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COMPACT OPERATORS ON THE BERGMAN SPACE OF MULTIPLY-CONNECTED DOMAINS

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ABSTRACT. If Ω is a smoothly bounded multiply-connected domain in the complex plane and $A = \sum_{j=1}^{m} \prod_{k=1}^{m_j} T_{\varphi_{j,k}}$, where $\varphi_{j,k} \in L^\infty(\Omega, d\nu)$, we show that A is compact if and only if its Berezin transform vanishes at the boundary.

1. INTRODUCTION

Let Ω be a bounded multiply-connected domain in the complex plane $\mathbf{C}$, whose boundary $\partial\Omega$ consists of finitely many simple closed smooth analytic curves. For $d\nu = \frac{1}{\pi}dxdy$ we consider the usual L^2-space $L^2(\Omega) = L^2(\Omega, d\nu)$. The Bergman Space $H^2(\Omega, d\nu)$, consisting of all holomorphic functions which are L^2-integrable, is a closed subspace of $L^2(\Omega, d\nu)$. The Bergman Projection is the orthogonal projection $P{:}L^2(\Omega, d\nu) \longrightarrow H^2(\Omega, d\nu)$. It is well known that for any $f \in L^2(\Omega, d\nu)$ we have

$$Pf(w) = \int_\Omega f(z)\overline{K^\Omega(z, w)}d\nu(z),$$

where K^Ω is the Bergman reproducing kernel of Ω. For $\varphi \in L^\infty(\Omega, d\nu)$ the Toeplitz operator T_φ from $H^2(\Omega, d\nu)$ to itself is defined by $T_\varphi = PM_\varphi$, where M_φ is the standard multiplication operator.

Axler and Zheng have proved (see [2]) that if D is the disk $S = \sum_j^m \prod_k^{m_j} T_{\varphi_{i,k}}$, where $\varphi_{i,k} \in L^\infty(D)$, then S is compact if and only if its Berezin transform vanishes at the boundary of the disk. Moreover, they asked if their result could hold if the disk is replaced by other domains in $\mathbf{C}$ which "lack appropriate symmetry." The aim of this paper is to prove, using new techniques, that the same characterization holds when Ω is a general smoothly bounded multiply-connected planar domain. Such multiply-connected domains "lack appropriate symmetry".

2. PRELIMINARIES

Let Ω be a bounded multiply-connected domain in the complex plane $\mathbf{C}$, whose boundary $\partial\Omega$ consists of finitely many simple closed smooth analytic curves γ_j $(j = 1, 2, \dots, n)$, where γ_j are positively oriented with respect to Ω and $\gamma_j \cap \gamma_i = \varnothing$ if $i \neq j$. We also assume that γ_1 is the boundary of the unbounded component of $\mathbf{C}\backslash\Omega$. Let Ω_1 be the bounded component of $\mathbf{C}\backslash\gamma_1$, and Ω_j $(j = 2, \dots, n)$ the unbounded component of $\mathbf{C}\backslash\gamma_j$, respectively, so that $\Omega = \bigcap_{j=1}^n \Omega_j$. We use the

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symbol Δ to indicate the punctured disk $\{z \in \mathbf{C} | 0 < |z| < 1\}$. Let Γ be any one of the domains Ω, Δ, Ω_j $(j = 2, \ldots, n)$. We call $K^\Gamma(z, w)$ the reproducing kernel of Γ and we use the symbol $k^\Gamma(z, w)$ to indicate the normalized reproducing kernel, i.e. $k^\Gamma(z, w) = K^\Gamma(z, w)/K^\Gamma(w, w)^{\frac{1}{2}}$.

For any $A \in \mathcal{B}(H^2(\Gamma, d\nu))$ we denote $\tilde{A}$, the Berezin transform of A (see [3] and [2]), where $\tilde{A}(w) = \langle Ak_w^\Gamma, k_w^\Gamma \rangle = \int_\Gamma Ak_w^\Gamma(z)\overline{k_w^\Gamma(z)}d\nu(z)$. With these definitions in mind we can state the main theorem:

Theorem 2.1. *Let A be an operator in $B(H^2(\Omega, d\nu))$ which can be written as $A = \sum_{j=1}^m \prod_{k=1}^{m_j} T_{\varphi_{j,k}}$, where $\varphi_{j,k} \in L^\infty(\Omega, d\nu)$. Then A is compact if and only if $\tilde{A}(z) \to 0$ as $z \to \partial\Omega$.*

We remind the reader (see [6]) that any bounded multiply-connected domain whose boundary consists of finitely many simple closed smooth analytic curves, i.e. a *regular domain,* is conformally equivalent to a canonical bounded multiply-connected domain whose boundary consists of finitely many circles. Moreover, it is possible to prove (see [1]) the following

Theorem 2.2. *Let Ω be a regular domain and let ψ be a conformal mapping from Ω onto D. Then $K^D(\psi(z), \psi(w))\psi'(z)\overline{\psi'(w)} = K^\Omega(z, w)$ and the operator $V_\psi f = \psi' \cdot f \circ \psi$ is an isometry from $L^2(D)$ onto $L^2(\Omega)$.*

If $A \in B(H^2(\Omega))$ and we define $A_D \in B(H^2(D))$ as $V_{\psi^{-1}}AV_\psi$, where ψ is a conformal mapping from Ω onto D, then we can prove the following

Proposition 2.3. $\tilde{A}(z) = \tilde{A}_D(\psi(z))$.

Proof. We have, by definition, $\tilde{A}(z) = \langle Ak_z^\Omega, k_z^\Omega \rangle = \int_\Omega Ak_z^\Omega(w)\overline{k_z^\Omega(w)}dw$, where $k_z^\Omega(\cdot) = K_z^\Omega(\cdot, z)K_z^\Omega(z, z)^{-\frac{1}{2}}$. Let us take $\psi^{-1} : D \to \Omega$. Since $(J_\mathbf{R}\psi^{-1})(\beta)$ is $|(\psi^{-1})'(\beta)|^2$ and there exists $\zeta \in D$ such that $\psi(z) = \zeta$ we obtain

$$\tilde{A}(z) = \int_D ((V_{\psi^{-1}}AV_\psi)(V_{\psi^{-1}}k^\Omega_{\psi^{-1}(\varsigma)}))(\beta)\overline{k^\Omega_{\psi^{-1}(\varsigma)}(\psi^{-1}(\beta))(\psi^{-1})'(\beta)}d\beta.$$

Since Theorem 1.2 implies that

$$k^\Omega_{\psi^{-1}(\zeta)}(\psi^{-1}(\beta)) = |(\psi^{-1})'(\zeta)|((\psi^{-1})'(\beta)\overline{(\psi^{-1})'(\zeta)})^{-1} \cdot k_\zeta^D(\beta),$$

then

$$\tilde{A}(z) = \int_D (A_D k_\zeta^D)(\beta)\overline{k_\zeta^D(\beta)}d\beta;$$

it follows that $\tilde{A}(z) = \tilde{A}_D(\psi(z))$.

Since every conformal map is an open map, we can conclude that it is enough to prove the theorem when the domain is a canonical bounded multiply-connected domain whose boundary consists of finitely many circles.

3. The structure of $H^2(\Omega)$ and some estimates about the Bergman kernel

In this section we state well-known facts which we will use in our work; for the proofs the reader can see [1] or [7] and [4].

From now on we will assume that $\Omega = \bigcap_{j=1}^n \Omega_j$, where $\Omega_1 = \{z \in \mathbf{C} : |z| < 1\}$ and $\Omega_j = \{z \in \mathbf{C} : |z - a_j| > r_j\}$ for $j = 2, \ldots, n$. Here $a_j \in \Omega_1$ and $0 < r_j < 1$ with $|a_j - a_k| > r_j + r_k$ if $j \neq k$ and $1 - |a_j| > r_j$. We will indicate with the symbol

Δ the punctured disk $\Omega_1\backslash\{0\}$. With the symbols $K^{\Omega_j}(z,w)$, $K^{\Omega}(z,w)$, $K^{\Delta}(z,w)$ we denote the Bergman kernel on Ω_j, Ω, and Δ respectively.

Theorem 3.1. *There exists an isomorphism $\mathcal{I} : L^2(\Delta) \longrightarrow L^2(\Omega_1)$ such that $H^2(\Omega_1) = \mathcal{I}(H^2(\Delta))$. Moreover, the Bergman kernels K^{Δ} and K^{Ω_1} satisfy the equation $K^{\Delta}(z,w) = K^{\Omega_1}(z,w)$.*

We observe that the previous Theorem and the well-known fact that the reproducing kernel of the unit disk is given by $(1-z\bar{w})^{-2}$ imply that, for $j = 2, ..., n$, $K^{\Omega_j}(z,w) = r_j^2(r_j^2 - (z-a_j)\cdot\overline{(w-a_j)})^{-2}$. Moreover, Theorem 3.1 and Theorem 2.3 and the result of Axler and Zheng imply that if $A = \sum_{j=1}^m \prod_{k=1}^{m_j} T_{\varphi_{j,k}}$, where $\varphi_{j,k} \in L^\infty(\Omega_\ell)$ and $\ell = 2, ..., n$, then A is compact if and only if its Berezin transform vanishes at the boundary.

We remind the reader that it has been proved that the kernel of Ω looks like the kernel of Ω_s near $\partial\Omega_s$. In fact, if we define $K_0^{\Omega}(z,w) = K^{\Omega}(z,w) - \sum_{\ell=1}^n K^{\Omega_\ell}(z,w)$, where for any $z,w \in \Omega\times\Omega$ $K_\ell^{\Omega}(z,w) = K^{\Omega_\ell}(z,w)$, we have the following

Lemma 3.2. *1. K_0^{Ω} is conjugate symmetric about z and w. For each $w \in \Omega$, $K_0^{\Omega}(\cdot,w)$ is conjugate analytic on Ω and $K_0^{\Omega} \in \mathcal{C}^\infty(\overline{\Omega}\times\Omega)$.*

2. There are neighborhoods U_j of $\partial\Omega_j$ $(j = 1,\dots,n)$ and a constant $C > 0$ such that $U_j \cap U_k$ is empty if $j \neq k$ and $\left|K^{\Omega}(z,w) - K_j^{\Omega}(z,w)\right| < C$ for $z \in \Omega$ and $w \in U_j$.

3. $K_0^{\Omega} \in L^\infty(\Omega\times\Omega)$.

4. There are constants $D > 0$ and $M > 0$ such that for any $(z,w) \in G_i\times\Omega \cup \Omega\times G_i$ we have $|K^{\Omega}(z,w)| < D|K_j^{\Omega}(z,w)|$ and $|K_j^{\Omega}(z,w)| < |K^{\Omega}(z,w)| + M$.

5. For any $z \in \Omega$ we have $K_j^{\Omega}(z,z) < K^{\Omega}(z,z)$.

The next Lemma gives us a good insight into the structure of the space $H^2(\Omega)$. This lemma will play a special role in our construction.

Lemma 3.3. *For $f \in H^2(\Omega)$, we can write it uniquely as $f(z) = \sum_{j=1}^n (P_jf)(z) + (P_0f)(z)$ with $P_jf \in H^2(\Omega_j)$, $P_0f \in H^2(\Omega)\cap\mathcal{C}^\infty(\overline{\Omega})$, $P_k(P_jf) = 0$ if $j \neq k$ and there exists a constant M_1 such that, for $j = 0, 1, \dots, n$, we have $||P_jf||_{\Omega} \leq ||P_jf|||_{\Omega_j} \leq M_1||f||_{\Omega}$. In particular, if $f \in H^2(\Omega_i)$, then $P_if = f$ and $||f||_{\Omega_i} \leq M_1||f||_{\Omega}$ for $i = 1,\dots,n$.*

Moreover, if $\{f_n\}$ is a bounded sequence in $H^2(\Omega)$ and $f_n \to 0$ weakly in $H^2(\Omega)$, then $P_jf_n \to 0$ weakly on $H^2(\Omega_j)$ for $j = 1,\dots,n$ and $P_0f_n \to 0$ uniformly on Ω.

4. Multiply-connected domains

The goal of this section is to prove Theorem 2.1. We start with the following

Definition. Let $\Omega = \bigcap_{i=1}^n \Omega_i$ be a bounded canonical multiply-connected domain. We say that the set of n functions $\mathcal{P} = \{p_1, ..., p_n\}$ is a ∂-partition for Ω if

1. For every $j = 1,\dots,n$, $p_j : \Omega \to [0,1]$ is a Lipschitz, $\mathcal{C}^\infty$-function.
2. For every $j = 1,\dots,n$ there exists an $\epsilon_j > 0$ such that if $U_{\epsilon_j} = \{\zeta \in \Omega : r_j < |\zeta - a_j| < r_j + \epsilon_j\}$, then $p_j(\zeta) = 1$ $\forall\zeta \in U_{\epsilon_j}$ and $p_j(\zeta) = 0$ if $\zeta \in U_{\varepsilon_k}$ and $k \neq j$.
3. For any $\zeta \in \Omega$, $\sum_{i=1}^n p_i(\zeta) = 1$.

From now on, in order to make our notation a little simpler, when we use a kernel operator we will denote it by the name of its kernel function. For example, the Bergman Projection will be denoted by the symbol K^Ω. Finally, we define the operators $Q_\ell : L^2(\Omega) \longrightarrow L^2(\Omega)$, for $\ell = 1, 2, \dots, n$, in the following way:

$$Q_\ell f(z) = \int_\Omega f(\zeta)|K_\ell^\Omega(\zeta, z)|dv(\zeta).$$

It is important to notice that $\forall \ell = 1, \dots, n$, Q_ℓ is a bounded operator (see [7]).

We start with the following

Lemma 4.1. *Let* $A = \sum_{j=1}^N \left(T_{\phi_{j,1}} \dots T_{\phi_{j,n_j}}\right)$, *where* $\phi_{j,k} \in L^\infty(\Omega)$ *for any* j *and* k. *If* $\mathcal{P} = \{p_1, \dots, p_n\}$ *is a* ∂*-partition for* Ω *and* $R_s \in \mathcal{B}(H^2(\Omega))$, *for* $s = 1, \dots, n$, *is defined as* $\sum_{j=1}^N \left(K_s^\Omega M_{\phi_{j,1} p_s} \dots K_s^\Omega M_{\phi_{j,n_j} p_s}\right)$, *then the following are equivalent:*

1. *A is a compact operator;*
2. *The operator $R_\ell P_\ell$ is compact for $\ell = 1, \dots, n$.*

Proof. If we define A_j as $A_j = T_{\phi_{j,1}} \dots T_{\phi_{j,n_j}}$ it follows that $A = \sum_j^N A_j$. Since the Bergman kernel of Ω is given by $\sum_{s=0}^n K_s^\Omega$ and $\mathcal{P} = \{p_1, \dots, p_n\}$ is a ∂-partition for Ω, then we have

$$A_j = (\sum_{s=0}^n K_s^\Omega)(\sum_{t=1}^n M_{\phi_{j,1} p_t}) \dots (\sum_{s=0}^n K_s^\Omega)(\sum_{t=1}^n M_{\phi_{j,n} p_t}).$$

A simple calculation shows that there exists a compact operator $\mathcal{K}_j$ such that $A_j = \mathcal{K}_j + \sum_{s=1}^n K_s^\Omega M_{\phi_{j,1} p_s} \dots K_s^\Omega M_{\phi_{j,n_j} p_s}$. This implies that

$$A = \sum_{j=1}^N \sum_{s=1}^n \prod_{k=1}^{n_j} K_s^\Omega M_{\phi_{j,k} p_s} + \sum_{j=1}^N \mathcal{K}_j$$

and, if we define $\mathcal{K} = \sum_{j=1}^N \mathcal{K}_j$ and $R_s = \sum_{j=1}^N \prod_{k=1}^{n_j} K_s^\Omega M_{\phi_{j,k} p_s}$, we have $A = \sum_{s=1}^n R_s + \mathcal{K}$. Since $\mathcal{K}$ is a sum of compact operators then it is a compact operator. Hence this implies that A is compact if and only if $\sum_{s=1}^n R_s$ is compact.

We observe that $I = \sum_{\ell=0}^n P_\ell$ implies

$$\sum_{t=1}^n R_t = \sum_{t=1}^n R_t \sum_{\ell=0}^n P_\ell = \sum_{t=1}^n R_t P_0 + \sum_{t,\ell=1}^n R_t P_\ell,$$

where the P_ℓ's are the operators studied in Lemma 3.3.

Claim I. The operator $R_t P_0$ is compact for $t = 1, \dots, n$.

Proof. Since $R_t P_0 = \sum_{j=1}^N K_t^\Omega M_{\phi_{j,1} p_t} \dots K_t^\Omega M_{\phi_{j,n_j} p_t} P_0$, it is enough to prove that for any n_j with $j = 1, \dots, N$ the operator $K_t^\Omega M_{\phi_{j,n_j} p_t} P_0$ is compact. We observe that if $g \in L^2(\Omega)$, then

$$(*) \qquad K_t^\Omega M_{\phi_{j,n_j} p_t} P_0 g(z) = \int_\Omega K_t^\Omega(w, z) \phi_{j,n_j}(w) p_t(w) P_0 g(w) = Q_t(\phi_{j,n_j} p_t P_0 g).$$

Let $\{f_n\}$ be a bounded sequence in $H^2(\Omega)$ such that $f_n \to 0$ weakly. Lemma 3.3 implies that $P_0 f_n \to 0$ uniformly on Ω; it follows that $||P_0 f_n||_\infty \to 0$. Therefore $(*)$ and the fact that Q_t is bounded imply that

$$||K_t^\Omega M_{\phi_{j,n_j} p_t} P_0 f_n||_2 = ||Q_t(\phi_{j,n_j} p_t P_0 f_n)||_2 \le ||P_0 f_n||_\infty ||Q_t \phi_{j,n_j} p_t||_2 \to 0.$$

This inequality together with a well-known theorem (see [5, Chap. 5]) implies that the operator is compact.

Claim II. The operator R_sP_ℓ is compact if $s \neq \ell$ and $s, \ell = 1, \dots, n$.

Proof. Since $R_sP_\ell = \sum_{j=1}^N K_s^\Omega M_{\phi_{j,1}p_s} \dots K_s^\Omega M_{\phi_{j,n_j}p_s} P_\ell$, it is enough to prove that for any n_j, with $j = 1, \dots, N$, the operator $R_{s,\ell} = K_s^\Omega M_{\phi_{j,n_j}p_s} P_\ell$ is compact. To do so we consider a bounded sequence $\{f_n\}$ in $L^2(\Omega)$ such that $f_n \to 0$ weakly and we prove that $||R_{s,\ell}f_n||_2 \to 0$. We know that the continuity of P_ℓ implies that $P_\ell f_k \to 0$ weakly on $H^2(\Omega_l)$ and $\{||P_\ell f_k||_{\Omega_\ell}\}$ is bounded by Lemma 3.3. Since it is a sequence of holomorphic functions we know that $\{P_\ell f_k\}$ is uniformly bounded on any compact subset of Ω_ℓ. Therefore the sequence $\{P_\ell f_k\}$ is a normal family of functions. Since $P_\ell f_k(\zeta) \to 0$ for any $\zeta \in \Omega_\ell$, then $P_\ell f_k$ converges uniformly on any compact subset of Ω_ℓ and consequently on $F = \text{supp}(p_s)$. Now we observe that

$$|R_{s,\ell} f_k(\zeta)| \leq Sup\{|P_\ell f_k(\zeta)| : \zeta \in F\} \cdot |Q_\ell(|\mathcal{X}_F \phi_{j,n_j} p_s|)(\zeta)|.$$

Then, by using the fact that Q_ℓ is bounded, we have

$$||R_{s,\ell} f_k||_\Omega \leq Sup\{|P_\ell f_k(\zeta)| : \zeta \in F\} \cdot M \cdot ||\phi_{j,1} p_s||_{\Omega,2} \to 0$$

and this completes the proof of our claim.

Finally, we observe that Claim I and Claim II imply that if, for any t, R_tP_t is compact, then A is compact. Moreover, since $P_t^2 = P_t$, $P_tP_s = 0$ if $s \neq t$, and I, the identity operator on $H^2(\Omega)$, is equal to $\sum_0^n P_t$, then A compact implies that R_tP_t is compact for any t and this completes the proof of the theorem. □

Remark. In the last Theorem we started with the operator

$$A = \sum_{j=1}^N \left(T_{\phi_{j,1}} \dots T_{\phi_{j,n_j}} \right)$$

and we have constructed the operator $R_s = \sum_{j=1}^N K_s^\Omega M_{\phi_{j,1}p_s} \dots K_s^\Omega M_{\phi_{j,n_j}p_s}$. Since for any $\zeta, z \in \Omega$ $K_s^\Omega(\zeta, z) = K^{\Omega_s}(\zeta, z)$, we know that R_s is an element of $B(H^2(\Omega_s))$. We also know that by construction $\phi_{j,i}p_s \in L^\infty(\Omega_s)$, so we can construct the following bounded operator

$$A_{\Omega_s} = \sum_{j=1}^N K^{\Omega_s} M_{\phi_{j,1}p_s} \dots K^{\Omega_s} M_{\phi_{j,n_j}p_s}$$

which acts on $H^2(\Omega_s)$.

We observe that $R_s - A_{\Omega_s}$ is compact. In fact, since

$$A_{\Omega_s} = \sum_{j=1}^N \prod_{k=1}^{n_j} K^{\Omega_s} M_{(\mathcal{X}_\Omega + \mathcal{X}_{\Omega_s - \Omega})\phi_{j,k}p_s},$$

it follows that $A_{\Omega_s} = R_s + \sum_{j=1}^N \sum_{k=1}^{2^{n_j}-1} T_{j,k}$, where each $T_{j,k}$ is a product of n_j Toeplitz operators and at least one of them has the form $K^{\Omega_s} M_{\mathcal{X}_{\Omega_s - \Omega}} M_{\phi_{j,i}p_s}$. Since $\int_{\Omega_s} \int_{\Omega_s} |K^{\Omega_s}|^2 |\mathcal{X}_{\Omega_s - \Omega}|^2 |\phi_{j,i}p_s|^2 dzdw < \infty$, $T_{j,k}$ is compact for any j and k. Therefore $R_s - A_{\Omega_s}$ is compact. With the definition of A_{Ω_s} in mind, the fact that $P_s(H^2(\Omega)) = H^2(\Omega_s)$ and the result proved in the last theorem, we can state the following.

Corollary 4.2. *If A is a member of $B(H^2(\Omega))$ which can be written as*

$$\sum_{j=1}^{N}\prod_{k=1}^{m_j} T_{\phi_{j,k}},$$

then A is compact if and only if $A_{\Omega_s} \in B(H^2(\Omega_s))$ is compact for $s = 1, \dots, n$.

To prove the main theorem it is important to observe that

Lemma 4.3. *Suppose that $\phi \in \mathcal{C}^\infty(\bar\Omega)$ and it is a Lipschitz function; then the operator $[P, M_\phi]$ is compact.*

Proof. We observe that $[P, M_\phi]$ is an integral operator, the kernel of which has the form $F(z,w) = (\phi(z) - \phi(w))K^\Omega(z,w)$. Using the same decomposition of Ω as in Lemma 3.3 we can write that $\Omega = H \cup (\bigcup_{\ell=}^{n} G_\ell)$, where $\forall \ell = 1, \dots, n$, $G_\ell = \{z | r_\ell < |z - a_\ell| \le r_\ell + \epsilon\}$, ϵ is such that $\forall j \ne k$ $\bar G_k \cap \bar G_j = \varnothing$ and $H = \Omega - (\bigcup_{j=1}^{n} G_{j.})$. Observe that $H \ne \varnothing$. Given such a partition of Ω we can conclude that $[P, M_\phi]$ is compact if and only if the operator $\sum_{\ell=1}^{n} \mathcal{X}_{G_\ell}(z)\mathcal{X}_{G_\ell}(w)(\phi(z) - \phi(w))K^\Omega(z,w)$ is a compact operator.

Claim. The operator $S_\ell(z,w) = \mathcal{X}_{G_\ell}(z)\mathcal{X}_{G_\ell}(w)(\phi(z) - \phi(w))K^\Omega(z,w)$ is in the Schatten class C_q, where $q = p(p-1)^{-1}$, $p \in (1,2)$ and $\ell = 1, \dots, n$.

Proof. We know (see [9]) that to prove our claim it is enough to show that $\mathcal{S}_\ell \in \mathrm{L}^p(G_\ell \times G_\ell)$ for $1 < p < 2$. By Lemma 3.2 we know that in $G_\ell \times G_\ell$ the inequality $|K^\Omega(z,w)|^p |K^{\Omega_\ell}(z,w)|^{-p} \le D^p$ holds and since ϕ is a Lipschitz function, there exists a constant M such that

$$\int_{G_\ell}\int_{G_\ell} |\phi(w) - \phi(z)|^p |K^\Omega(z,w)|^p dzdw \le \int_{G_\ell}\int_{G_\ell} \frac{|z-w|^p D^p M^p r_\ell^{2p} dzdw}{|r_\ell^2 - \overline{(z-a_\ell)}(w-a_\ell)|^{2p}}.$$

If we define $\varsigma = (z - a_\ell)r_\ell^{-1}$ and $\omega = (w - a_\ell)r_\ell^{-1}$, then we can write

$$\int_{G_\ell}\int_{G_\ell} D^p M^p r_\ell^{2p} \frac{|z-w|^p dzdw}{|r_\ell^2 - \overline{(z-a_\ell)}(w-a_\ell)|^{2p}} = \int_{\tilde G_\ell}\int_{\tilde G_\ell} D^p M^p r_\ell^{p} \frac{|\zeta - \omega|^p}{|1 - \bar\zeta\omega|^{2p}} d\zeta d\omega,$$

where $\tilde G_\ell = \{\zeta | 1 < |\zeta| \le 1 + \epsilon r_\ell^{-1}\}$. Since $|\ (\zeta - \omega)(1 - \bar\zeta\omega)^{-1}\ |$ is bounded in $\tilde G_\ell \times \tilde G_\ell$, we can conclude that $\int_{G_\ell}\int_{G_\ell} |\phi(w) - \phi(z)|^p |K^\Omega(z,w)|^p dzdw < \infty$ if and only if

$$I = \int_{\tilde G_\ell}\int_{\tilde G_\ell} \frac{d\zeta d\omega}{|1 - \bar\zeta\omega|^p} < \infty.$$

A simple calculation shows that I is finite if $p < 2$; then we can conclude that the operator S_ℓ is a C_q operator. Therefore the operator is a compact operator and this implies that $[P, M_\phi]$ is compact. □

Lemma 4.4. *Let $\Omega = \bigcap_{t=1}^{n} \Omega_t$ be a canonical multiply-connected domain. Suppose that $A \in \mathcal{B}(H^2(\Omega))$ is an operator which can be written as $A = T_{\phi_1} \dots T_{\phi_m}$, where $\phi_i \in L^\infty(\Omega)$ for $i = 1, \dots, m$ and $p_s \in \mathcal{P}$, where $\mathcal{P} = \{p_1, \cdots, p_n\}$ is a ∂-partition. If we define A_{p_s} to be $T_{\phi_1 p_s} \dots T_{\phi_m p_s}$, then $\lim_{\zeta \to \partial\Omega_s} \tilde A(\zeta) = \lim_{\zeta \to \partial\Omega_s} \tilde A_{p_s}(\zeta)$.*

Proof. We observe that $A = (T_{\phi_1(1-p_s)} + T_{\phi_1 p_s}) \dots (T_{\phi_m(1-p_s)} + T_{\phi_m p_s})$. If we perform all the products we obtain 2^m terms. One of them is A_{p_s}. If we write A_j for the others $j = 1, \dots, (2^m - 1)$ summands, this means that $A = A_{p_s} + \sum_{j=1}^{2^m-1} A_j$. Then it is easy to see that each A_j can be written as $T_{\psi_1^j}...T_{\psi_\ell^j(1-p_s)}...T_{\psi_m^j}$, where $\psi_k^j \in L^\infty(\Omega)$ for any j,k. The point here is that one of the Toeplitz operators which appears in the product has as its symbol a function which vanishes in a neighborhood of $\partial\Omega_s$. We claim that for any $j = 1,\ldots,(2^m-1)$ there exists a compact operator K_j such that $T_{\psi_1^j}...T_{\psi_\ell^j(1-p_s)}T_{\psi_{\ell+1}^j}...T_{\psi_m^j} = T_{\psi_1^j}...T_{\psi_\ell^j}T_{\psi_{\ell+1}^j(1-p_s)}...T_{\psi_m^j} + K_j$. Let us prove this claim. We observe that to prove this claim it is enough to show that the operator $B = T_{\psi_\ell^j(1-p_s)}T_{\psi_{\ell+1}^j}$ - $T_{\psi_\ell^j}T_{\psi_{\ell+1}^j(1-p_s)}$ is compact. Since Lemma 4.3 implies that $B_1 = T_{\psi_\ell^j}T_{(1-p_s)} - T_{\psi_\ell^j(1-p_s)}$ and $B_1 = T_{(1-p_s)}T_{\psi_{\ell+1}^j} - T_{\psi_\ell^j(1-p_s)}$ are both compact operators, then we can conclude that $B_1T_{\psi_{\ell+1}^j}$ and $T_{\psi_\ell^j}B_2$ are both compact and this fact proves our claim. If we repeat this process we can conclude that for any $j = 1,\ldots,(2^m-1)$ there exists a compact operator K_j such that $A_j = T_{\psi_1^j}...T_{\psi_\ell^j}...T_{\psi_m^j(1-p_s)} + K_j$. If we define $A_j^r = T_{\psi_1^j}...T_{\psi_\ell^j}...T_{\psi_m^j(1-p_s)}$, then $A - A_{p_s} = \sum_{j=1}^{2^m-1} A_j^r + \sum_{j=1}^{2^m-1} K_j$. Therefore to prove our lemma it is enough to show that $\lim_{\zeta\to\partial\Omega_s} \| T_{\psi_m^j(1-p_s)}k_\zeta^\Omega \| = 0$. Since $K_0^\Omega(w,z) = K^\Omega(w,z) - \sum_{\ell=1}^n K^{\Omega_\ell}(w,z)$ simple calculations imply that

$$\lim_{z\to\partial\Omega_s} \sum_{\ell\neq s}^n \frac{||\psi_m^j(1-p_s)K_z^{\Omega_\ell}||_2}{||K_z^\Omega||_2} = 0 \text{ and } \lim_{z\to\partial\Omega_s} \frac{||\psi_m^j(1-p_s)K_{0,z}^\Omega||_2}{||K_z^\Omega||_2} = 0.$$

To complete our proof we need to prove that

$$\lim_{z\to\partial\Omega_s} ||T_{\psi_m^j(1-p_s)}\frac{K_z^{\Omega_s}}{||K_z^\Omega||_2}||_2 = 0.$$

We have $||T_{\psi_m^j(1-p_s)}K_z^{\Omega_s}||K_z^\Omega||_2^{-1}\;||_2^2 \leq ||M_{\psi_m^j(1-p_s)}K_z^{\Omega_s}||K_z^\Omega||_2^{-1}||_2^2$ and this implies that

$$||T_{\psi_m^j(1-p_s)}\frac{K_z^{\Omega_s}}{||K_z^\Omega||_2}||_2^2 \leq ||\psi_m^j||_\infty^2 \frac{||K_z^{\Omega_s}||_2^2}{||K_z^\Omega||_2^2}\int_{\Omega_s} |(1-p_s)(w)|^2 |k_z^{\Omega_s}(w)|^2.$$

Using the definition of the Berezin Transform we can write that

$$||\psi_m^j||_\infty^2 \frac{||K_z^{\Omega_s}||_2^2}{||K_z^\Omega||_2^2}\int_{\Omega_s} |(1-p_s)(w)|^2 \frac{|K_z^{\Omega_s}(w)|^2}{||K_z^{\Omega_s}||_2^2} = ||\psi_m^j||_\infty^2 \frac{||K_z^{\Omega_s}||_2^2}{||K_z^\Omega||_2^2}\tilde{T}_{|1-p_s|^2}(z).$$

To complete our proof, we observe that $||K_z^{\Omega_s}||_2^2||K_z^\Omega||_2^{-2} \leq 1 \;\forall z\in\Omega$ and $\tilde{T}_{|1-p_s|^2}(z) \to 0$ as $z\to\partial\Omega_s$ since $T_{|1-p_s|^2} \in B(H^2(\Omega_s))$ is compact and this implies that

$$\lim_{z\to\partial\Omega_s} ||T_{\psi_m^j(1-p_s)}\frac{K_z^{\Omega_s}}{||K_z^\Omega||_2}||_2 = 0.$$

Remark 1. We observe that $A = (\sum_0^n K^{\Omega_\ell})M_{\psi_1 p_s}...(\sum_0^n K^{\Omega_\ell})M_{\psi_m p_s}$. Moreover, since $\forall \ell \neq s$ and $j = 1,\ldots,m$ we have that $K_\ell^\Omega M_{\psi_j p_s}$ and $M_{\psi_j p_s}K_\ell^\Omega$ are compact, therefore $A_{p_s} = K_s^\Omega M_{\psi_1 p_s}\ldots K_s^\Omega M_{\psi_m p_s} + \mathbf{k}$, where $\mathbf{k} \in B(H^2(\Omega))$ is a compact operator.

If we define $A_s = K_s^\Omega M_{\psi_1 p_s}\ldots K_s^\Omega M_{\psi_m p_s}$, then we obtain that $\tilde{A}_{p_s}(z) = \tilde{A}_s(z) + \tilde{\mathbf{k}}(z)$. Therefore $\lim_{z\to\partial\Omega_s}\tilde{A}_{p_s}(z) = \lim_{z\to\partial\Omega_s}\tilde{A}_s(z)$.

Claim. If $A_{\Omega_s} \in B(H^2(\Omega_s))$, where $A_{\Omega_s} = \prod_{\ell=1}^m (K^{\Omega_s}M_{\psi_\ell p_s})$, and $\tilde{A}_{p_s}(z) \to 0$ as $z\to\partial\Omega_s$, then $\lim_{z\to\partial\Omega_s}\tilde{A}_{\Omega_s}(z) = 0$.

Before we prove the claim we observe that in the first limit we take the Berezin Transform in Ω and in the second limit we take the Berezin Transform in Ω_s.

Proof. We know that $\lim_{z\to\partial\Omega_s}\tilde{A}_{p_s}(z) = \lim_{z\to\partial\Omega_s}\tilde{A}_s(z)$ and that $\tilde{A}_s(z) = \langle A_s k_z^\Omega, k_z^\Omega\rangle$. A direct calculation implies that $\lim_{z\to\partial\Omega_s}||(\sum_{\ell\neq s}K_{\ell,z}^\Omega)||K_z^\Omega||_2^{-1}|| = 0$; therefore

$$\begin{aligned}\lim_{z\to\partial\Omega_s} A_s(z) &= \lim_{z\to\partial\Omega_s}\langle A_s\frac{K_{s,z}^\Omega}{||K_z^\Omega||_2}, \frac{K_{s,z}^\Omega}{||K_z^\Omega||_2}\rangle\\ &= \lim_{z\to\partial\Omega_s}\left(\frac{||K_z^{\Omega_s}||_2}{||K_z^\Omega||_2}\right)^2\langle A_s k_z^{\Omega_s}, k_z^{\Omega_s}\rangle.\end{aligned}$$

By Lemma 3.2 we know that there exists a constant $D > 0$ such that $0 < D^{-1} \leq ||K_z^{\Omega_s}||_2^2||K_z^\Omega\ ||_2^{-2}\leq 1$ if z is close enough to $\partial\Omega_s$. Therefore we have proved that $\lim_{z\to\partial\Omega_s}\tilde{A}_{p_s}(z) = 0$ implies $\lim_{z\to\partial\Omega_s}\langle A_s k_z^{\Omega_s}, k_z^{\Omega_s}\rangle = 0$. Now we observe that $A_s - A_{\Omega_s} = K_s^\Omega M_{\phi_1 p_s}\dots K_s^\Omega M_{\phi_n p_s} - K^{\Omega_s}M_{\phi_1 p_s}\dots K^{\Omega_s}M_{\phi_n p_s}$ is a compact operator. In fact, for any j we have

$$K_s^\Omega M_{\phi_j p_s}f(w) = K^{\Omega_s}M_{\phi_j p_s}f(w) - K_s^\Omega M_{\mathcal{X}_{\Omega_s-\Omega}}M_{\phi_j p_s}f(w)$$

and

$$\int_{\Omega_s}\int_{\Omega_s}|K^{\Omega_s}(z,w)|^2|\mathcal{X}_{\Omega_s-\Omega}(z)|^2dzdw < \infty.$$

It follows that $\lim_{z\to\partial\Omega_s}\langle A_s k_z^{\Omega_s}, k_z^{\Omega_s}\rangle = \lim_{z\to\partial\Omega_s}\langle A_{\Omega_s}k_z^{\Omega_s}, k_z^{\Omega_s}\rangle$ and this completes the proof of the claim.

Remark 2. If A is an operator on $H^2(\Omega)$ which can be written as $\sum_{j=1}^m\prod_{k=1}^{m_j}T_{u_{j,k}}$, where $u_{j,k}\in L^\infty(\Omega)$, $p_s\in\mathcal{P}$, where $\mathcal{P} = \{p_1,\cdots,p_n\}$ is a ∂-partition and $A_{p_s} = \sum_{j=1}^m\prod_{k=1}^{m_j}T_{u_{j,k}p_s}$, then $\lim_{\zeta\to\partial\Omega_s}\tilde{A}(\zeta) = \lim_{\zeta\to\partial\Omega_s}\tilde{A}_{p_s}(\zeta)$. Moreover, if A_{Ω_s} is defined as before, then it follows that $\lim_{z\to\partial\Omega_s}\tilde{A}_{p_s}(z) = 0$ implies $\lim_{z\to\partial\Omega_s}\tilde{A}_{\Omega_s}(z) = 0$.

Proof. Since the operator A can be written as a *finite* sum of finite products of Toeplitz operators it is enough to rewrite Lemma 4.4 and Remark 1 with the obvious modifications.

Theorem 4.5. *Let Ω be a bounded multiply-connected domain in the complex plain* $\mathbf{C}$. *If A is an operator on $H^2(\Omega)$ which can be written as $\sum_{j=1}^m\prod_{k=1}^{m_j}T_{u_{j,k}}$, where $u_{j,k}\in L^\infty(\Omega)$, then A is a compact operator if and only if $\tilde{A}(\zeta)\to 0$ as $z\to\partial\Omega$.*

Proof. Since $k_\lambda^\Omega\to 0$ weakly in $L^2(\Omega)$ and A is compact we have $||Ak_\lambda^\Omega||_2\to 0$ and this implies that $\lim_{\zeta\to\partial\Omega}\tilde{A}(\zeta) = 0$. Now let us suppose that $\tilde{A}(\zeta)\to 0$; then for any $s = 1,\dots,n$ we have $\lim_{\zeta\to\partial\Omega_s}\tilde{A}(\zeta) = 0$. Then Lemma 4.4 implies that $\lim_{\zeta\to\partial\Omega_s}\tilde{A}_{p_s}(\zeta) = 0$, where $p_s\in\mathcal{P}$ and $\mathcal{P} = \{p_1,\dots,p_n\}$ is a ∂-partition of Ω.

Using the remarks which follow Lemma 4.4 and the definition of A_{Ω_s} we can conclude that $\lim_{\zeta\to\partial\Omega_s}\tilde{A}_{\Omega_s}(\zeta) = 0$. Since $A_{\Omega_s}\in B(H^2(\Omega_s))$, the observation after Thorem 3.1 implies that A_{Ω_s} is compact. Since this is true for $s = 1,\dots,n$, Corollary 4.2 implies that A is compact and we are done. ◻

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