.2}$$

and

$$\liminf_{t\to\infty} s^{2/(p-1)}(t) |u(t)|_\infty > 0, \tag{3.3}$$

hence, in particular,

$$\liminf_{t\to\infty} t^{4/(3(p-1))} |u(t)|_\infty > 0. \tag{3.4}$$

The next result shows that possibility (i), i.e. exponential decay, occurs for suitably small data.

Theorem 3.2. *There exists $K > 0$ depending only on p, such that, if*

$$|u_0|_\infty \leq K \min(1, s_0^{-2/(p-1)}),$$

then $T^* = \infty$ and (i) *in Theorem 3.1 occurs. More precisely one may take $\alpha = \frac{1}{32 s_0^2}$ and $C = 2|u_0|_\infty$ in* (3.1)*, and one then has $s_\infty \leq 4 s_0$.*

It is an open problem whether possibility (ii) actually occurs.[1] However, if instead of (SP), we consider the problem (SP)$'$, where the free-boundary condition $(\mathrm{SP})_4$ is replaced with

$$\lambda s'^k(t) = -u_x(t, s(t)), \qquad k = (p+1)/(p-1), \ \lambda > 0,$$

[1]Note added in proof: This has been proved recently (see M. Fila and Ph. Souplet, article in preparation).

then the following Proposition 3.3 shows that there exist some global solutions of (SP)$'$ such that $s_\infty = \infty$. Note that k becomes arbitrarily close to 1 when p goes to infinity. We conjecture that there exist some solutions of (SP) which satisfy (ii) in Theorem 3.1, but that these solutions are unstable.

As a matter of fact, by using similar techniques, one can generalize the results of Theorems 3.1 and 3.2 to the modified problem (SP)$'$ for $k > 1$, with different exponents in the estimates (3.2)–(3.4). This will appear in a forthcoming publication.

Proposition 3.3. *There exists $\lambda > 0$, such that for all $s_0 > 0$, there exists a global solution (u, s) of problem* (SP)$'$ *such that $\lim_{t\to\infty} \uparrow s(t) = \infty$. More specifically, u can be found under the self-similar form*

$$u(t,x) = (t+T)^{-\frac{1}{p-1}} u_0(x/\sqrt{t+T}), \qquad s(t) = s_0\sqrt{1+\frac{t}{T}}$$

for all $t \geq 0$, $0 \leq x \leq s(t)$, and for some $T > 0$.

Remark 3.1. We note that for the solutions u of Proposition 3.3, $|u(t)|_\infty$ decays exactly like $s^{-2/(p-1)}(t) \sim Ct^{-1/(p-1)}$. This suggests that the estimate (3.3) in Theorem 3.1 is sharp.

For convenience, we first prove Theorem 3.2, whose result will be used in part in the proof of Theorem 3.1.

Proof of Theorem 3.2. The proof relies on the construction of a suitable supersolution. The idea is inspired from [RT]. For γ, α and $\varepsilon > 0$ to be fixed, we define

$$\sigma(t) = 2s_0(2 - e^{-\gamma t}), \quad t \geq 0, \qquad V(y) = 1 - y^2, \quad 0 \leq y \leq 1,$$

and

$$v(t,x) = \varepsilon e^{-\alpha t} V(x/\sigma(t)), \qquad t \geq 0,\ 0 \leq x \leq \sigma(t).$$

An easy computation yields:

$$\begin{aligned} Pv \equiv v_t - v_{xx} - v^p &= \varepsilon e^{-\alpha t}[-\alpha V - x\sigma'\sigma^{-2}V' - \sigma^{-2}V'' - \varepsilon^{p-1}e^{-(p-1)\alpha t}V^p] \\ &\geq \varepsilon e^{-\alpha t}\left[-\alpha + \frac{1}{8s_0^2} - \varepsilon^{p-1}\right] \end{aligned}$$

for all $t > 0$ and $0 < x < \sigma(t)$. On the other hand, we have $\sigma'(t) = 2\gamma s_0 e^{-\gamma t} > 0$ and $-v_x(t,\sigma(t)) = 2\varepsilon\sigma^{-1}(t)e^{-\alpha t}$. If we choose $\alpha = \gamma = (16s_0^2)^{-1}$, and $\varepsilon \leq \varepsilon_0 \equiv \min(\frac{1}{8}, (16s_0^2)^{-1/(p-1)})$, it follows that

$$\begin{cases} Pv \geq 0, \quad t > 0, \quad 0 < x < \sigma(t), \\ \sigma'(t) > -v_x(t,\sigma(t)), \quad t > 0, \\ v(t,\sigma(t)) = v_x(t,0) = 0, \quad t > 0, \\ \sigma(0) = 2s_0 > s_0. \end{cases}$$

Now let $K = \frac{1}{2}\min(\frac{1}{8}, 16^{-1/(p-1)})$, and assume that $|u_0|_\infty \leq K\min(1, s_0^{-2/(p-1)})$. Choosing $\varepsilon = 2|u_0|_\infty \leq \varepsilon_0$, we also get $u_0(x) < v(0,x)$ for $0 \leq x \leq s_0$. By using the maximum principle, one then shows that $s(t) < \sigma(t)$ and that $u(t,x) < v(t,x)$ for $0 \leq x \leq s(t)$, as long as u exists. In particular, it follows from the continuation property (2.1) that u exists globally. The proof is complete. □

Let us now prove Theorem 3.1. We first consider the case $s_\infty < \infty$. To prove boundedness and decay to 0, we adapt the apriori estimate method of Fila [Fi]. This argument was originally designed to prove boundedness (but not decay) of solutions of problems in fixed bounded domains. We shall need the following two lemmas.

Lemma 3.4. *Assume that $T^* = \infty$ and $s_\infty < \infty$. Then we have*

$$\liminf_{t\to\infty} |u(t)|_{L^\infty} < \infty \quad \textit{and} \quad \liminf_{t\to\infty} |u(t)|_{H^1} < \infty.$$

Proof of Lemma 3.4. Assume that $\lim_{t\to\infty} |u_x(t)|_{L^2} = \infty$. Using the notation of Theorem 2.1 and equality (2.7), it follows that

$$F''(t) > 2(p+1)\int_0^t \int_0^{s(\tau)} u_t^2(\tau, x)\, dx\, d\tau,$$

for all t sufficiently large, that is, (2.8) holds. But we may then apply the concavity argument exactly as in the proof of Theorem 2.1 to deduce that $T^* < \infty$. This contradiction proves that actually $\liminf_{t\to\infty} |u_x(t)|_{L^2} < \infty$.

On the other hand, since $u(t, s(t)) = 0$ and $s(t) \le s_\infty < \infty$, it follows that $|u(t)|_{L^\infty}^2 \le s_\infty |u(t)|_{H^1}^2 < s_\infty(1 + s_\infty^2)|u_x(t)|_{L^2}^2$, hence the conclusion. □

Lemma 3.5. *Assume that $T^* = \infty$ and $s_\infty < \infty$. Then, for all $A > 0$, there exists $\tau = \tau(A) > 0$ such that for all $t \ge 0$,*

$$|u(t)|_{L^{p+1}} \le A \Rightarrow (|u(s)|_{L^{p+1}} \le 2A, \ \textit{for all } s \in [t, t+\tau]).$$

The proof of Lemma 3.5 is essentially similar to that of [Fi, Lemma 1.6] and is hence omitted.

It is now convenient to introduce the following change of variables (which is classical in free-boundary problems): $y = x/s(t)$, $V(t, y) = u(t, ys(t))$, $0 \le y \le 1$. We then have

$$\begin{aligned} &V_y(t, y) = s(t)u_x(t, ys(t)), \qquad V_{yy}(t, y) = s^2(t)u_{xx}(t, ys(t)), \\ &V_t(t, y) = [u_t + ys'(t)u_x](t, ys(t)), \end{aligned} \tag{3.5}$$

and V satisfies the equation:

$$\begin{aligned} &V_t - s^{-2}V_{yy} = V^p + ys's^{-1}V_y, \qquad t > 0, \quad 0 < y < 1, \\ &V_y(t, 0) = 0, \quad V(t, 1) = 0, \qquad t > 0. \end{aligned}$$

Proof of Theorem 3.1 (i). We suppose that $T^* = \infty$ and $s_\infty < \infty$. Assume for contradiction that $\limsup_{t\to\infty} |u(t)|_\infty > 0$. Then, using Lemma 3.4, we deduce that there exist $A > 0$ and a sequence $\tau_n \to \infty$, such that

$$A \le |V(\tau_n)|_{L^\infty} = |u(\tau_n)|_{L^\infty} \le 2A.$$

In particular, we have $|u(\tau_n)|_{L^{p+1}} \le s_\infty^{1/(p+1)}|u(\tau_n)|_{L^\infty} \le K \equiv s_\infty^{1/(p+1)}2A$. Therefore, by Lemma 3.5, there exists $\tau > 0$ such that, for all $s \in [\tau_n, \tau_n + \tau]$, $|u(s)|_{L^{p+1}} \le 2K$. Since $\widetilde{E}(t)$ is nonincreasing, it follows that

$$\begin{aligned} |u(s)|_{L^\infty} \le s_\infty^{1/2}|u(s)|_{H^1} &\le s_\infty^{1/2}(1 + s_\infty)|u_x(s)|_{L^2} \\ &\le s_\infty^{1/2}(1 + s_\infty)(2\widetilde{E}(0) + (2/(p+1))|u(s)|_{L^{p+1}}^{p+1})^{1/2}, \end{aligned}$$

so that

$$|u(s)|_{L^\infty}, \ |u(s)|_{H^1}, \ |V(s)|_{H^1} \le \text{Const}. \tag{3.6}$$

for all $s \in [\tau_n, \tau_n + \tau]$. In particular, since the embedding $H^1(0,1) \subset C([0,1])$ is compact, it follows that some subsequence of $V(\tau_n)$ (which we still denote $V(\tau_n)$) converges in $C([0,1])$ to some function $Z \not\equiv 0$.

On the other hand, since $T^* = \infty$, (2.3) and Theorem 2.1 imply that

$$\int_0^\infty \int_0^{s(t)} u_t^2 + \int_0^\infty \frac{s'^3}{2} < \infty. \tag{3.7}$$

We deduce the existence of $t_n \in [\tau_n, \tau_n + \tau]$ such that

$$\lim_{n\to\infty} \int_0^{s(t_n)} u_t^2(t_n) = \lim_{n\to\infty} s'(t_n) = 0. \tag{3.8}$$

This combined with (3.6) implies that

$$\begin{aligned}\int_0^1 V_{yy}^2(t_n)dy &= s^3(t_n) \int_0^{s(t_n)} u_{xx}^2(t_n)\, dx \\ &\le 2s_\infty^3 \left[\int_0^{s(t_n)} u_t^2(t_n) + \int_0^{s(t_n)} u^{2p}(t_n) \right] \le C^{\mathrm{te}},\end{aligned}$$

so that $V(t_n)$ is bounded in $H^2(0,1)$. It follows that $V(t_n)$ converges in $C^1([0,1])$ to some function W (up to a subsequence). Also, by combining (3.5), (3.6) and (3.8), we get

$$\lim_{n\to\infty} |V_t(t_n)|_{L^2} = 0.$$

Therefore, W satisfies the equation

$$-W_{yy} = s_\infty^2 W^p \quad \text{in } \mathcal{D}'(0,1). \tag{3.9}$$

But since $W \in C^1([0,1])$, then (3.9) is actually satisfied in $[0,1]$ in the classical sense. Next,

$$V_y(t_n, 1) = s(t_n) u_x(t_n, s(t_n)) = -s(t_n)s'(t_n) \to 0, \quad \text{as } n \to \infty,$$

by (3.8). Therefore $W_y(1) = W(1) = 0$, so that $W \equiv 0$ by local uniqueness.

But on the other hand, we have

$$\begin{aligned}\int_0^t |V(t_n, y) - V(\tau_n, y)|^2\, dy &\le \int_0^1 \int_\tau^{\tau_n+1} V_t^2(\sigma, y)\, dy\, d\sigma \\ &\le \int_{\tau_n}^{\tau_n+1} \int_0^1 (u_t(\sigma, ys(\sigma)) + ys'(\sigma) u_x(\sigma, ys(\sigma)))^2\, dy\, d\sigma \\ &\le 2s_0^{-1} \int_{t_n}^{t_n+1} \int_0^{s(\sigma)} u_t^2\, dy\, d\sigma + 2s_0^{-1} \int_{t_n}^{t_n+1} s'^2(\sigma) \left(\int_0^{s(\sigma)} u_x^2\, dy \right) d\sigma \\ &\le C \int_{t_n}^{t+n+1} \int_0^{s(\sigma)} u_t^2\, dy\, d\sigma + C \left(\int_{t_n}^{t_n+1} s'^3\, d\sigma \right)^{\frac{2}{3}} \underset{n\to\infty}{\longrightarrow} 0,\end{aligned}$$

where we have used (3.6), (3.7) and Hölder's inequality. It follows that $V(\tau_n) \to W$ in $L^2(0,1)$, hence $W = Z \not\equiv 0$. This contradiction proves that $\lim_{t\to\infty} |u(t)|_\infty = 0$. Since $s_\infty < \infty$, the estimate (3.1) is then an easy consequence of Theorem 3.2.

We now turn to the case $s_\infty = \infty$ of Theorem 3.1. In this case, the previous compactness argument does not work any longer since the size of the domain increases without bound. To overcome this difficulty, we shall use a variant of the

rescaling argument introduced by Gidas and Spruck [GS] to obtain apriori bounds for elliptic problems. This argument was adapted by Giga [G] to prove boundedness of global solutions of (1.1) in bounded domains (also in higher dimensions), and then modified by the second author [S1] to obtain boundedness and decay to 0 of global solutions for the Cauchy problem.

Proof of Theorem 3.1 (ii). We suppose that $T^* = \infty$ and $s_\infty = \infty$. Assume for contradiction that $\limsup_{t\to\infty} |u(t)|_\infty > 0$. Then there exists a real $\varepsilon > 0$ and a sequence $t_n \to \infty$ such that $|u(t_n)|_\infty = \sup_{t\in[t_0,t_n]} |u(t)|_\infty = \sigma_n \geq \varepsilon$. Let $x_n \in [0, s(t))$ be such that $u(t_n, x_n) = \sigma_n$, and set

$$\lambda_n = \sigma_n^{-(p-1)/2} \leq M = \varepsilon^{-(p-1)/2}.$$

If $\lim_{n\to\infty} x_n = \infty$, we define the rescaled function

$$v_n(s, y) = \lambda_n^{2/(p-1)} u(t_n + \lambda_n^2 s, x_n - \lambda_n y),$$

for

$$-\lambda_n^{-2} t_n \leq t \leq 0, \qquad \lambda_n^{-1}(x_n - s(t_n)) \leq y \leq L_n \equiv \lambda_n^{-1} x_n.$$

Otherwise, we may assume (up to extracting a subsequence) that $\lim_{n\to\infty} x_n = x_0 \in [0, \infty)$, and we set

$$v_n(s, y) = \lambda_n^{2/(p-1)} u(t_n + \lambda_n^2 s, x_n + \lambda_n y),$$

for

$$-\lambda_n^{-2} t_n \leq t \leq 0, \qquad -\lambda_n^{-1} x_n \leq y \leq L_n \equiv \lambda_n^{-1}(s(t_n) - x_n).$$

In each case, one has $v_n(0, 0) = 1$, $\lim_{n\to\infty} L_n = \infty$, and $0 \leq v_n(s, y) \leq 1$ on $D_n = [-t_n/M^2, 0] \times [0, L_n]$. Moreover, the equation being invariant under this rescaling, the function v_n satisfies

$$\partial_s v_n - \partial_y^2 v_n = v_n^p, \quad \text{in } D_n.$$

For each compact $Q_m = [-m, 0] \times [0, m]$ (m an integer ≥ 1), we have $Q_m \subset D_n$ for all sufficiently large n, say $n \geq N(m)$. By classical parabolic regularity theory (see, e.g., [LSU]), one then has the estimate

$$|v_n|_{C^{\delta/2,\delta}(Q_m)} \leq C(m)|v_n|_{W_q^{1,2}(Q_m)} \leq C'(m), \quad \text{for all } n \geq N(m+1)$$

for some large $q > 1$ and small $\delta \in (0, 1)$. By diagonal procedure, it follows that (some subsequence of) v_n converges uniformly on any compact subset of $Q = (-\infty, 0] \times [0, \infty)$ to a function $v(s, y) \geq 0$, continuous on Q, which is a solution of $v_s - v_{yy} = v^p$ in Q in the sense of distributions, and satisfies $v(0, 0) = 1$.

On the other hand, for all $m > 0$, one has

$$\int_{-m}^{0} \int_0^m (\partial_s v_n(s, y))^2 dy\, ds \leq \lambda_n^{2(p+1)/(p-1)-1} \int_{t_n - m\lambda_n^2}^{t_n} \int_0^{s(t_n)} u_t^2(t, x)\, dx\, dt.$$

Using (3.7), $2(p + 1)/(p - 1) > 1$, and the boundedness of λ_n, it follows that $\partial_s v_n$ tends to 0 in $L^2_{\text{loc}}(Q)$ as $n \to \infty$, hence in $\mathcal{D}'(Q)$. Therefore, $v_s = 0$ and $v \equiv v(y)$ satisfies $0 \leq v \leq 1$, $v(0) = 1$, and

$$-v_{yy} = v^p \tag{3.10}$$

in $\mathcal{D}'(0, \infty)$. It follows that $v \in C^2([0, \infty))$. Since v must then be concave and bounded, it follows easily from (3.10) that $v \equiv 0$, which contradicts $v(0) = 1$. This contradiction proves that $\lim_{t\to\infty} |u(t)|_\infty = 0$.

The estimate (3.2) follows from Hölder's inequality and (3.7), by writing $s(t) - s_0 = \int_0^t s' \le (\int_0^t s'^3)^{1/3} t^{2/3}$. The estimate (3.3) is an immediate consequence of Theorem 3.2, and (3.4) then follows by combining (3.2) and (3.3). □

Proof of Proposition 3.3. Set $s(t) = s_0(1+t/T)^{1/2}$, $a = T^{-1/2}s_0$, $y = (t+T)^{-1/2}x = ax/s(t)$, and $u(t,x) = (t+T)^{-1/(p-1)}V(y)$ for $t \ge 0$ and $0 \le x \le s(t)$. An easy computation shows that the problem is equivalent to finding $a > 0$ and a function $V \equiv V(y) \in C^2([0,a])$, such that

$$V^p + \tfrac{1}{p-1}V + \tfrac{y}{2}V_y + V_{yy} = 0, \qquad 0 < y < a, \tag{3.11}$$

$$V > 0 \quad \text{in } [0,a), \qquad V(a) = 0, \qquad V_y(0) = 0, \tag{3.12}$$

$$-V_y(a) = \lambda(\tfrac{a}{2})^k. \tag{3.13}$$

By [HW, Propositions 3.8 and 3.9], there exists $a > 0$ and a function V satisfying (3.11), (3.12). Since $V_y(a) < 0$ by local uniqueness, the condition (3.13) is then fulfilled for a suitable choice of $\lambda > 0$. □

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