PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 3, Pages 677–684
S 0002-9939(00)05706-3
Article electronically published on September 20, 2000

WEAKLY ABELIAN LATTICE-ORDERED GROUPS

A. M. W. GLASS

(Communicated by Stephen D. Smith)

Respectfully dedicated (with gratitude) to W. Charles Holland on his 65th Birthday

Abstract. Every nilpotent lattice-ordered group is weakly Abelian; i.e., satisfies the identity $x^{-1}(y \vee 1)x \vee (y \vee 1)^2 = (y \vee 1)^2$. In 1984, V. M. Kopytov asked if every weakly Abelian lattice-ordered group belongs to the variety generated by all nilpotent lattice-ordered groups [The Black Swamp Problem Book, Question 40]. In the past 15 years, all attempts have centred on finding counterexamples. We show that two constructions of weakly Abelian lattice-ordered groups fail to be counterexamples. They include all preiously considered potential counterexamples and also *many* weakly Abelian ordered free groups on finitely many generators. If *every* weakly Abelian ordered free group on finitely many generators belongs to the variety generated by all nilpotent lattice-ordered groups, then every weakly Abelian lattice-ordered group belongs to this variety. This paper therefore redresses the balance and suggests that Kopytov's problem is even more intriguing.

1. Basic definitions and facts

A group equipped with a partial order $\leq$ is called a *p.o.group* if $axb \leq ayb$ whenever $x \leq y$. If, further, the partial order is a lattice (any pair of elements x & y have a least upper bound (denoted by $x \vee y$) and a greatest lower bound (denoted by $x \wedge y$)), then the p.o. group is called a *lattice-ordered group*. If the partial order is total (i.e., for any pair of elements x & y, either $x \leq y$ or $y \leq x$), then the p.o. group is said to be an *ordered group*. We will write $G \overrightarrow{\otimes} H$ for the direct product of the ordered groups G and H ordered by: $(g_1, h_1) > (g_2, h_2)$ if $g_1 > g_2$ or both $g_1 = 1 = g_2$ & $h_1 > h_2$ where $g_j \in G$ & $h_j \in H$ $(j = 1, 2)$; it is an ordered group. If G is an ordered group and H is a lattice-ordered group, then $G \overrightarrow{\otimes} H$ is a lattice-ordered group under the same ordering ($(g_1, h_1) > (g_2, h_2)$ if $g_1 > g_2$ or both $g_1 = 1 = g_2$ & $h_1 > h_2$).

Lattice-ordered groups are torsion-free [1, Corollary 2.1.3] but there are torsion-free groups which cannot be made into lattice-ordered groups (see, e.g., the proof in [1, Theorem 2.A]). Indeed, the class of groups that can be made into lattice-ordered groups cannot be defined in the first order language of group theory [6] (or see [1, Theorem 2.A]).

Received by the editors May 21, 1999.
2000 *Mathematics Subject Classification.* Primary 06F15, 20E05, 20F19; Secondary 20F18, 08B15.

Let G be a lattice-ordered group and $G^+ = \{g \in G : g \geq 1\}$. As is well known (see, e.g., [1, Chapter 2]) every element of G can be written in the form ab^{-1} for some $a, b \in G^+$; moreover, $\{|g| : g \in G\} = G^+ = \{g \vee 1 : g \in G\}$ where we write $|g|$ for $g \vee g^{-1}$. Therefore G^+ completely determines the order on G.

As is also standard, we will use the shorthand $\mathbb{Z}$ for the group of integers under addition with the standard ordering, $\mathbb{N}$ for the set of non-negative integers, and $f \ll g$ for: $f^n \leq g$ for all $n \in \mathbb{Z}$ (and write that f is *very much less* than g). In any ordered group, $f_1 f_2 \ll g$ whenever $f_1 \ll g$ & $f_2 \ll g$ (see, e.g., [1, Chapter 4]).

If G is a partially ordered group and $C \subseteq G$, we will call C *convex* if $g \in C$ whenever there are $c_1, c_2 \in C$ such that $c_1 \leq g \leq c_2$. If $X \subseteq G$, then the intersection of all convex subsets of G that contain X is convex and is called the *convex closure* of X in G; it is $\{g \in G : (\exists x_1, x_2 \in X)(x_1 \leq g \leq x_2)\}$. If X is a normal subgroup, so is its convex closure (as is easily verified).

A subgroup of a lattice-ordered group that is also a sublattice will be called an ℓ-subgroup. A group homomorphism between lattice-ordered groups that preserves the lattice operations is called an ℓ-homomorphism, an injective such is called an ℓ-embedding, and a bijective one whose inverse is also an ℓ-homomorphism is called an ℓ-isomorphism. The kernels of ℓ-homomorphisms are prcisely the normal convex ℓ-subgroups (see, e.g., [1, Section 3.2]); they are called *ℓ-ideals*. Note that a group embedding ϕ of an ordered group G in an ordered group H is an ℓ-embedding if $G^+\phi \subseteq H^+$.

2. Weakly Abelian lattice-ordered groups

A lattice-ordered group G is called *weakly Abelian* if $g^{-1}|f|g \leq |f|^2$ for all $f, g \in G$ [4] (or see [1, Section 6.4]). Every weakly Abelian lattice-ordered group is residually ordered; i.e., a subdirect product of (weakly Abelian) ordered groups [1].

Any locally nilpotent lattice-ordered group is weakly Abelian ([5], or see [1, Theorem 6.D]); so the variety of lattice-ordered groups generated by all nilpotent lattice-ordered groups is contained in the variety of all weakly Abelian lattice-ordered groups. In 1984, V. M. Kopytov asked if the converse were true [The Black Swamp Problem Book, Question 40]. Since weakly Abelian lattice-ordered groups are residually ordered, the question is equivalent to:

> *Does every weakly Abelian ordered group belong to the variety (of lattice-ordered groups) generated by all nilpotent lattice-ordered groups?*

All attempts have centred on finding counterexamples (see, e.g., [2]). In contrast, we show that all the examples considered to date are *approximately nilpotent*; i.e., belong to the variety (of lattice-ordered groups) generated by all nilpotent lattice-ordered groups.

Throughout we will use the following well-known result [4] (or see, e.g., [1, Lemma 6.4.1]):

Lemma 2.1. *A lattice-ordered group G is weakly Abelian if and only if $[f, g] \ll |f| \wedge |g|$ for all $f, g \in G$.*

As is standard (see, e.g., [3, Chapter 11]), we write $[f, g]$ for $f^{-1}g^{-1}fg$ and $[f, g, h]$ as a shorthand for $[[f, g], h]$, etc. for G a group and $f, g, h \in G$. If H and K are subgroups of G, we write $[H, K]$ for the subgroup generated by $\{[h, k] : h \in H,\ k \in K\}$ and define the lower central series $\gamma_m(G)$ of G inductively:

$$\gamma_0(G) = G,\ \gamma_{m+1}(G) = [\gamma_m(G), G]$$

(for all $m \in \mathbb{N}$). Each $\gamma_m(G)$ is an invariant subgroup of G (and hence a normal subgroup of G). Moreover, G is nilpotent class c if and only if $\gamma_{c+1}(G) = \{1\}$; i.e., if $[g_1, \dots, g_{c+1}] = 1$ for all $g_1, \dots, g_{c+1} \in G$. Thus $G/\gamma_{c+1}(G)$ is nilpotent of class c for all groups G.

If G is a group and $\bigcap_{m=0}^{\infty} \gamma_m(G) = \{1\}$, then the map $g \mapsto \hat{g} \in \prod_{m=0}^{\infty} G/\gamma_m(G)$ given by $\hat{g}_m = \gamma_m(G)g$ for all $m \in \mathbb{N}$ is an embedding of G into a direct product of nilpotent groups. Since $\bigcap_{m=0}^{\infty} \gamma_m(F) = \{1\}$ if F is any free group (see, e.g., [3, Theorem 11.2.4]), all free groups belong to the variety (of groups) generated by all nilpotent groups; hence this variety is the variety of all groups.

For each $m \in \mathbb{N}$, we will write $C_m(G)$ for the convex closure of $\gamma_m(G)$. Since $C_m(G) \supseteq \gamma_m(G)$ we get that $G/C_m(G)$ is nilpotent class m.

3. Extensions of centrally ordered groups

If G is an ordered group and $\bigcap_{m=0}^{\infty} C_m(G) = \{1\}$, then we will call G *centrally ordered*. As above, any centrally ordered group is approximately nilpotent (and so weakly Abelian). There is a standard way to order the free group F. For each $m \in \mathbb{N}$, $\gamma_m(F)/\gamma_{m+1}(F)$ is a free Abelian group. We can choose a basis of basic commutators in F of weight m whose cosets by $\gamma_{m+1}(F)$ are free generators of the Abelian group $\gamma_m(F)/\gamma_{m+1}(F)$. Let $c_{m,1}, c_{m,2}, \dots$ be these generators. Each element of F can be written uniquely in the form $w_1 \cdots w_r$ where each $w_i \neq 1$ has the form $c_{i,1}^{t_1} \cdots c_{i,s(i)}^{t_{s(i)}}$ with $t_{s(i)} \neq 0$, and if F is finitely generated, then so is each $\gamma_m(F)/\gamma_{m+1}(F)$; indeed, $s(i)$ is given by the Witt Formula and these $c_{i,j}$ can be chosen to be basic commutators (see, e.g., [3, Chapter 11]). If we impose the order generated by

$$c_{0,1} \gg c_{0,2} \gg \cdots \gg c_{1,1} \gg c_{1,2} \gg \cdots \gg c_{2,1} \gg c_{2,2} \gg \cdots,$$

then $C_m(F) = \gamma_m(F)$ for each $m \in \mathbb{Z}$. This makes F a centrally ordered group, whence it is approximately nilpotent.

The easiest place to look for a weakly Abelian ordered group that is not approximately nilpotent is to put a cyclic ordered group below a centrally ordered group (whence $\bigcap_{m=0}^{\infty} C_m(G) \neq \{1\}$). The first example considered was $F \overrightarrow{\otimes} \mathbb{Z}$ where F is the weakly Abelian centrally ordered free group above on 2 generators. This was far too naïve as we now show. Indeed, we now prove that central extensions of certain approximately nilpotent groups are still approximately nilpotent.

Theorem A. *Let G be a centrally ordered group and H be any approximately nilpotent lattice-ordered group. Then $G \overrightarrow{\otimes} H$ is an approximately nilpotent lattice-ordered group.*

Proof. Let $L = G \overrightarrow{\otimes} H$, $m \in \mathbb{N}$ and $L_m = G/C_m(G) \overrightarrow{\otimes} H$. Then there are nilpotent lattice-ordered groups $H(n)$ $(n \in \mathbb{N})$ with each $H(n)$ nilpotent of class n, an ℓ-subgroup D of $\prod\{H(n) : n \in \mathbb{N}\}$ and an ℓ-ideal K of D such that H is ℓ-isomorphic to D/K. Then each $G/C_m(G) \overrightarrow{\otimes} H(n)$ is nilpotent of class $\max\{m, n\}$ and we may regard $D^* = G/C_m(G) \overrightarrow{\otimes} D$ as an ℓ-subgroup of $\prod\{G/C_m(G) \overrightarrow{\otimes} H(n) : n \in \mathbb{N}\}$ by $(C_m(G)g, d)(n) = (C_m(G)g, d(n))$ $(n \in \mathbb{N})$. Thus D^* is approximately nilpotent. Moreover, $K^* = \{1\} \overrightarrow{\otimes} K$ is an ℓ-ideal of D^* and D^*/K^* is ℓ-isomorphic to L_m. Hence L_m is approximately nilpotent. If $C_r(G) = \{1\}$ for some $r \in \mathbb{N}$, then $L = L_r$ is approximately nilpotent. So assume that $C_m(G) \neq \{1\}$ for all $m \in \mathbb{N}$.

Consider the group homomorphism ϕ_m of L onto L_m that is given by $(g,h)\phi_m = (C_m(G)g, h)$. Then ϕ_m has kernel $C_m(G)$. (Caution: ϕ_m is not order preserving.) Hence the group homomorphism ϕ of L into $\prod\{L_m : m \in \mathbb{N}\}$ induced by $\{\phi_m : m \in \mathbb{N}\}$ is an embedding. Furthermore, if $g \in G\backslash\{1\}$, then there is $\ell(g) \in \mathbb{N}$ such that $g \notin C_\ell(G)$ for all $\ell \geq \ell(g)$ (since $\bigcap_{m=0}^\infty C_m(G) = \{1\}$). Hence the map $\phi\nu$ from L into $\prod\{L_m : m \in \mathbb{N}\}/\sum\{L_m : m \in \mathbb{N}\}$ is an embedding, where ν is the natural ℓ-homomorphism from $\prod\{L_m : m \in \mathbb{N}\}$ onto $\prod\{L_m : m \in \mathbb{N}\}/\sum\{L_m : m \in \mathbb{N}\}$. Since the obstructions to order-preserving of $(g,h), 1$ are at only an initial set of values of m (dependent on the particular $g \in G$ — viz: $\{m \in \mathbb{N} : m \leq \ell(g)\}$), the map $\phi\nu$ is an ℓ-homomorphism. Since each L_m is approximately nilpotent, so is $\prod\{L_m : m \in \mathbb{N}\}$. Therefore so is $\prod\{L_m : m \in \mathbb{N}\}/\sum\{L_m : m \in \mathbb{N}\}$. Thus $L\phi\nu$ is approximately nilpotent and, consequently, so is L (since $\phi\nu$ is an ℓ-embedding). $\square$

Corollary 3.1. *Let F be a centrally ordered free group and H be any approximately nilpotent lattice-ordered group. Then $F \overrightarrow{\otimes} H$ is approximately nilpotent.*

4. Fully tiered free groups

For any free group F on a finite set of generators and $m \in \mathbb{N}$, there are $a_{m,1}, \dots, a_{m,s(m)} \in \gamma_m(F)$ such that $a_{m,1}\gamma_{m+1}(F), \dots, a_{m,s(m)}\gamma_{m+1}(F)$ freely generate the free Abelian group $\gamma_m(F)/\gamma_{m+1}(F)$. If F is an ordered free group, we will always always choose these generators so that $a_{m,1} > a_{m,2} > \dots > a_{m,s(m)} > 1$.

As is standard, if G is an ordered group and $f, g \in G$, then we will write $f \sim g$ if there are $m, n \in \mathbb{Z}$ such that $f \leq g^m$ & $g \leq f^n$. If $f \sim g$, then f & g are said to be *Archimedean equivalent*. Observe that $\sim$ is an equivalence relation and that $g \sim g^{-1}$ for all $g \in G$. Moreover, $g \sim 1$ if and only if $g = 1$; and if $f, g \in G$ with $f, g > 1$, then exactly one of $f \ll g$, $f \sim g$, or $g \ll f$ holds. So, in the case of ordered free groups, for each $m, i \in \mathbb{N}$ (with $i < s(m)$ if the free group is finitely generated) $a_{m,i} \sim a_{m,i+1}$ or $a_{m,i} \gg a_{m,i+1}$.

Another considered potential counterexample (to the conjecture that every weakly Abelian ordered group is approximately nilpotent) is the free group F on generators a_1, a_2, a_3 as before but now put the free subgroup G generated by a_1 & a_2 above a_3. So if $c_{i,1}, \dots, c_{i,s(i)}$ denote the basic commutators of weight i involving only a_1, a_2 and $d_{i,1}, \dots, d_{i,s^*(i)}$ denote the basic commutators of weight i in which a_3 (or its inverse) occurs, then we can order F via:

$$a_1 \gg a_2 \gg c_{1,1} \gg \dots \gg c_{1,s(1)} \gg c_{2,1} \gg \dots \gg c_{2,s(2)}$$

$$\gg \dots \gg a_3 \gg d_{1,1} \gg \dots \gg d_{1,s^*(1)} \gg \dots .$$

Then G is centrally ordered, and $a_3 \in \bigcap_{m=0}^\infty C_m(F)$, but F is not a central extension by G. However, this weakly Abelian ordered group is also approximately nilpotent as we will show in the next theorem.

The key ingredient in the above example is that for each $m \in \mathbb{N}$ there is a basis $b_1, \dots, b_{r(m)}$ for $\gamma_m(F)/\gamma_{m+1}(F)$ with $b_1 \gg b_2 \gg \dots \gg b_{r(m)}$; i.e., for each $m \in \mathbb{N}$, the m^{th} level can be tiered. More precisely, an ordered free group F is said to be *tiered* if, for each $m \in \mathbb{N}$, there are $a_{m,1}, \dots, a_{m,s(m)} \in \gamma_m(F)$ such that $a_{m,1}\gamma_{m+1}(F), \dots, a_{m,s(m)}\gamma_{m+1}(F)$ freely generate $\gamma_m(F)/\gamma_{m+1}(F)$ and $a_{m,1} \gg a_{m,2} \gg \dots \gg a_{m,s(m)}$. Note that if F is a tiered ordered free group, then (for each $m, i \in \mathbb{N}$ with $i < s(m)$) $a_{m,i}^{k_i} \cdots a_{m,s(m)}^{k_{s(m)}} \sim a_{m,i}$ if $k_i \neq 0$; but there are no assumptions about how the generators from the m^{th} level compare with those of any other level. However, if the tiered order is weakly Abelian, then $a_{0,1} \gg a_{1,1} \gg a_{2,1} \gg \dots$, but no assumption is made between $a_{0,3}$ and $a_{2,1}$ if, e.g., $a_{2,1} = [a_1, a_2, a_1]$. So the ordered free group on 3 generators in the previous paragraph is tiered. Indeed, it is *fully tiered*; that is, for each $m, n, i, j \in N$ with $i \leq s(m),\ j \leq s(n),\ a_{m,i} \sim a_{n,j}$ if and only if $m = n$ & $i = j$.

Theorem B. *Every finitely generated free group equipped with a fully tiered weakly Abelian order is approximately nilpotent.*

The free weakly Abelian lattice-ordered group on a finite set of generators X is a subdirect product of the free group on X equipped with all possible weakly Abelian (total) orders. Moreover, the free weakly Abelian lattice-ordered group on any infinite set of generators is an ultraproduct of the free weakly Abelian lattice-ordered groups on finitely many generators [1, Theorem 1.A]. Hence to prove that every weakly Abelian lattice-ordered group is approximately nilpotent, it suffices to show that any finitely generated free group with a weakly Abelian (total) order is approximately nilpotent. Theorem B can be viewed as the first step in that direction.

The main idea in the proof of Theorem B is the use of an appropriate right transversal of each $\gamma_m(G)$ in G to obtain an ordering of $G/\gamma_m(G)$ that approximates the original ordering on G. For technical reasons we require the "fully tiered" hypothesis (but I have no idea how intrinsic it is for anything other than the particular method where it is critical). We will need some notation and lemmata to achieve the proof of the theorem.

5. The proof of Theorem B

Let G be a finitely generated free group equipped with a weakly Abelian order and for each $m \in \mathbb{N}$, let $a_{m,1} > \dots > a_{m,s(m)} > 1$ yield generators of $\gamma_m(G)/\gamma_{m+1}(G)$ as above. Clearly each $g \in G\backslash\{1\}$ can be uniquely written in the form $w_0 \cdots w_{r(g)}$, where each $w_j = a_{j,1}^{k_{j,1}} \cdots a_{j,s(j)}^{k_{j,s(j)}}$ for some $k_{j,1}, \dots, k_{j,s(j)} \in \mathbb{Z}$ and $w_{r(g)} \neq 1$. Let $w_n(g) = 1$ for all $n > r(g)$. We will write $t_m(g)$ for $w_1 \cdots w_m$. Note that $t_m(g) = g$ if $m \geq r(g)$ and $t_m(g) = 1$ if and only if $g \in \gamma_{m+1}(G)$; furthermore, $\gamma_{m+1}(G)g = \gamma_{m+1}(G)t_m(g)$ for each $g \in G$ and $m \in \mathbb{N}$.

Throughout the rest of the proof, we will use this notation and will also reserve the symbols v_j for $a_{j,1}^{\ell_{j,1}} \cdots a_{j,s(j)}^{\ell_{j,s(j)}}$ for some $\ell_{j,1}, \dots, \ell_{j,s(j)} \in \mathbb{Z}$ $(j \in \mathbb{N})$.

If G is a tiered ordered free group and $j \in \mathbb{N}$ and $i = \min\{k \colon k_{j,\ell} \neq 0\}$ we call $a_{j,i}^{k_{j,i}}$ the *dominant term* of $w_j \neq 1$. By default, we will call 1 the dominant term of 1!

Lemma 5.1. *If G is a tiered weakly Abelian ordered free group and $a_{j,i}^{k_{j,i}}$ is the dominant term of $w_j \neq 1$, then $w_j = a_{j,i}^{k_{j,i}} c$ where $c \ll a_{j,i}$. Hence $t_m(w_j) > 1$ if and only if $k_{j,i} > 0$ (for $m \geq j$).*

Proof. The form is obvious with $c = a_{j,i+1}^{k_{j,i+1}} \cdots a_{j,s(j)}^{k_{j,s(j)}}$. Since each of $a_{j,n} \ll a_{j,i}$ $(n = i+1, \dots, s(j))$, we have $c \ll a_{j,i}$. Thus, for $m \geq j$, we have $t_m(w_j) > 1$ if $k_{j,i} > 0$ and $t_m(w_j) < 1$ if $k_{j,i} < 0$. □

Throughout the rest of the paper we will use the following commutator identities:

$$xy = yx[x, y],\ [xy, z] = [x, z][x, z, y][y, z],\ [x, yz] = [x, z][x, y][x, y, z].$$

Corollary 5.2. *If G is a tiered weakly Abelian ordered free group, $m \geq j$, $a_{j,i_1}^{k_{j,i_1}}$ is the dominant term of $w_j \neq 1$ and $a_{j,i_2}^{\ell_{j,i_2}}$ is the dominant term of $v_j \neq 1$, then the dominant term of $t_m(w_j v_j)$ is*
$a_{j,i_1}^{k_{j,i_1}}$ if $i_1 < i_2$; is
$a_{j,i_1}^{k_{j,i_1}+\ell_{j,i_1}}$ if $i_1 = i_2$; and is
$a_{j,i_2}^{\ell_{j,i_2}}$ if $i_1 > i_2$.
Hence $t_m(w_j v_j) \geq 1$ whenever $t_m(w_j), t_m(v_j) \geq 1$.

Proof. By Lemma 5.1, we have $w_j v_j = a_{j,i_1}^{k_{j,i_1}} c a_{j,i_2}^{k_{j,i_2}} d$ where $c \ll a_{j,i_1}$ and $d \ll a_{j,i_2}$. Using the above commutator identities we obtain $w_j v_j = \prod_{n=1}^{n=s(j)} a_{j,n}^{k_{j,n}+\ell_{j,n}} \cdot c^*$, where c^* is a finite product of commutators of weight at least $j+1$. Moreover, since the ordering is weakly Abelian, each of these constituent commutators is certainly very much less than $a_{j,i}$ where $i = \min\{i_1, i_2\}$. Thus the dominant term in $w_j v_j$ is $a_{j,i}^{k_{j,i}+\ell_{j,i}}$; and this is also the dominant term$_m$ if $m \geq j$ (see below). □

If G is a fully tiered ordered free group and $g = w_0 \cdots w_{r(g)} \neq 1$, let $a_{j,i(j)}^{k_{j,i(j)}}$ be the dominant term of w_j $(j = 1, \ldots, r(g))$. Let $j_0 \in \mathbb{N}$ be such that $a_{j_0,i(j_0)} \gg a_{j,i(j)}$ for all $j \neq j_0$. Then $a_{j_0,i(j_0)}$ is said to be the *dominant term* of g. Given any $m \in \mathbb{N}$, the *dominant term$_m$* of g will be the dominant term of $w_0 \cdots w_m$, i.e., the dominant term of $t_m(g)$.

Lemma 5.3. *Let G be a fully tiered weakly Abelian ordered free group, $f = v_0 \cdots v_{r(f)} \neq 1$ and $g = w_0 \cdots w_{r(g)} \neq 1$ have dominant term$_m$ $a_{j_1,i_1}^{\ell_{j_1,i_1}}$ and $a_{j_2,i_2}^{k_{j_2,i_2}}$, respectively. Then the dominant term of $t_m(fg)$ is*
$a_{j_1,i_1}^{\ell_{j_1,i_1}}$ if $a_{j_1,i_1} \gg a_{j_2,i_2}$; is
$a_{j_1,i_1}^{\ell_{j_1,i_1}+k_{j_1,i_1}}$ if $(j_1,i_1) = (j_2,i_2)$; and is
$a_{j_2,i_2}^{k_{j_2,i_2}}$ if $a_{j_1,i_1} \ll a_{j_2,i_2}$.

Proof. We prove the lemma by induction on $r(g)$ (with $r(f)$ arbitrary).

First observe that if $j' < j$, then $w_j v_{j'} = v_{j'} w_j [w_j, v_{j'}]$ and that $c_{j,j'} = [w_j, v_{j'}] \in \gamma_{j+1}(G)$ with $c_{j,j'} \ll a_{j,i(j)}^{k_{j,i(j)}}, a_{j',i(j')}^{\ell_{j',i(j')}}$. Hence $w_j v_{j'} = v_{j'} w_j c_{j,j'}$.

Using the commutator identities, we obtain that $v_0 \cdots v_{r(f)} \cdot w_0$ has normal form $u_0 \cdots u_t$ where the dominant term d_j of u_j is the dominant term of $w_0 v_0$ (as given by Corollary 5.2) if $j = 0$, and is either the dominant term of w_j if $j \in \{1, \ldots, r(f)\}$ (and this is not very much less than $a_{0,i_2}^{k_{0,i_2}}$) or is very much less than $a_{0,i_2}^{k_{0,i_2}}$. Hence the dominant term$_m$ of $v_0 \cdots v_{r(f)} \cdot w_0$ is as required and the basic step of the induction is complete.

Assume that the dominant term of $t_m(v_0 \cdots v_{r(f)} \cdot w_0 \cdots w_r)$ is as prescribed and let $r(g) = r+1$. We may assume that $m \geq r+1$ (otherwise the proof is already complete). Then $fg = (v_0 \cdots v_{r(f)} \cdot w_0 \cdots w_r) w_{r+1}$. Thus $fg = (u_0 \cdots u_t) \cdot w_{r+1}$ where the dominant term of $u_0 \cdots u_t$ is as above. Using the commutator identities and the observation at the beginning of the proof we get that fg has normal form $(u_0 \cdots u_r) \cdot u_{r+1}^* \cdots u_{t^*}^*$ where $u_{r+1}^* = u_{r+1} w_{r+1} c_{r+1}$ written in normal form (with c_{r+1} a finite product of commutators of weight at least $r+2$ and all very much less than the dominant term of w_{r+1}) has dominant term as given by Corollary 5.2 and each of $u_{r+2}^*, \ldots, u_{t^*}^*$ has dominant term$_m$ very much less than the dominant term of $t_m(f)$ and of $t_m(g) = g$. This completes the induction step, and the lemma now follows.

Lemma 5.4. *Let G be a fully tiered weakly Abelian ordered free group and $g = w_0 \cdots w_{r(g)} \neq 1$ have dominant term$_m$ $a_{j,i}^{k_{j,i}}$. Then the dominant term of $t_m(g^{-1})$ is $a_{j,i}^{-k_{j,i}}$.*

Proof. Observe that if $w_j = \prod_{n=1}^{s(j)} a_{j,n}^{k_{j,n}}$, then $w_j^{-1} = \prod_{n=1}^{s(j)} a_{j,n}^{-k_{j,n}} \cdot c_j$ where c_j is a finite product of commutators each of weight at least $j+1$ with only those $a_{j,n}$ appearing for which $k_{j,n} \neq 0$; so $c_j \ll a_{j,i(j)}$. Thus the dominant term in w_j^{-1} is $a_{j,i(j)}^{-k_{j,i(j)}}$. Thus $g^{-1} = w_{r(g)}^* \cdots w_0^*$ where $w_j^* = \prod_{n=1}^{s(j)} a_{j,n}^{-k_{j,n}} \cdot c_j$ $(j = 0, \dots, r(g))$ and c_j is as above. Caution: this is not the normal form for g^{-1} and w_j^* need not be a product of powers of $a_{j,1}, \dots, a_{j,s(j)}$ in that order. However, we can apply the commutator identities to this form to get that g^{-1} has normal form $u_1 \cdots u_t$ where the dominant term d_j in each u_j satisfies:

either $d_j \ll a_{j_1,i(j_1)}$ for some $j_1 < j$ or $d_j = a_{j,i(j)}^{-k(j,i(j))}$.

The second possibility can only arise if $j = 0, \dots, r(g)$; and it actually does occur if $a_{j,i(j)} \gg a_{j_1,i(j_1)}$ for all $j_1 < j$. Thus the dominant term of $t_m(g^{-1})$ is just the inverse of the dominant term of $t_m(g)$. □

By Lemmata 5.3 and 5.4 we get that $t_m(fg) \geq 1$ & $t_m(h^{-1}fh) \geq 1$ whenever $t_m(f), t_m(g) \geq 1$; that is:

Corollary 5.5. *Let G be a fully tiered weakly Abelian ordered free group and $m \in \mathbb{N}$. Define $\gamma_{m+1}(G)g > 1$ if and only if $t_m(g) > 1$ $(g \in G)$. Then $G/\gamma_{m+1}(G)$ is an ordered group.*

We can now prove Theorem B.

Proof. Consider the natural homomorphism $\phi_m : G \to G/\gamma_{m+1}(G)$ where the image group is ordered as in Corollary 5.5. Note that $\phi_m(g) > 1$ if $g > 1$ in G and $m \geq r(g)$. Let ϕ be the homomorphism of G into the lattice-ordered group $\prod_{m=0}^{\infty} G/\gamma_{m+1}(G)$ induced by $\{\phi_m : m \in \mathbb{N}\}$; it is an embedding since $\bigcap_{m=0}^{\infty} \gamma_{m+1}(G) = \{1\}$. Then $\phi\nu$ is an embedding of G into

$$\left(\prod_{m=0}^{\infty} G/\gamma_{m+1}(G)\right) \Big/ \left(\sum_{m=0}^{\infty} G/\gamma_{m+1}(G)\right)$$

(since $\phi_m(g) \neq 1$ if $m \geq r(g)$). Moreover, as in the proof of Theorem A, $\phi\nu$ preserves order. Since $\prod_{m=0}^{\infty} G/\gamma_{m+1}(G)$ is approximately nilpotent, so is

$$\left(\prod_{m=0}^{\infty} G/\gamma_{m+1}(G)\right) \Big/ \left(\sum_{m=0}^{\infty} G/\gamma_{m+1}(G)\right).$$

Hence G is approximately nilpotent. □

6. Concluding remarks

(1) The example in [2] is obtained by using the lower central series of a finitely generated group that is free metabelian. If we put a fully tiered order thereon (as is permissible), then the resulting group is approximately nilpotent (even though it does not belong to the variety of lattice-ordered groups that is generated by those

nilpotent lattice-ordered groups that belong to a variety $\mathcal{W}_p$; see [2] or [1, Chapter 11]).

(2) It should be noted that if the free group G is fully tiered (or even tiered), then, since $a_{j,1} \gg \ldots \gg a_{j,s(j)}$, we have that $a_{j,1}a_{j,2}, \ldots, a_{j,1}a_{j,s(j)}, a_{j,1}$ are all Archimedeanly equivalent and their $\gamma_{j+1}(G)$ cosets also freely generate $\gamma_j(G)/\gamma_{j+1}(G)$. So the existence of a fully tiered basis can occur when the original generators are Archimedeanly equivalent. This may suggest that Theorem B includes most (?) weakly Abelian total orders on finitely generated free groups.

(3) In contrast, let G be a weakly Abelian ordered free group on a finite set of generators such that $a_{j,1}, \ldots, a_{j,s(j)}$ are Archimedeanly equivalent ($j \in \mathbb{N}$). Let H be a non-trivial ultrapower of G. Then H is a weakly Abelian ordered group. Choose s strictly increasing functions $f_1, \ldots, f_s$ from $\mathbb{N}$ into $\mathbb{N}$ so that $\lim_{n\to\infty} f_i(n)/f_{i+1}(n) = \infty$ for $i = 1, \ldots, s-1$. We embed G in H via the map which sends $a_{0,i}$ to the equivalence class of $b_{0,i}$, the function which is $a_{0,i}^{f_i(n)}$ on the n^{th}-coordinate. Then $b_{0,i} \gg \ldots \gg b_{0,s(0)}$. Repeat the process inductively and one may obtain a fully tiered weakly Abelian order on the resulting free group (which necessarily belongs to any variety of lattice-ordered groups containing the original weakly Abelian ordered free group on the same set of generators). This might suggest that fully tiered weakly Abelian finitely generated ordered free groups would be most likely to be approximately nilpotent (among all weakly Abelian finitely generated ordered free groups).

References

[1] A. M. W. Glass, *Partially Ordered Groups*, Series in Algebra vol. 7, World Scientific Pub. Co., Singapore, 1999.

[2] S. A. Gurchenkov, *On varieties of weakly Abelian ℓ-groups*, Mat. Slovaca **42**(1992), 437-441. MR **94a:**20067

[3] M. Hall, *The Theory of Groups*, Macmillan Co., New York, 1959. MR **21:**1996

[4] J. Martinez, *Varieties of lattice-ordered groups*, Math. Z. **137**(1974), 265-284.

[5] N. R. Reilly, *Nilpotent, weakly Abelian and Hamiltonian lattice-ordered groups*, Czech. Math. J. **33**(1983), 348-353. MR **85m:**06035

[6] A. A. Vinogradov, *Non-axiomatizability of lattice-ordered groups*, Siberian Math. J. **13**(1971), 331-332.

Department of Pure Mathematics & Mathematical Statistics, 16 Mill Lane, Cambridge CB2 1SB, England

E-mail address: amwg@dpmms.cam.ac.uk