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COMPACT RANGE PROPERTY AND OPERATORS ON C^*-ALGEBRAS

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ABSTRACT. We prove that a Banach space E has the compact range property (CRP) if and only if, for any given C^*-algebra $\mathcal{A}$, every absolutely summing operator from $\mathcal{A}$ into E is compact. Related results for p-summing operators ($0 < p < 1$) are also discussed as well as operators on non-commutative L^1-spaces and C^*-summing operators.

1. INTRODUCTION

A Banach space E is said to have the compact range property (CRP) if every E-valued countably additive measure of bounded variation has compact range. It is well known that every Banach space with the Radon-Nikodym property (RNP) has the (CRP) and for dual Banach spaces, the (CRP) were completely characterized as those whose predual do not contain any copies ℓ^1. For more in depth discussions on Banach spaces with the (CRP), we refer to [9].

The following characterization can be found in [9]: A Banach space E has the (CRP) if and only if every 1-summing operator from $C[0,1]$ into E is compact. Since $C[0,1]$ is a (commutative) C^*-algebra, it is a natural question whether $C[0,1]$ can be replaced by any C^*-algebras. Let us recall that in [7] it was shown that if X is a Banach space that does not contain any copies of ℓ^1, then any 1-summing operator from any given C^*-algebra into X^* is compact, hinting that, as in commutative case, the (CRP) is the right condition to provide compactness. The present note is an improvement of [7]. Our main result confirms that, if $\mathcal{A}$ is a C^*-algebra and E is a Banach space that has the (CRP), then every 1-summing from $\mathcal{A}$ into E is compact. Our proof relies on factorizations of summing operators used in [7] and properties of integral operators.

There is another well known characterization of spaces with the (CRP) in terms of operators defined on $L^1[0,1]$: a Banach space E has the (CRP) if and only every operator T from $L^1[0,1]$ into E is Dunford-Pettis (completely continuous). Thus the (CRP) is also referred to as the *complete continuity property* (CCP). Unlike the 1-summing operators on C^*-algebras, operators defined on non-commutative L^1-spaces do not behave the same way as those defined on $L^1[0,1]$ do. In the

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last section of this note, we will discuss these operators along with C^*-summing operators studied by Pisier in [6].

Our terminology and notation are standard as may be found in [2] and [4] for Banach spaces, [5] and [8] for C^*-algebras and operator algebras.

2. Preliminaries

In this section, we recall some definitions.

Definition 1. Let X and Y be Banach spaces and $0 < p < \infty$. An operator $T : X \to Y$ is said to be p-summing if there is a constant C such that for any finite sequence $(x_1, x_2, \dots, x_n)$ of X, one has

$$\left(\sum_{i=1}^{n} \|Tx_i\|^p\right)^{\frac{1}{p}} \leq C \sup\left\{\left(\sum_{i=1}^{n} |\langle x_i, x^*\rangle|^p\right)^{\frac{1}{p}} ;\ x^* \in X^*, \|x^*\| \leq 1\right\}.$$

The smallest constant C for which the above inequality holds is denoted by $\pi_p(T)$ and is called the p-summing norm of T.

Definition 2. We say that an operator $T : X \to Y$ is an integral operator if it admits a factorization:

$$\begin{array}{ccc} X & \xrightarrow{i\circ T} & Y^{**} \\ \Big\downarrow{\scriptstyle \alpha} & & \Big\uparrow{\scriptstyle \beta} \\ L^{\infty}(\mu) & \xrightarrow{J} & L^{1}(\mu) \end{array}$$

where i is the natural inclusion from Y into Y^{**}, μ is a probability measure on a compact space K, J is the natural inclusion and α and β are bounded linear operators.

We define the integral norm $i(T) := \inf\{\|\alpha\| \cdot \|\beta\|\}$ where the infinum is taken over all such factorizations.

Similarily, we shall say that T is *strictly integral* if T is integral and on the factorization above β takes its values in Y.

It is well known that integral operators are 1-summing but the converse is not true.

If $X = C(K)$ where K is a compact Hausdorff space, then it is well known that every 1-summing operator from X into Y is integral.

For more details on the different properties of the classes of operators involved, we refer to [3].

The following simple fact will be needed in the sequel.

Proposition 3. *Let $T : X \to Y$ be a strictly integral operator. If Y has the (CRP), then T is compact.*

Proof. The operator T has a factorization $T = \beta \circ J \circ \alpha$ where $\alpha : X \to L^{\infty}(\mu)$, $J : L^{\infty}(\mu) \to L^{1}(\mu)$ and $\beta : L^{1}(\mu) \to Y$ are as in the above definition. Note that J is 1-summing so $\beta \circ J : L^{\infty}(\mu) \to Y$ is 1-summing and since $L^{\infty}(\mu)$ is a $C(K)$-space and Y has the (CRP), $\beta \circ J$ (and hence T) is compact. □

We recall that a von Neumann algebra $\mathcal{M}$ is said to be σ-finite if the identity is countably decomposable, equivalently if there exists a faithful state $\varphi \in \mathcal{M}_*$. As is customary, for every functional $\varphi \in \mathcal{M}_*$ and $x \in \mathcal{M}$, $x\varphi$ (resp. φx) denotes the normal functional $y \to \varphi(yx)$ (resp. $y \to \varphi(xy)$).

3. Main result

Theorem 4. *For a Banach space E, the following are equivalent:*

(1) *E has the CRP;*

(2) *Every 1-summing operator $T : C[0,1] \to E$ is compact;*

(3) *For any given C^*-algebra $\mathcal{A}$, every 1-summing operator $T : \mathcal{A} \to E$ is compact.*

The equivalence (1) $\Leftrightarrow$ (2) is well known; we refer to [4], [9] for more details. Clearly (3) $\Rightarrow$ (2) so what we need to show is (1) $\Rightarrow$ (3). For this, it is enough to consider the following particular case (see [7] for this reduction).

Proposition 5. *Let E be a Banach space with the (CRP) and $\mathcal{M}$ be a σ-finite von Neumann algebra. If $T : \mathcal{M} \to E$ is 1-summing and is weak* to weakly continuous, then T is compact.*

Proof. The proof is a refinement of the argument used in Proposition 3.2 of [7]. We will include most of the details for completeness. Without loss of generality, we can and do assume that E is separable.

Let $\delta > 0$. From Lemma 2.3 of [7],

$$\|Tx\| \leq 2(1+\delta)\pi_1(T)\|xf + fx\|_{\mathcal{M}_*} \quad \text{for every } x \in \mathcal{M},$$

where f is a faithful normal state in $\mathcal{M}_*$. If $L^2(f)$ is completion of the prehilbertian space $(\mathcal{M}, \langle \cdot, \cdot \rangle)$ where $\langle x, y \rangle = f(\dfrac{xy^* + y^*x}{2})$, then we have the following factorization:

$$\begin{array}{ccccc} \mathcal{M} & \xrightarrow{T} & E & & \\ \downarrow J & & & \nwarrow L & \\ L^2(f) & \xrightarrow{\theta} & L^2(f)^* & \xrightarrow{J^*} & \mathcal{M}_* \end{array}$$

where J is the inclusion map, $\theta(Jx) = \langle \cdot, J(x^*) \rangle$ for every $x \in \mathcal{M}$ and $L(\frac{xf+fx}{2}) = Tx$. We recall that L is a well defined bounded linear map since $\{xf + fx;\ x \in \mathcal{M}\}$ is dense in $\mathcal{M}_*$ and $\|L(xf+fx)\| \leq 4(1+\delta)\pi_1(T)\|xf+fx\|_{\mathcal{M}_*}$. Let $S := J^* \circ \theta \circ J$.

Claim: *$J \circ L^* : E^* \to L^2(f)$ is compact.*

For this, let us consider $L^* : E^* \longrightarrow \mathcal{M}$. Since E is separable, it is isometric to a subspace of $C[0,1]$. Let I_E be the isometric embedding of E in $C[0,1]$ and i be the natural inclusion of $C[0,1]$ into $C[0,1]^{**}$.

Define the following map $\widetilde{T}$ from $\mathcal{M}$ into $C[0,1]$ by setting $\widetilde{T} = \overline{I_E \circ T(x^*)}$ for every $x \in \mathcal{M}$. (Here, $\overline{f}$ is the map $t \to \overline{f(t)}$ for $f \in C[0,1]$ with $\overline{f(t)}$ being the conjugate of the complex number $f(t)$.)

Clearly, $\widetilde{T}$ is linear and bounded and it can be shown that $\widetilde{T}$ is 1-summing and is weak* to weakly continuous. In fact, if $(x_1, x_2, \dots, x_n)$ is a finite sequence in $\mathcal{M}$, then

$$\begin{aligned}\sum_{i=1}^{n} \|\widetilde{T}x_i\| &= \sum_{i=1}^{n} \|\overline{I_E \circ T(x_i^*)}\| \\ &= \sum_{i=1}^{n} \|I_E \circ T(x_i^*)\| \\ &= \sum_{i=1}^{n} \|T(x_i^*)\| \\ &\leq \pi_1(T) \sup\left\{\sum_{i=1}^{n} |\langle x_1^*, \varphi\rangle|,\ \varphi \in \mathcal{M}^*,\ \|\varphi\| \leq 1\right\} \\ &\leq \pi_1(T) \sup\left\{\sum_{i=1}^{n} |\langle x_i, \varphi^*\rangle|,\ \varphi \in \mathcal{M}^*,\ \|\varphi\| \leq 1\right\},\end{aligned}$$

so $\widetilde{T}$ is 1-summing with $\pi_1(\widetilde{T}) \leq \pi_1(T)$. Moreover if $(x_\alpha)_\alpha$ is a net that converges to zero weak* in $\mathcal{M}$, so does the net $(x_\alpha^*)_\alpha$ and since T is weak* to weakly continuous, $(T(x_\alpha^*))_\alpha$ converges to zero weakly in E and hence $(\widetilde{T}(x_\alpha))_\alpha$ is weakly null which shows that $\widetilde{T}$ is weak* to weakly continuous.

To complete the proof, consider

$$E^* \xrightarrow{L^*} \mathcal{M} \xrightarrow{i\circ\widetilde{T}} C[0,1]^{**}.$$

Since $C[0,1]^{**}$ has the Hahn-Banach extension property and $i \circ \widetilde{T}$ is 1-summing, $i \circ \widetilde{T}$ is an integral operator. Let $K : C[0,1]^* \longrightarrow \mathcal{M}_*$ such that $K^* = i \circ \widetilde{T}$ (such an operator exists since $i \circ \widetilde{T}$ is weak* to weakly continuous); K is integral ([3]) and since $\mathcal{M}_*$ is a complemented subspace of its bidual $\mathcal{M}^*$ (see for instance [8]), K is strictly integral and therefore $L \circ K : C[0,1]^* \to E$ is strictly integral and by Proposition 3, $L \circ K$ (and hence $(L \circ K)^* = i \circ \widetilde{T} \circ L^*$) is compact.

Let (U_n) be a bounded sequence in E^*. There exists a subsequence (U_{n_k}) so that $(i \circ \widetilde{T} \circ L^*(U_{n_k}))_k$ is norm convergent in $C[0,1]^{**}$. Since i and I_E are isometries, we get that $(T \circ L^*(U_{n_k}))_k$ is norm convergent so

$$\lim_{k,m} \|T(L^*(U_{n_k})^*) - T(L^*(U_{n_m})^*)\| = 0.$$

As in [7], we get

$$\begin{aligned}&\lim_{k,m} \langle T(L^*(U_{n_k})^*) - T(L^*(U_{n_m})^*), U_{n_k} - U_{n_m}\rangle \\ &\quad = \lim_{k,m} \|J \circ L^*(U_{n_k} - U_{n_m})\|^2_{L^2(f)} = 0\end{aligned}$$

which proves that $(J \circ L^*(U_{n_k}))_k$ is norm-convergent in $L^2(f)$. The proof is complete. □

Theorem 6. *Let $\mathcal{A}$ be a C^*-algebra, E be a Banach space and $0 < p < 1$. Every p-summing operator from $\mathcal{A}$ into E is compact.*

Proof. Let $T : \mathcal{A} \to E$ be an operator with $\pi_p(T) < \infty$. One can choose, by the Pietsch Factorization Theorem, a probability space (Ω, Σ, μ) such that

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{T} & E \\ \downarrow J & & \uparrow \widetilde{T} \\ S & \xrightarrow{i_p} & S_p \\ \downarrow & & \downarrow \\ L^\infty(\mu) & \xrightarrow{j_p} & L^p(\mu) \end{array}$$

where S is a subspace of $L^\infty(\mu)$, S_p is the closure of S in $L^p(\mu)$ and i_p is the restriction of the natural inclusion j_p.

Denote by S_1 the closure of S in $L^1(\mu)$, by i_1 the restriction of the natural inclusion and $i_{1,p}$ the natural inclusion of S_1 into S_p.

Claim: $\widetilde{T} \circ i_{1,p} : S_1 \to E$ is weakly compact.

To see this, let $(f_n)_n$ be a bounded sequence in $S_1 \subset L^1(\mu)$. By Komlòs's Theorem, there exist a subsequence $(f_{n_k})_k$ and a function $f \in L^1(\mu)$ such that $\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^m f_{n_k}(\omega) = f(\omega)$ for a.e. $\omega \in \Omega$. Since $0 < p < 1$,

$$\lim_{m\to\infty} \| \frac{1}{m} \sum_{k=1}^m f_{n_k} - f \|_p = 0.$$

This shows that $f \in S_p$ and $\left(\widetilde{T} \circ i_{1,p} \left(\frac{1}{m} \sum_{k=1}^m f_{n_k} \right) \right)_m$ converges to $\widetilde{T}(f)$ in E and the claim follows.

Using the factorization of a weakly compact operator [1], $i_{1,p} \circ \widetilde{T}$ factors through a reflexive space and since $i_1 \circ J$ is 1-summing, the theorem follows from Theorem 4. □

4. Concluding remarks

Let us recall some definitions

Definition 7. Let X and Y be Banach spaces. An operator $T : X \to Y$ is called Dunford-Pettis if T sends weakly compact sets into norm compact sets.

The following class of operators was introduced by Pisier in [6] as extension of the p-summing operators in the setting of C^*-algebras.

Definition 8. Let $\mathcal{A}$ be a C^*-algebra and E be a Banach space, $0 < p < \infty$. An operator $T : \mathcal{A} \to E$ is said to be p-C^*-summing if there exists a constant C such that for any finite sequence $(A_1, \dots, A_n)$ of Hermitian elements of $\mathcal{A}$, one has

$$\left(\sum_{i=1}^n \|T(A_i)\|^p \right)^{\frac{1}{p}} \leq C \Big\| \left(\sum_{i=1}^n |A_i|^p \right)^{1/p} \Big\|_{\mathcal{A}}.$$

Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful tracial state τ and let J be the canonical inclusion map from $\mathcal{M}$ into $L^1(\mathcal{M}, \tau)$. As in the commutative case, we have the following:

Proposition 9. *Let E be a Banach space and $T : L^1(\mathcal{M},\tau) \to E$ a bounded linear map. Then the following are equivalent:*

(i) *T is Dunford-Pettis;*
(ii) *$T \circ J$ is compact.*

Proof. (i) $\implies$ (ii) is trivial. For the converse, let $(a_n)_n$ be a weakly null sequence in the unit ball of $L^1(\mathcal{M},\tau)$. It is clear that $(a_n^*)_n$ is also weakly null so without loss of generality, we can assume that $(a_n)_n$ is a sequence of self-adjoint operators. For each $n \geq 1$, set $a_n = \int_{-\infty}^{\infty} t\ de_t^{(n)}$, the spectral decomposition of a_n, and for every $N \geq 1$, let

$$p_{n,N} = \int_{-N}^{N} 1\ de_t^{(n)}.$$

It is clear that for every $n \geq 1$ and $N \geq 1$,

$$\tau(\mathbf{1} - p_{n,N}) = \tau(\int_{\{|t|>N\}} 1\ de_t^{(n)}) \leq \frac{1}{N}\tau(|a_n|).$$

By Akeman's characterization of relatively weakly compact subset in $L^1(\mathcal{M},\tau)$ (see for instance [8], Theorem 5.4, p.149), we conclude that for any given $\epsilon > 0$, there is $N_0 \geq 1$ such that for every $n \geq 1$, $\|a_n(\mathbf{1} - p_{n,N_0})\| \leq \epsilon$. Moreover $(a_n p_{n,N_0})_n$ is a bounded sequence in $\mathcal{M}$ and since $T \circ J$ is compact, there is a compact subset K_ϵ of E such that $\{T(a_n);\ n \in \mathbb{N}\} \subset K_\epsilon + \epsilon B_E$. The proof is complete. □

Fix a type II_1 von Neumann algebra $\mathcal{M}$ such that $\mathcal{M}$ contains a complemented copy of a Hilbert space H. The space H is reflexive (and therefore has (CRP)) but the projection map P from $L^1(\mathcal{M},\tau)$ onto H cannot be Dunford-Pettis.

A very well known property of p-summing operators is that they are Dunford-Pettis. This in not the case for C^*-summing operators in general. By Proposition 9, $P \circ J$ is not compact. We remark that the argument used in [7] requires only that the operator is C^*-summing and Dunford-Pettis; hence since J is clearly C^*-summing and $P \circ J$ is not compact, $P \circ J$ should not be Dunford-Pettis.

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