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ON C*-EXTREME POINTS

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ABSTRACT. Each weak* compact C*-convex set in a hyperfinite factor (in particular in B($\mathcal{H}$)) is the weak* closure of the C*-convex hull of its C*-extreme points.

1. Introduction and main results

A subset K of a unital C*-algebra R is called *R-convex* (or *C*-convex* if R is clear from the context) if $\sum_{j=1}^n a_j^* x_j a_j \in K$ whenever $x_j \in K$ and $a_j \in R$ for all j and $\sum_{j=1}^n a_j^* a_j = 1$. As defined in [10], a point $x \in K$ is *C*-extreme* for K if the condition

$$x = \sum_{j=1}^n a_j^* x_j a_j, \quad \sum_{j=1}^n a_j^* a_j = 1,\ x_j \in K,\ a_j \text{ invertible in } R,\ n \in \mathbb{N}, \tag{1.1}$$

implies that all x_j are unitarily equivalent to x in R. C*-extreme points for subsets K of $R = \mathbb{M}_n := \mathbb{M}_n(\mathbb{C})$ are extreme in the usual sense by [10], but the converse is not true (see [8] or [6]). It was conjectured already in [10] that a variant of the Krein-Milman theorem should hold for compact C*-convex sets, and indeed, much later for subsets of $\mathbb{M}_n$ such a theorem was established by Morenz [12] using some previous work of Farenick and Morenz (see [4], [5] and [6]). Another Krein-Milman type theorem for the so-called matricially convex sets in locally convex spaces has been proved recently by Webster and Winkler [15]. Methods in [12] and [15] (although different) both used in an essential way the finite dimensionality of $\mathbb{M}_n$. In this note we shall prove a variant of the Krein-Milman theorem for C*-convex subsets of B($\mathcal{H}$) where $\mathcal{H}$ is a separable Hilbert space. But since the same proof is actually valid for subsets in hyperfinite factors, we shall formulate the result for such factors. For this we need a special kind of C*-extreme points.

A point $x \in K$, where K is a C*-convex subset in a C*-algebra R, is *R-extreme for K* if the condition

$$x = \sum_{j=1}^n a_j x_j a_j, \quad \sum_{j=1}^n a_j^2 = 1,\ x_j \in K,\ a_j \text{ invertible and positive in } R, \tag{1.2}$$

implies that $x_j = x$ and $a_j x = x a_j$ for all j. Using the polar decomposition of the coefficients a_j in (1.1) it is easy to show that each R-extreme point is C*-extreme.

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It suffices to check the condition (1.2) for an R-extreme point in the case $n = 2$. (Indeed, suppose that x satisfies the condition for $n = 2$. For a general n put $y = \bigoplus_{j=2}^{n} x_j$, $a = (a_2, \ldots, a_n)^T$ and let $a = u|a|$ be the polar decomposition of the column a. Then, writing (1.2) as $x = a_1 x_1 a_1 + |a|(u^* y u)|a|$, it follows that $x_1 = x$ and a_1 commutes with x. The same argument can be applied to $j = 2, \ldots, n$.) If R is commutative, then it is not difficult to show that the R-extreme points (and the C*-extreme points) are just the usual extreme points. Also, for a general R, all R-extreme points are extreme in the usual sense.

Now we can state the main result of this note.

Theorem 1.1. *Each weak* compact C^*-convex set K in a hyperfinite factor R is the weak* closure of the C^*-convex hull of the set $\mathrm{ext}_R(K)$ of all R-extreme points of K.*

To prove Theorem 1.1 we shall first consider for each $x \in R$ the weak* closure $\overline{\mathrm{co}}_R(x)$ of the C*-convex hull $\mathrm{co}_R(x)$ of x, where

$$\mathrm{co}_R(x) = \{\sum_{j=1}^{n} a_j^* x a_j : \ a_j \in R,\ \sum_{j=1}^{n} a_j^* a_j = 1,\ n \in \mathbb{N}\}.$$

Recall also that for each $n \in \mathbb{N}$ the matricial range $W_n(x)$ of an element $x \in R$ is the set of all $\phi(x)$, where ϕ is any unital completely positive map from R to $\mathbb{M}_n$ (see [1] or [14]).

The following two results will be used in the proof of Theorem 1.1.

Theorem 1.2. *Let R be any factor and $A \subseteq R$ a subfactor (containing the unit of R) isomorphic to $\mathbb{M}_n$ for some $n \in \mathbb{N}$. Then*

$$\overline{\mathrm{co}}_R(x) \cap A = W_n(x)$$

for each $x \in R$, where $W_n(x)$ is regarded as a subset of A by identifying A with $\mathbb{M}_n$ (using any C^*-isomorphism). Moreover, $\phi(K) \subseteq K$ for each unital completely positive map $\phi : R \to A$ and each weak compact C^*-convex subset K of R.*

Lemma 1.3. *Let R be a unital C^*-algebra and A a C^*-subalgebra containing the unit of R such that for each non-zero $x \in R$ there exists a conditional expectation $E : R \to A$ satisfying $E(x) \neq 0$. If K is a C^*-convex subset of R such that $\phi(K) \subseteq K$ for each unital completely positive map $\phi : R \to A$, then $\mathrm{ext}_A(K \cap A) \subseteq \mathrm{ext}_R(K)$.*

Theorem 1.2 and Lemma 1.3 will be proved in Section 2. In the case $R = \mathbb{M}_n$ Theorem 1.1 will be proved in Section 3 by using some results of [5] and [12] (or [15]). Assuming this, we can now deduce Theorem 1.1 for a general hyperfinite factor R.

Proof of Theorem 1.1. Since R is hyperfinite there exists a sequence of finite dimensional C*-subalgebras $B_0 \subseteq B_1 \subseteq B_2 \ldots$ of R and (normal) conditional expectations $E_n : R \to B_n$ such that the sequence $(E_n x)$ converges to x in the strong operator topology for each $x \in R$. Moreover, except in the case R is of type III_0, B_n can be chosen to be factors since by the classification of injective factors ([3] and [7]) all such factors are infinite tensor products of finite dimensional factors (see also [9, 13.1.15]). In case R is of type III_0 the algebras B_n cannot always be chosen to be factors, but their existence together with the expectations E_n follows by [2]. In any case (since all non-zero projections in a countably decomposable type III

factor are equivalent) there exists for each n a subfactor $A_n \cong \mathbb{M}_{m_n}$ ($m_n \in \mathbb{N}$) in R such that $A_n \supseteq B_n$ for all n.

Let K be as in the statement of the theorem and $x \in K$. Regarding E_n as a unital completely positive map from R into A_n, we have $E_n(x) \in K \cap A_n$ by Theorem 1.2. Hence, assuming the finite dimensional version of Theorem 1.1 (proved in Section 3) it follows that

$$E_n(x) \in \overline{\text{co}}_{A_n}(\text{ext}_{A_n}(K \cap A_n)). \tag{1.3}$$

Since $R \cong A_n \otimes R_n$ for some subfactor $R_n \subseteq R$, the (normal) conditional expectations from R to A_n (namely the slice maps) separate points of R, hence by Lemma 1.3 $\text{ext}_{A_n}(K \cap A_n) \subseteq \text{ext}_R(K)$. Now (1.3) implies that $E_n(x) \in \overline{\text{co}}_R(\text{ext}_R(K))$, hence $x \in \overline{\text{co}}_R(\text{ext}_R(K))$ since $E_n(x)$ converges to x. □

We remark that the existence of C*-extreme points in a weak* compact C*-convex subset K of a general von Neumann algebra is proved in [11], but the extreme points obtained in [11] are in general not sufficient to generate K. So for general von Neumann algebras the problem if each weak* compact C*-convex subset is generated by its C*-extreme points remains open.

2. Proofs of Theorem 1.2 and Lemma 1.3

To prove Theorem 1.2 we need the following result from [11].

Lemma 2.1. *Let A be a unital C^*-algebra, $a_1, \dots, a_m$ elements of A and ρ a state on A in the weak* closure of the pure states. Then for each $\varepsilon > 0$ there exists an element $h \in A$ such that $\|h\| = 1$ and $\|h^*(a_i - \rho(a_i))h\| < \varepsilon$ for $i = 1, \dots, m$.*

Proof of Theorem 1.2. Let $y \in \overline{\text{co}}_R(x) \cap A$ and choose a net $(y_\nu) \subseteq \text{co}_R(x)$ converging to y in the strong operator topology. Since the map $w \mapsto \sum_{j=1}^n a_j^* w a_j$ on R is completely positive and unital for any $a_j, \dots, a_n$ in R satisfying $\sum_{j=1}^n a_j^* a_j = 1$, it follows that $W_n(y_\nu) \subseteq W_n(x)$ for all ν and consequently $W_n(y) \subseteq W_n(x)$ (since unital completely positive maps into $\mathbb{M}_n$ can be approximated by normal such maps in the point-norm topology). But $y \in W_n(y)$ since $y \in A \cong \mathbb{M}_n$, hence $y \in W_n(x)$. This proves the inclusion $\overline{\text{co}}_R(x) \cap A \subseteq W_n(x)$.

To prove the reverse inclusion, let $y \in W_n(x)$ and let $\phi : R \to A = \mathbb{M}_n$ be a unital completely positive map such that $\phi(x) = y$. By the Stinespring representation theorem (see [13]) $\phi(w) = V^*\sigma(w)V$ ($w \in R$), where σ is a representation of R on a Hilbert space $\mathcal{K}$ and $V : \mathbb{C}^n \to \mathcal{K}$ is an isometry. Since $R \cong \mathbb{M}_n(B)$, where $B = A' \cap R$, there is a representation π of B on a Hilbert space $\mathcal{H}$ such that $\mathcal{K} = \mathcal{H}^n$ and σ is (unitarily equivalent to) $\pi_n : \mathbb{M}_n(B) \to \mathbb{M}_n(\mathrm{B}(\mathcal{H}))$, where $\pi_n([b_{ij}]) = [\pi(b_{ij})]$ for each $[b_{ij}] \in \mathbb{M}_n(B)$. Let ε_j ($j = 1, \dots, n$) be the standard basic vectors in $\mathbb{C}^n$ and put $\xi_j = V\varepsilon_j \in \mathcal{H}^n$. Let $y_{ij} \in \mathbb{C}$ be the entries of y. Then

$$y_{ij} = \langle y\varepsilon_j, \varepsilon_i\rangle = \langle \phi(x)\varepsilon_j, \varepsilon_i\rangle = \langle V^*\pi_n(x)V\varepsilon_j, \varepsilon_i\rangle = \langle \pi_n(x)\xi_j, \xi_i\rangle. \tag{2.1}$$

Suppose first that σ (and hence π) is irreducible and let $\xi \in \mathcal{H}$ be any unit vector. Then by the Kadison transitivity theorem there exists a unitary $u \in R$ such that $\pi_n(u)(\varepsilon_j \otimes \xi) = \xi_j$ for $j = 1, \dots, n$, where $\varepsilon_j \otimes \xi \in \mathcal{H}^n$ denotes the vector with the j-th component equal to ξ and the remaining components equal to 0. Now (2.1) can be rewritten as

$$y_{ij} = \langle \pi_n(u^*xu)(\varepsilon_j \otimes \xi), \varepsilon_i \otimes \xi\rangle = \langle \pi((u^*xu)_{ij})\xi, \xi\rangle = \omega((u^*xu)_{ij}),$$

where $(u^*xu)_{ij}$ is the entry of $u^*xu \in \mathbb{M}_n(B)$ on the position (i,j) and ω is the state $b \mapsto \langle \pi(b)\xi, \xi\rangle$ on B. Thus, denoting by $\omega_n : \mathbb{M}_n(B) \to \mathbb{M}_n$ the map induced by ω, we have $y = \omega_n(z)$, where $z = u^*xu$. Since $\overline{\text{co}}_R(z) = \overline{\text{co}}_R(x)$, it suffices now to prove that $y \in \overline{\text{co}}_R(z)$ (for then we will have $y \in \overline{\text{co}}_R(x) \cap A$). Let $\epsilon > 0$. Since ω is pure and $\omega(z_{ij}) = y_{ij}$ (where z_{ij} are the entries of z), by Lemma 2.1 there exists a positive element e $(= hh^*)$ in B such that $\|e\| = 1$ and

$$\|e(z_{ij} - y_{ij})e\| < \epsilon \quad (i,j = 1, \dots, n). \tag{2.2}$$

Moreover, replacing e by a suitable spectral projection of e in R, we may assume that e itself is a projection. Since B is a factor, there is a family of partial isometries $u_k \in B$ such that $u_k u_k^* \leq e$ and $\sum_k u_k^* u_k = 1$. Now (2.2) implies that

$$\|\sum_k u_k^* z_{ij} u_k - y_{ij}\| = \|\sum_k u_k^*(z_{ij} - y_{ij})u_k\| \leq \epsilon \ \ (i,j = 1, \dots, n),$$

hence $\|\sum_k u_k^{*(n)} z u_k^{(n)} - y\| \leq n\epsilon$, where $u_k^{(n)} \in \mathbb{M}_n(B)$ is the diagonal matrix with u_k on the main diagonal. This implies that the distance of y to $\overline{\text{co}}_R(z) = \overline{\text{co}}_R(x)$ is dominated by $n\epsilon$ for all $\epsilon > 0$, hence $y \in \overline{\text{co}}_R(x)$.

Consider now a more general situation, when σ is a direct sum of a family of irreducible representations $\sigma_k : R \to \mathrm{B}(\mathcal{K}_k)$. Then, with respect to the decomposition $\mathcal{K} = \bigoplus_k \mathcal{K}_k$, V can be represented as a (possibly infinite) column $V = (v_k)$, where each $v_k \in \mathrm{B}(\mathbb{C}^n, \mathcal{K}_k)$ has the polar decomposition $v_k = u_k|v_k|$ with $|v_k| = \sqrt{v_k^* v_k}$ and $u_k \in \mathrm{B}(\mathbb{C}^n, \mathcal{K}_k)$ an isometry. (This is possible since $\dim \mathcal{K}_k \geq n$ even if R is finite dimensional, since $R \supseteq A \cong \mathbb{M}_n$ and σ_k is injective if the factor R is finite dimensional.) Now $y = V^*\sigma(x)V = \sum_k |v_k|(u_k^*\sigma_k(x)u_k)|v_k|$. Since each σ_k is irreducible and u_k an isometry, $u_k^*\sigma_k(x)u_k \in \overline{\text{co}}_R(x)$ by the argument from the previous paragraph, hence it follows easily that $y \in \overline{\text{co}}_R(x)$ since $\sum_k |v_k|^2 = V^*V = 1$.

Finally, in general we can approximate the map $\phi(\cdot) = V^*\sigma(\cdot)V$ in the point-norm topology by the maps of the form $V_\nu^*\sigma_\nu(\cdot)V_\nu$, where each σ_ν is a direct sum of irreducible representations. To get such an approximation one can use the fact that each state on $\mathbb{M}_n(R)$ can be approximated in the weak* topology by convex combinations of pure states and the well known connection between completely positive maps from R to $\mathbb{M}_n$ and positive functionals on $\mathbb{M}_n(R)$ (see [13, Chapter 5], we leave the details to the reader). Thus, in any case we have $y \in \overline{\text{co}}_R(x)$, hence $W_n(x) \subseteq \overline{\text{co}}_R(x) \cap A$ (since by definition $W_n(x) \subseteq A \cong \mathbb{M}_n$).

Let now K be any weak* compact C*-convex subset of R and $\phi : R \to A$ a unital completely positive map. Then $\phi(x) \in W_n(x) \subseteq \overline{\text{co}}_R(x) \subseteq K$ for each $x \in K$. Thus $\phi(K) \subseteq K$. ∎

Proof of Lemma 1.3. Let $x \in \text{ext}_A\,(K \cap A)$ and suppose that

$$x = \sum_{j=1}^n a_j x_j a_j,$$

where $x_j \in K$, $a_j \in R$ is positive and invertible for each j and $\sum_{j=1}^n a_j^2 = 1$. Then for each conditional expectation $E : R \to A$ we have

$$x = \sum_{j=1}^n E(a_j x_j a_j) = \sum_{j=1}^n E(a_j^2)^{1/2}\phi_j(x_j)E(a_j^2)^{1/2}, \tag{2.3}$$

where $\phi_j : R \to A$, $\phi_j(w) := E(a_j^2)^{-1/2}E(a_j w a_j)E(a_j^2)^{-1/2}$ are unital completely positive mappings. Since $\phi_j(x_j) \in K \cap A$ by the hypothesis and $x \in \text{ext}_A\,(K \cap A)$, (2.3) implies that $\phi_j(x_j) = x$ and $E(a_j^2)x = xE(a_j^2)$ for all j. By the definition of ϕ_j it follows now that $E(a_j x_j a_j) = E(a_j^2)^{1/2} x E(a_j^2)^{1/2} = E(a_j^2)x = xE(a_j^2)$, hence

$$E(a_j x_j a_j - a_j^2 x) = 0 \quad \text{and} \quad E(a_j^2 x - x a_j^2) = 0 \tag{2.4}$$

since a conditional expectation from R to A is A-linear. By hypothesis the conditional expectations from R to A separate points of R, hence (2.4) implies that $a_j x_j a_j = a_j^2 x$ and $a_j^2 x = x a_j^2$, but since a_j is invertible and positive it follows that $a_j x = x a_j$ and $x_j = x$ for all $j = 1, \dots, n$. Thus $x \in \mathrm{ext}_R(K)$. □

3. The finite dimensional case

In this section we shall prove Theorem 1.1 in the case $R = \mathbb{M}_n$ ($n \in \mathbb{N}$). For convenience we recall now some results from [12] in a form suitable for our application.

Observe first that if $x \in \mathrm{ext}_R(K)$ for some C*-convex set K in a unital C*-algebra R and if

$$x = \sum_{j=1}^{n} a_j x_j a_j, \text{where } x_j \in K,\ a_j \geq 0,\ \sum_{j=1}^{n} a_j^2 = 1,$$

then $x_1 = x$ and a_1 commutes with x if a_1 is invertible. (To see this, note that $x = \frac{1}{2} a_1 x_1 a_1 + a^*(\bigoplus_{j=1}^n x_j)a$, where a is the column $(\frac{1}{\sqrt{2}} a_1, a_2, \dots, a_n)^T$, and use the polar decomposition of a to write x in the form $x = \frac{1}{2} a_1 x_1 a_1 + |a| y |a|$, where $y \in K$, a_1 and $|a|$ are invertible and $\frac{1}{2} a_1^2 + |a|^2 = 1$.)

The following lemma is a variation on [6, Theorem 4.1], but we do not require finite dimensionality of R.

Lemma 3.1. *Let K be a norm closed C^*-convex subset in any von Neumann algebra R. Suppose that*

$$x = \sum_{j=1}^{n} a_j x_j a_j,$$

where $x_j \in K$, $a_j \in R$, $a_j \geq 0$ and $\sum_{j=1}^n a_j^2 = 1$. If $x \in \mathrm{ext}_R(K)$, then the range projection p of a_1 reduces x and $xp = px_1p$. In particular, if x is irreducible in R (that is, the commutant of $\{x, x^*\}$ in R consists of scalars only) and $a_1 \neq 0$, then $x_1 = x$.

Proof. By the Dixmier approximation theorem ([9, 8.3.5]) K contains a central element, so translating by a central element, we may assume that $0 \in K$. Then $a^* y a \in K$ for all $y \in K$ and $a \in R$ with $\|a\| \leq 1$. Using the polar decomposition of the column $(a_2, \dots, a_n)^T$ we may write x in the form $x = a_1 x_1 a_1 + aya$, where $y \in K$ and $a = \sqrt{1 - a_1^2}$. For each $\alpha \in [0, 1]$ we have

(3.1)
$$x = (1 - \alpha)aya + (\alpha aya + a_1 x_1 a_1) = (1 - \alpha)aya + |t(\alpha)| u(\alpha)^* (y \oplus x_1) u(\alpha) |t(\alpha)|,$$

where $t(\alpha) = \binom{\alpha^{1/2} a}{a_1}$ and $t(\alpha) = u(\alpha)|t(\alpha)|$ is the polar decomposition of $t(\alpha)$. Since $|t(\alpha)|^2 = \alpha a^2 + a_1^2$ is invertible if $\alpha > 0$, (3.1) implies that $u(\alpha)^*(y \oplus x_1) u(\alpha) = x$ if $\alpha \in (0, 1]$ by the observation preceding the lemma. But from

$$u(\alpha) = t(\alpha)|t(\alpha)|^{-1} = \begin{pmatrix} \alpha^{1/2} a (\alpha a^2 + a_1^2)^{-1/2} \\ a_1 (\alpha a^2 + a_1^2)^{-1/2} \end{pmatrix} \quad (\alpha \in (0, 1])$$

we see that $\lim_{\alpha\to 0} u(\alpha) = \binom{p^\perp}{p}$ in the strong operator topology, hence by continuity $x = \lim_{\alpha\to 0} u(\alpha)^*(y \oplus x_1)u(\alpha) = px_1p + p^\perp yp^\perp$ (where the convergence is in the weak operator topology). This implies that $px = xp = px_1p$. If x is irreducible and $a_1 \neq 0$, then $p = 1$, hence $x = x_1$. □

It can be proved [11] that each weak* compact C*-convex subset K of a factor R contains a scalar multiple of 1 as an R-extreme point (in the case $R = \mathbb{M}_n$, which is the only case that will be needed in this section, this also follows from [4]). Thus, translating, we may assume without loss of generality that $0 \in \mathrm{ext}_R(K)$.

From now on let K be a C*-convex compact subset of $\mathbb{M}_n$ with $0 \in \mathrm{ext}_{\mathbb{M}_n}(K)$.

Notation. For each $k \leq n$ let $P_k \in \mathbb{M}_n$ be the orthogonal projection onto the first k coordinates. Put

$$K_k = P_k K P_k$$

and regard K_k as a subset of $\mathbb{M}_k$ by identifying $\mathbb{M}_k$ with $P_k\mathbb{M}_nP_k$. It is easy to see that K_k is C*-convex. For each $x \in \mathbb{M}_k$ we shall denote by $\tilde{x}$ the element $x \oplus 0 \in \mathbb{M}_n$.

Remark 3.2. If x is an irreducible C*-extreme point of K, then $x \in \mathrm{ext}_{\mathbb{M}_n}(K)$ by [12, Corollary 1.8]. In our present situation this can also be seen in a more straightforward way as follows. Suppose that $x = \sum_{j=1}^2 a_jx_ja_j$, where $x_j \in K$ and all $a_j \neq 0$ are positive with $\sum_{j=1}^2 a_j^2 = 1$. Then by Lemma 3.1 $x_j = x$ for all j, hence

$$x = \sum_{j=1}^{2} a_j x a_j. \tag{3.2}$$

Writing a_1 (and hence also $a_2 = \sqrt{1-a_1^2}$) as a diagonal matrix, $a_1 = \bigoplus_{i=1}^m \alpha_i 1_{n_i}$, where $\alpha_i \neq \alpha_k$ if $i \neq k$ and writing x correspondingly as a block matrix, $x = [x_{ik}]$ ($x_{ik} \in \mathbb{M}_{n_i,n_k}$), it follows from (3.2) by a straightforward computation that $x_{ik} = 0$ if $i \neq k$. But, since x is irreducible, this implies that $m = 1$ and a_1, a_2 are scalar multiples of 1.

In general, K may have no irreducible C*-extreme points and one has to consider also the possible irreducible C*-extreme points of the compressions K_k. Note that if x is an irreducible C*-extreme point of K_k ($k < n$) such that x is a compression of an irreducible C*-extreme point y of K_j for some $k < j \leq n$, say $\tilde{x} = p\tilde{y}p$ for a projection $p \in \mathbb{M}_n$, then $\tilde{x}$ can be expressed as the C*-convex combination $\tilde{x} = p\tilde{y}p + p^\perp 0 p^\perp$.

Notation. Denote by S_n the set of all irreducible C*-extreme points of K and (for $k < n$) by S_k the set of all irreducible C*-extreme points of K_k which are not compressions of irreducible C*-extreme points of K_j for $k < j \leq n$. (The elements of K_k are called structural elements in [12].)

From [12, Theorem 4.5] or [15, Theorem 4.3] and [5] we have the following.

Theorem 3.3 ([12]). *Each $x \in K$ can be expressed as a finite sum*

$$x = \sum_{i=1}^{l} t_i^* x_i t_i,$$

where $x_i \in S_{n_i}$ $(n_i \in \mathbb{N},\ n_i \leq n)$, $t_i \in \mathbb{M}_{n_i,n}$ *and* $\sum_{i=1}^{l} t_i^* t_i = 1$. *Thus K is equal to the C^*-convex hull in $\mathbb{M}_n$ of the set* $\tilde{S} := \bigcup_{k=1}^{n} \tilde{S}_k$, *where* $\tilde{S}_k = \{\tilde{y} :\ y \in S_k\}$.

We will show (Proposition 3.5) that elements of the form $\bigoplus_{i=1}^r x_i$, where $x_i \in S_{n_i}$ $(\sum_{i=1}^r n_i = n)$ are $\mathbb{M}_n$-extreme in K. Since each $\tilde{x}_i$ can obviously be expressed as a C*-convex combination of $\bigoplus_{i=1}^r x_i$ and 0, it follows then from Theorem 3.3 (since $0 \in \mathrm{ext}_{\mathbb{M}_n}(K)$) that $K = \mathrm{co}_{\mathbb{M}_n}(\mathrm{ext}_{\mathbb{M}_n}(K))$. This will prove Theorem 1.1 in the case $R = \mathbb{M}_n$.

We need the following slight variation of [12, Lemma 4.6]. For convenience of the reader we shall present a simple proof.

Lemma 3.4 ([12])**.** *Let $x \in K_k$ $(k \leq n)$ be of the form $x = \bigoplus_{i=1}^r x_i$, where $x_i \in S_{m_i}$ $(m_i \in \mathbb{N})$. Suppose that $x = v^* y v$ for some $y \in K$ and some isometry $v \in \mathbb{M}_{n,k}$. Then there exist a unitary $U \in \mathbb{M}_n$ and a matrix $z \in K_{n-k}$ such that $y = U(x \oplus z)U^*$ and $v = U\binom{1}{0}$, where $1 \in \mathbb{M}_k$ is the identity matrix.*

Proof. Assume first that $r = 1$ (that is, $x \in S_k$) and $v = \binom{1}{0}$ (that is, v is the inclusion of $\mathbb{C}^k$ into $\mathbb{C}^n$). By Theorem 3.3

$$y = \sum_{j=1}^{l} t_j^* y_j t_j, \tag{3.3}$$

where $y_j \in S_{n_j}$, $t_j \in \mathbb{M}_{n_j,n}$ and $\sum_{j=1}^l t_j^* t_j = 1$. Let $t_j v = u_j |t_j v|$ be the polar decomposition of $t_j v$ with $u_j \in \mathbb{M}_{n_j,k}$ an isometry or a coisometry. Then

$$x = v^* y v = \sum_{j=1}^{l} |t_j v| (u_j^* y_j u_j) |t_j v|,$$

where $u_j^* y_j u_j \in K$ (since K is C*-convex and $0 \in K$). Put $\alpha_j = |t_j v|$. Since $x \in S_k$ (hence x is irreducible and C*-extreme in K_k), $x \in \mathrm{ext}_{\mathbb{M}_k}(K_k)$ by Remark 3.2, hence if $\alpha_j \neq 0$, then Lemma 3.1 and the irreducibility of x imply that $u_j^* y_j u_j = x$ and α_j is a scalar. Decompose the set $\mathbb{L} := \{1, \ldots, l\}$ as $\mathbb{L} = \mathbb{L}_1 \cup \mathbb{L}_2$, where $\mathbb{L}_1 = \{j \in \mathbb{L} :\ \alpha_j \neq 0\}$ and $\mathbb{L}_2 = \mathbb{L} \setminus \mathbb{L}_1$. Since u_j is an isometry or a coisometry, the identity $u_j^* y_j u_j = x$ means that one of x, y_j is unitarily equivalent to the compression of the other; but since $y_j \in S_{n_j}$ and $x \in S_k$ this is possible only if $y_j \cong x$ (where $\cong$ denotes unitary equivalence). Hence u_j must be unitary and $n_j = k$ for all $j \in \mathbb{L}_1$. The identity $t_j v = \alpha_j u_j$ then implies (since $v = \binom{1}{0}$) that $t_j = (\alpha_j u_j, b_j)$ for some $b_j \in \mathbb{M}_{k,n-k}$. If $j \in \mathbb{L}_2$, then $t_j v = u_j \alpha_j = 0$ implies that t_j is of the form $t_j = (0, b_j)$. Thus from (3.3) we have (since $\sum_{j \in \mathbb{L}_1} \alpha_j^2 = \sum_{j=1}^l |t_j v|^2 = v^* \sum_{j=1}^l t_j^* t_j v = 1$)

$$y = \sum_{j \in \mathbb{L}_1} \begin{pmatrix} \alpha_j u_j^* \\ b_j^* \end{pmatrix} u_j x u_j^* (\alpha_j u_j, b_j) + \sum_{j \in \mathbb{L}_2} \begin{pmatrix} 0 \\ b_j^* \end{pmatrix} y_j (0, b_j) = \begin{pmatrix} x & g \\ h & z \end{pmatrix},$$

where $z \in \mathbb{M}_{n-k}$,

$$g = \sum_{j \in \mathbb{L}_1} \alpha_j x u_j^* b_j \quad \text{and} \quad h = \sum_{j \in \mathbb{L}_1} \alpha_j b_j^* u_j x. \tag{3.4}$$

But, from

$$1 = \sum_{j=1}^{l} t_j^* t_j = \sum_{j \in \mathbb{L}_1} \begin{pmatrix} \alpha_j u_j^* \\ b_j^* \end{pmatrix} (\alpha_j u_j, b_j) + \sum_{j \in \mathbb{L}_2} \begin{pmatrix} 0 \\ b_j^* \end{pmatrix} (0, b_j)$$

we see (looking at the entries on the position (1,2)) that $\sum_{j\in\mathbb{L}_1}\alpha_j u_j^* b_j = 0$, hence (3.4) implies that $g=0$. Similarly $h=0$ and y is of the form $x\oplus z$.

Assume now that $x=\bigoplus_{i=1}^r x_i$, where $r>1$, but v is still of the form $\binom{1}{0}$, so that y can be represented as a matrix of the form

$$y=\begin{pmatrix} x & g \\ h & z \end{pmatrix}.$$

Applying (to elements x_i) what we have just proved above, it follows inductively that the successive block rows and columns of the matrices g and h (respectively) corresponding to the summands x_i of x are all 0. Thus $g=0$, $h=0$ and $y=x\oplus z$.

Finally, a general isometry $v:\mathbb{C}^k\to\mathbb{C}^n$ can be extended to a unitary operator U on $\mathbb{C}^n$, so that $v=U\binom{1}{0}$ and the proof is completed by applying the above argument to U^*yU in place of y. □

Proposition 3.5. *If $x\in K$ is of the form $x=\bigoplus_{i=1}^r x_i$, where $x_i\in S_{n_i}$ $(n_i\in\mathbb{N})$, then $x\in \operatorname{ext}_{\mathbb{M}_n}(K)$.*

Proof. Suppose first that $x\cong y^{(r)}$ (= the direct sum of r copies of y) for some $y\in S_k$, where $rk=n$. Since y is an irreducible C*-extreme point of K_k, y is $\mathbb{M}_k$-extreme in K_k (see Remark 3.2). Observe that the canonical isomorphism $\mathbb{M}_k\cong\mathbb{M}_k^{(r)}\subseteq\mathbb{M}_n$ identifies K_k with $K\cap\mathbb{M}_k^{(r)}$. (Note that $w^{(r)}=\sum_{j=1}^r e_{j1}we_{1j}\in K$ for each $w\in K_k$ since K is C*-convex and $0\in K$, where $\{e_{ij}\}_{i,j=1}^r$ is the appropriate matrix unit in $\mathbb{M}_r(\mathbb{M}_k)$, namely the standard matrix unit in $\mathbb{M}_r$ tensored with $1\in\mathbb{M}_k$.) It follows then by Lemma 1.3 (and Theorem 1.2 which justifies the application of Lemma 1.3, as in Section 1) that $y^{(r)}\in\operatorname{ext}_{\mathbb{M}_n}(K)$.

Thus we may assume now that not all the summands x_i of x are mutually unitarily equivalent, hence the set $\{1,\dots,r\}$ can be partitioned into two parts $\mathbb{I}$ and $\mathbb{J}$ such that $x_i\not\cong x_j$ if $i\in\mathbb{I}$ and $j\in\mathbb{J}$. Put

$$y=\bigoplus_{i\in\mathbb{I}}x_i \quad\text{and}\quad z=\bigoplus_{j\in\mathbb{J}}x_j$$

and let $k=\sum_{i\in\mathbb{I}}n_i$, $l=\sum_{j\in\mathbb{J}}n_j$. By induction on the dimension we may assume that $y\in\operatorname{ext}_{\mathbb{M}_k}(K_k)$ and $z\in\operatorname{ext}_{\mathbb{M}_l}(K_l)$. To prove that $x\in\operatorname{ext}_{\mathbb{M}_n}(K)$, suppose that

$$x=\sum_{j=1}^m t_j^* w_j t_j, \tag{3.5}$$

where $w_j\in K$, $\sum_{j=1}^m t_j^2=1$ and each $t_j\in\mathbb{M}_n$ is positive and invertible. According to the decomposition $x=y\oplus z$, decompose each t_j as $t_j=(a_j,b_j)$, where $a_j\in\mathbb{M}_{n,k}$, $b_j\in\mathbb{M}_{n,l}$, and let $a_j=u_j|a_j|$, $b_j=v_j|b_j|$ be the polar decompositions. Then (3.5) implies (by considering the two diagonal blocks of sizes $k\times k$ and $l\times l$, respectively) that

$$y=\sum_{j=1}^m |a_j|(u_j^*w_ju_j)|a_j| \quad\text{and}\quad z=\sum_{j=1}^m |b_j|(v_j^*w_jv_j)|b_j|.$$

Since $y\in\operatorname{ext}_{\mathbb{M}_k}(K_k)$ and $z\in\operatorname{ext}_{\mathbb{M}_l}(K_l)$ it follows now that $|a_j|y=y|a_j|$, $|b_j|z=z|b_j|$ and

$$u_j^*w_ju_j=y,\qquad v_j^*w_jv_j=z\qquad (j=1,\dots,m). \tag{3.6}$$

Since t_j is invertible, a_j is injective, hence $|a_j|$ is invertible and u_j is an isometry. Similarly v_j is an isometry. Thus (3.6) implies by Lemma 3.4 that there exist unitaries $U_j, V_j \in \mathbb{M}_n$ such that $u_j = U_j\binom{1}{0}$, $v_j = V_j\binom{0}{1}$ and

$$w_j = U_j(y \oplus h_j)U_j^*, \qquad w_j = V_j(g_j \oplus z)V_j^* \tag{3.7}$$

for some matrices g, h. Putting $W_j = U_j^* V_j$, we have from (3.7) that $(y \oplus h_j)W_j = W_j(g_j \oplus z)$ and $(y \oplus h_j)^* W_j = W_j(g_j \oplus z)^*$. Since no irreducible direct summand x_i of y is unitarily equivalent to any irreducible direct summand x_j of z, the only operator intertwining the identity representations of $\mathrm{C}^*(y)$ and $\mathrm{C}^*(z)$ is 0, hence $W_j = c_j \oplus d_j$ for some unitary operators $c_j \in \mathbb{M}_k$ and $d_j \in \mathbb{M}_l$. Then $g_j = c_j^* y c_j$, $h_j = d_j z d_j^*$, $w_j = U_j(y \oplus d_j z d_j^*)U_j^*$ (from (3.7)), $u_j = U_j\binom{1}{0}$ and $v_j = V_j\binom{0}{1} = U_j W_j \binom{0}{1} = U_j(c_j \oplus d_j)\binom{0}{1} = U_j\binom{0}{d_j}$. Thus $a_j = u_j|a_j| = U_j\binom{|a_j|}{0}$, $b_j = v_j|b_j| = U_j\binom{0}{d_j|b_j|}$ and therefore

$$t_j = (a_j, b_j) = U_j(|a_j| \oplus d_j|b_j|) = U_j(1 \oplus d_j)(|a_j| \oplus |b_j|).$$

Since t_j and $|a_j| \oplus |b_j|$ are positive, while $U_j(1 \oplus d_j)$ is unitary, it follows now by the uniqueness of the polar decomposition that $U_j(1 \oplus d_j) = 1$ and $t_j = |a_j| \oplus |b_j|$. This clearly implies that t_j commutes with $x = y \oplus z$ (since $|a_j|$ and $|b_j|$ commute with y and z, respectively) and $w_j = U_j(y \oplus d_j z d_j^*)U_j^* = U_j(1 \oplus d_j)(y \oplus z)(1 \oplus d_j^*)U_j^* = y \oplus z = x$. □

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