

## SEMIDIRECT SUM OF GROUPS IN WHICH ENDOMORPHISMS ARE GENERATED BY INNER AUTOMORPHISMS

FENG-KUO HUANG

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ABSTRACT. An I–E group is a group  $G$  in which every endomorphism is finitely generated by its inner automorphisms. In this paper a characterization for a semidirect sum of I–E groups to be an I–E group is obtained and some well-known results are generalized. We then use this characterization to prove that a semidirect sum of finite I–E groups will again be an I–E group if the normal semidirect summand is unique and fully invariant. Conditions for a group to be an I–E group are also given.

### 1. INTRODUCTION

A group  $G$  is called an I–E group if all endomorphisms of  $G$  are generated by inner automorphisms.  $\text{Inn}(G)$  and  $\text{End}(G)$  denote the set of all inner automorphisms and endomorphisms of  $G$ , respectively. The group operation is denoted additively even when  $G$  is not necessarily abelian. Consequently, we use semidirect sum instead of semidirect product. Herein we use the right-hand mapping convention:  $a(fg) = (af)g$  for all  $a \in G$ . The terminology used in this paper follows Meldrum [18] and Robinson [21].

The study of I–E groups can be traced back to at least two different origins. In 1963, L. Fuchs [9] had raised the following question: *For which abelian groups  $G$  do their automorphism groups  $\text{Aut}(G)$  generate the endomorphism ring  $\text{End}(G)$ ?* R. S. Pierce [20], R. W. Stringall [23], H. Freedman [7] and F. Castagna [3] gave certain results on both the positive and the negative sides of this question. During the same period, A. Fröhlich [8] had shown that finite simple groups are I–E groups. The next step in this direction was taken by J. J. Malone and C. G. Lyons; they showed that a dihedral group  $D_n$  of order  $2n$  is an I–E group only if  $n$  is odd [12, 13]. In a further step, J. J. Malone had shown that the generalized quaternion groups  $Q_n$  are not I–E groups [15]. These investigations were recently generalized by C. G. Lyons and G. Mason in [14]; they proved that dicyclic groups of order  $4n$  with  $n$  odd are I–E groups. Next Y. Fong and J. D. P. Meldrum proved that symmetric groups  $S_n$  with  $n > 4$  are I–E groups [5, 6]. In [22], G. Saad, M. J. Thomsen, and S. A. Syskin claimed alternating groups  $A_n$  with  $n \neq 4$ , and special linear groups  $SL(n, q)$ , except  $SL(2, 3)$ , are all I–E groups. For a detailed history of I–E groups, refer either to [16] or [22].

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The above results primarily determine some concrete examples for which a group  $G$  is I-E. Recently, questions concerning the structure of I-E groups have been considered. J. J. Malone and G. Mason [17] have shown: *a semidirect sum of cyclic groups of relatively prime order is I-E when the cyclic normal semidirect summand is the commutator subgroup*. C. G. Lyons and G. L. Peterson [10] then made the following improvement: *a semidirect sum of cyclic groups of relatively prime order is I-E*. Observing that a finite abelian group is I-E if and only if it is cyclic, S. A. Syskin [25] proves: *a semidirect sum of two I-E groups of relatively prime orders is an I-E group*.

Their results depend heavily on [10, Theorem 2.1], which characterizes when a semidirect product of I-E groups of relatively prime orders is an I-E group. In this paper, not assuming  $G$  to be finite, this characterization theorem is generalized and, at the same time, a fairly concise proof is provided in Theorem 2.1. We then use this result, in Theorem 2.11, to prove that a semidirect sum of I-E groups will be an I-E group if the normal semidirect summand is fully invariant and is a unique minimal normal subgroup. Some conditions to ensure that a group will be an I-E group are also given in Proposition 2.10 and in Theorem 3.4. Moreover, we briefly discuss the direct sum of I-E groups. Using Theorem 3.4, we generalize another result by C. G. Lyons and G. L. Peterson in Corollary 3.5. Examples are provided to illustrate and delimit our results.

The near-ring generated by the group of inner automorphisms  $\text{Inn}(G)$  is denoted by  $I(G)$ , and  $E(G)$  will denote the near-ring generated by the endomorphisms  $\text{End}(G)$ . For a group  $G$  and its subgroups  $H$  and  $K$ , the centralizer of  $H$  in  $K$  will be denoted as  $C_K(H)$ . Let  $I(G, H) = \{f \in I(G) \mid Gf \subseteq H\}$  and  $E(G, H) = \{f \in E(G) \mid Gf \subseteq H\}$ . Moreover  $\mathcal{J}_2(N)$  denotes the  $\mathcal{J}_2$  radical of the near-ring  $N$ . Details about the  $\mathcal{J}_2$  radical can be found in [18].

## 2. SEMIDIRECT SUMS OF I-E GROUPS

Recall the  $I(G)$ -subgroups [18, p. 157] are equivalent to normal subgroups of  $G$ , and the  $E(G)$ -subgroups are equivalent to the fully invariant subgroups of  $G$ . Therefore a necessary condition for  $G$  to be an I-E group is that each normal subgroup must be fully invariant. It can easily be shown that this condition is equivalent for a group to be an I-E group when  $G$  is finitely generated abelian. Unfortunately, this condition is not sufficient in general. For example, the group  $A_4$  satisfies this condition but fails to be an I-E group [22]. The following result characterizes the I-E property for a group, which is a semidirect sum of a fully invariant subgroup  $H$  and a subgroup  $K$ , in terms of the behavior of the projection map.

**Theorem 2.1.** *Let  $G$  be semidirect sum of a fully invariant subgroup  $H$  and a subgroup  $K$  of  $G$ . Suppose  $H$  and  $K$  are both I-E groups. Then the following are equivalent:*

- (1)  $G$  is an I-E group.
- (2)  $I(G, H) = E(G, H)$ .
- (3) The projection map  $p: G \rightarrow K$  is in  $I(G)$  and  $p\alpha(1-p) \in I(G)$  for all  $\alpha \in \text{End}(G)$ .

*Proof.* (1) implies (2) and (2) implies (3) are clear. We want to show that (3) implies (1).

Let  $\alpha \in \text{End}(G)$  and write  $\alpha = (1 - p)\alpha + p\alpha$  where 1 denotes the identity map of  $G$ . Since  $H$  is fully invariant,  $H\alpha \subseteq H$ . Thus

$$\alpha|_H = (1 - p)\alpha|_H \in \text{End}(H).$$

Since  $H$  is an I-E group, we have  $(1 - p)\alpha|_H = \tilde{r}$  for some  $\tilde{r} \in I(H)$ . Because  $\tilde{r} = \sum_{i=1}^n \varepsilon_i \rho_{h_i}$  where each  $\rho_{h_i}$  is an inner-automorphism induced by  $h_i \in H$  and  $\varepsilon_i \in \{1, -1\}$  for  $i \in \{1, 2, \dots, n\}$ , we may view  $\tilde{r} = r|_H$  where  $r = \sum_{i=1}^n \varepsilon_i \rho_{h_i} \in I(G)$ . Note that  $1 - p \in I(G)$  by our hypothesis and hence  $(1 - p)r \in I(G)$ . It follows that  $(1 - p)\alpha = (1 - p)r \in I(G)$ .

Now, we need to show  $p\alpha \in I(G)$ . Write  $p\alpha = p\alpha(1 - p) + p\alpha p$ . It suffices to show that  $p\alpha p \in I(G)$ . Note that  $p\alpha p$  is an endomorphism of  $G$  and  $Gp\alpha p \subseteq K$ . Therefore  $p\alpha p|_K \in \text{End}(K)$ .

By using an argument similar to that used above to prove  $(1 - p)\alpha \in I(G)$ , we obtain  $p\alpha p \in I(G)$ . Therefore  $p\alpha \in I(G)$  and  $E(G) = I(G)$  as desired.  $\square$

The following example shows that in Theorem 2.1 the requirement  $p\alpha(1 - p) \in I(G)$  is indeed necessary.

**Example 2.2.** Let  $G = A_5 \oplus \mathbb{Z}_5$ . Observe that both  $A_5$  and  $\mathbb{Z}_5$  are I-E groups and that  $A_5$  is a fully invariant subgroup of  $G$ . In fact,  $A_5$  is the commutator subgroup of  $G$ . From [25, Theorem 3], we know the projection map  $p: G \rightarrow \mathbb{Z}_5$  is in  $I(G)$ . However, the order of  $E(G)$  is  $59^{59} \cdot 5 \cdot 60^{240}$  and the order of  $I(G)$  is  $59^{59} \cdot 5$  by using the results presented in [24]. Therefore  $G$  is not an I-E group.

**Corollary 2.3.** *Let  $G$  be semidirect sum of a fully invariant subgroup  $H$  and a subgroup  $K$  of  $G$ . Let  $p: G \rightarrow K$  be the projection map. Suppose  $H$  and  $K$  are both I-E groups and  $\{p\} \cup \mathcal{J}_2(E(G)) \subseteq I(G)$ . Then  $G$  is an I-E group.*

*Proof.* It is routine to verify that  $p\alpha(1 - p)\beta$  is nilpotent for all  $\alpha, \beta \in E(G)$ . Since  $\mathcal{J}_2(E(G))$  contains all nil right  $E(G)$ -subgroups [18, Corollary 5.24],  $\mathcal{J}_2(E(G))$  must contain  $p\alpha(1 - p)E(G)$ , and in particular contains all the elements  $p\alpha(1 - p)$  for all  $\alpha \in \text{End}(G)$ . By Theorem 2.1,  $G$  is an I-E group.  $\square$

A 0-symmetric near-ring  $N$  is called 2-primal if the prime radical  $\mathcal{P}_0(N)$  is equal to the completely prime radical  $\mathcal{P}_2(N)$ . Examples for 2-primal near-rings are abundant. Let  $G$  be finite dihedral group with order not divisible by 4 or the generalized quaternion group. Then  $E(G)$  is 2-primal. For more details, please refer to [1, 2].

**Corollary 2.4.** *Let  $G$  be semidirect sum of a fully invariant subgroup  $H$  and a subgroup  $K$  of  $G$ . Let  $p: G \rightarrow K$  be the projection map. Suppose  $H$  and  $K$  are both I-E groups and  $E(G)$  is 2-primal. If  $\{p\} \cup \mathcal{P}_0(E(G)) \subseteq I(G)$ , then  $G$  is an I-E group.*

*Proof.* Since  $\mathcal{P}_0(E(G))$  contains all the nilpotent elements including  $p\alpha(1 - p)$  for all  $\alpha \in \text{End}(G)$ , the assertion follows from Theorem 2.1.  $\square$

The utility of Corollary 2.3 and Corollary 2.4 can be readily demonstrated by considering  $E(S_3)$  where  $S_3$  is the symmetric group of order 6. Observe that  $\{p\} \cup \mathcal{J}_2(E(S_3)) \subseteq I(S_3)$  and  $E(S_3)$  is 2-primal with  $\{p\} \cup \mathcal{P}_0(E(S_3)) \subseteq I(S_3)$  [11]. Hence  $E(S_3)$  illustrates both Corollary 2.3 and 2.4. As a corollary, we obtain one of the main results of C. G. Lyons and G. L. Peterson [10].

**Corollary 2.5** ([10, Theorem 2.1]). *Suppose that  $G$  is the semidirect sum of a normal subgroup  $H$  and a subgroup  $K$  where  $(|H|, |K|) = 1$  and  $H, K$  are both I-E groups. Then the following are equivalent:*

- (1)  $G$  is an I-E group.
- (2) The projection map  $p : G \rightarrow K \in I(G)$ .

*Proof.* Let  $\pi$  be the set of prime factors of  $|H|$ . The hypothesis that  $(|H|, |K|) = 1$  implies that  $G$  is a Hall  $\pi$ -separable group and  $H$  is the unique Hall  $\pi$ -subgroup of  $G$ . From Theorem 9.1.6 in [21] and the fact that the homomorphic image of a  $\pi$ -subgroup is a  $\pi$ -subgroup, we see that  $H$  is a fully invariant subgroup of  $G$ . Observe that  $p\alpha(1-p) = 0$  for all  $\alpha \in \text{End}(G)$  when the order of  $H$  and  $K$  are relatively prime. By applying Theorem 2.1 above, we obtain the result.  $\square$

The following example shows that Theorem 2.1 is a proper generalization of the Lyons-Peterson result [10, Theorem 2.1].

**Example 2.6.** Let  $G = A_8 \oplus PSL(3, 4)$  where  $A_8$  is the alternating group of degree 8 and  $PSL(3, 4)$  is the projective special linear group of order  $20160 = 8!/2$ . Note that  $PSL(3, 4)$  is not isomorphic to  $A_8$ ; for it can be demonstrated that  $PSL(3, 4)$  has no elements of order 15, unlike  $A_8$  which has  $(1, 2, 3, 4, 5)(6, 7, 8)$ , an element of order 15. Let  $p : G \rightarrow PSL(3, 4)$  be the projection map. Observe that both  $A_8$  and  $PSL(3, 4)$  are fully invariant subgroups of  $G$ . Hence  $p\alpha(1-p) = 0 \in I(G)$  for all  $\alpha \in \text{End}(G)$ . The restriction map  $p|_{PSL(3,4)}$  is an endomorphism of  $PSL(3, 4)$  which is an I-E group. Therefore  $p|_{PSL(3,4)} \in I(PSL(3, 4))$ . On the other hand,  $p|_{A_8} = 0 \in I(A_8)$  and so  $p \in I(G)$ . Hence  $G$  is an I-E group by Theorem 2.1. Note that the orders of  $A_8$  and  $PSL(3, 4)$  are not relatively prime. In fact, they have the same order.

Let  $G$  be a semidirect sum of a normal subgroup  $H$  and a subgroup  $K$ . From [19, Lemma 4.1], we know that  $K$  being an I-E group is necessary for  $G$  to be an I-E group. In Example 2.7, we exhibit a group  $G$  such that  $H$  is fully invariant in  $G$  and  $K$  is an I-E group with  $(|H|, |K|) = 1$ . Moreover, the projection map  $p : G \rightarrow K$  is in  $I(G)$ , but  $G$  is not an I-E group. Therefore assuming that  $H$  is an I-E group in Corollary 2.5 is not superfluous. However, Example 2.8 shows that, in general,  $G$  can be an I-E group with  $H$  fully invariant, but  $H$  is not an I-E group.

**Example 2.7.** We first quote a result from [22, Theorem 16]: Let  $G$  have a minimal normal subgroup  $H$  of order  $p^n$  for some prime  $p$  and  $n \geq 1$  such that  $C_G(H) = H$  and  $G/H$  is cyclic of order  $q$  prime to  $p$ . Then  $I(G) = E(G)$  if and only if  $n = 1$ .

Let  $H \cong \bigoplus_{i=1}^n \mathbb{Z}_2$  where  $n \neq 1$  and  $K = \mathbb{Z}_3$ . Let  $G$  be the semidirect sum of  $H$  with  $K$ .  $G$  is not an I-E group. Here it can be seen that  $H$  is a fully invariant subgroup of  $G$  and  $K$  is an I-E group. But  $H \cong \bigoplus_{i=1}^n \mathbb{Z}_2$  is not an I-E group when  $n \neq 1$ . Note that the projection map  $p : G \rightarrow K$  is in  $I(G)$  by [25, Theorem 2].

**Example 2.8.** Let  $G$  be the symmetric group  $S_4$  of degree 4 and consider  $S_4$  as the semidirect sum of the alternating group  $A_4$  and the cyclic group  $\mathbb{Z}_2$ .  $S_4$  is an I-E group but the fully invariant subgroup  $A_4$  is not an I-E group.

Using the condition of relatively prime on the order of a subgroup and its index, Proposition 2.10 shows that the I-E property can be lifted from a maximal normal subgroup. Note that Proposition 2.10 can be deduced from [25, Theorem 2]. We will provide a detailed constructive proof for easier reference and hopefully motivate

some clue to improve this result. The following lemma was quoted from [25] which we will use in Proposition 2.10.

**Lemma 2.9** ([25, Theorem 1]). *Let  $G$  be a finite group with a unique minimal normal subgroup  $H$ . Assume that  $G$  is a semidirect sum of  $H$  and a subgroup  $K$ . Then the projection map  $p: G \rightarrow K$  is in  $I(G)$ .*

**Proposition 2.10.** *Let  $G$  be a finite group with a maximal normal subgroup  $H$  such that the order of  $H$  is coprime to its index. If  $H$  is an I-E group, then  $G$  is an I-E group.*

*Proof.* By using the Schur–Zassenhaus theorem, we know the complement  $K$  of  $H$  in  $G$  exists. Since  $G/H$  is simple, it is an I-E group. From Corollary 2.5, we can conclude that  $G$  is an I-E group if the projection map  $p: G \rightarrow K$  is in  $I(G)$ .

Note that the centralizer of  $H$  in  $K$ , denoted  $C_K(H)$ , is 0 or  $K$  by the maximality of  $H$ . If  $C_K(H) = K$ , then  $G = H \oplus K$  with  $(|H|, |K|) = 1$ . Since both  $H$  and  $K$  are I-E groups,  $G$  is an I-E group by [10, Corollary 2.2].

So assume  $C_K(H) = 0$ . If  $H$  is also a minimal normal subgroup, then it is unique and so  $p \in I(G)$  by Lemma 2.9. Suppose  $H$  is not minimal. Let  $0 = H_n \subseteq H_{n-1} \subseteq \dots \subseteq H_0 = H$  be a principal series of  $H$ . Since  $H$  is an I-E group, each  $H_i$  is a fully invariant subgroup of  $H$  and thus a normal subgroup of  $G$  for all  $i = 0, 1, \dots, n$ . Therefore  $H_{n-1}$  is a minimal normal subgroup of  $G$ .

We now want to show that  $p_{n-1}: H_{n-1} + K \rightarrow K$  is in  $I(H_{n-1} + K)$ . Since  $H_{n-1}$  is a minimal normal subgroup of  $G$ ,  $H_{n-1} = \bigoplus_{i=1}^m S_i$  where the  $S_i$  are mutually isomorphic simple groups. Moreover  $K \cong G/H$  is also a simple group by maximality of  $H$ . Let  $G_{n-i} = H_{n-i} + K$  for all  $i = 1, 2, \dots, n$ . The centralizer  $C_K(H_{n-1})$  is a normal subgroup of  $K$ , and so must be 0 or  $K$ .

*Case I:* Assume  $C_K(H_{n-1}) = K$ .

Let  $s$  be the order of  $H_{n-1}$  and let  $t$  be the order of  $K$ . Since  $s, t$  are relatively prime, there exist integers  $u, v$  such that  $us + vt = 1$ . Now for all  $h \in H_{n-1}, k \in K$ ,  $us(h + k) = ush + usk = (1 - vt)k = k$ . Therefore  $p_{n-1} = us1 \in I(G_{n-1})$ , where 1 denotes the identity map, is the desired projection from  $G_{n-1}$  to  $K$ .

*Case II:* Assume  $C_K(H_{n-1}) = 0$ .

If  $H_{n-1}$  is a minimal normal subgroup of  $G_{n-1}$ , it is unique and so  $p_{n-1}: G_{n-1} \rightarrow K$  is in  $I(G_{n-1})$  by Lemma 2.9. If  $H_{n-1}$  is not minimal, then without loss of generality, we assume  $Q = S_1 \oplus S_2 \oplus \dots \oplus S_r$  with  $r < m$  a minimal normal subgroup of  $G_{n-1}$ . By repeating the argument in Case I, we may assume  $Q$  is a minimal normal subgroup of  $Q + K$  and  $C_K(Q) = 0$ . Therefore the projection  $q_1: Q + K \rightarrow K$  is in  $I(Q + K)$ .

Applying the above arguments inductively on the group  $\bar{G}_{n-1} = G_{n-1}/Q \cong S_{r+1} \oplus S_{r+2} \oplus \dots \oplus S_m + K$ , there exists  $q_2 \in I(G_{n-1})$  such that for all  $h \in H_{n-1}, k \in K$ , we have  $(h + k)q_2 = c + k$  for some  $c \in Q$ . A routine argument yields that  $p_{n-1} = q_2q_1 \in I(G_{n-1})$  is the desired projection from  $G_{n-1}$  to  $K$ .

Now, let  $\bar{G} = G/H_{n-1}$ . Then  $\bar{H}_{n-2}$  is a minimal normal subgroup of  $\bar{G}$ . Similar reasoning as in Cases I and II above yields that  $\bar{p}_{n-2}: G_{n-2}/H_{n-1} \rightarrow K/H_{n-1}$  is in  $I(G_{n-2}/H_{n-1})$ . Hence there exist maps  $p_{n-1}$  and  $\nu \in I(G_{n-2})$  such that for all  $x \in H_{n-1}, h \in H_{n-2}$ , and  $k \in K$ , we have  $(x + k)p_{n-1} = k$  and  $(h + k)\nu = y + k$  for some  $y \in H_{n-1}$ . Therefore  $(h + k)\nu p_{n-1} = (y + k)p_{n-1} = k$ . So  $p_{n-2} = \nu p_{n-1} \in I(G_{n-2})$  is the desired projection from  $G_{n-2}$  to  $K$ .

Inductively, we conclude that the projection  $p = p_0: G_0 = G \rightarrow K$  is in  $I(G)$ . This completes the proof. □

Note that when coprimeness is not assumed in Proposition 2.10, there is no obvious evidence to ensure the lifting of the I–E property from a maximal normal subgroup  $H$  to  $G$ . In Example 2.2,  $A_5$  is maximal in  $G$ , but  $G$  is not an I–E group. On the other hand, the I–E condition cannot be inherited by a maximal normal subgroup, in general, as shown in Example 2.8.

In the final result of this section, we do not assume the relatively prime condition on the order of  $H$  and  $K$ .

**Theorem 2.11.** *Let  $G$  be a finite group and let  $G$  be a semidirect sum of a fully invariant subgroup  $H$  and a subgroup  $K$ . Suppose  $H$  is a unique minimal normal subgroup of  $G$ . If  $H$  and  $K$  are both I–E groups, then  $G$  is an I–E group.*

*Proof.* Note that the assumption on  $H$  implies the projection map  $p \in I(G)$  by Lemma 2.9.

We first consider the case when  $H$  is nonabelian. Since  $H$  is a minimal normal subgroup of  $G$ ,  $H$  is characteristically simple and so the automorphism near-ring  $A(H) = M_0(H)$  by [18, Theorem 10.11]. Moreover, the hypothesis that  $H$  is an I–E group together with  $A(H) = M_0(H)$  implies  $I(H) = M_0(H)$  and thus  $H$  is a finite simple nonabelian group by a result of Frölich [8].

Since  $H$  is normal in  $G$ , the centralizer  $C_G(H)$  is a normal subgroup of  $G$ . By uniqueness of  $H$ ,  $C_G(H)$  must be contained in  $H$  and therefore  $C_G(H) = 0$  because  $H$  is simple nonabelian.

Let  $\alpha \in \text{End}(G)$  be arbitrary. If  $Gp\alpha(1-p) = K\alpha(1-p) = 0$ , then  $p\alpha(1-p) = 0 \in I(G)$ . If  $K\alpha(1-p) \neq 0$ , let  $W = \{k \in K \mid k\alpha(1-p) \neq 0\}$ . Observe that  $1 - \rho_a \in I(G, H)$  for all  $a \in H$  where  $\rho_a$  is the inner automorphism of  $G$  induced by element  $a$ . Let  $k \neq 0 \in K$ . If  $k(1 - \rho_a) = 0$  for all  $a \in H$ , then  $k \in C_G(H) = 0$  and thus there must exist some  $b \in H$  with respect to  $k$  such that  $k(1 - \rho_b) \neq 0$ . Hence the hypothesis for Theorem 10.24 in [18] holds for the set  $K \setminus \{0\}$ . By applying [18, Theorem 10.24], there exists  $\eta \in I(G)$  such that

$$(2.1) \quad k\eta = \begin{cases} k\alpha(1-p), & \text{if } k \in W; \\ 0, & \text{if } k \in K \setminus W. \end{cases}$$

It is now easy to verify that  $p\alpha(1-p) = p\eta \in I(G)$  for all  $\alpha \in \text{End}(G)$ . By Theorem 2.1,  $G$  is an I–E group.

Now consider the case when  $H$  is abelian. The assumption that  $H$  is I–E and finite abelian implies that  $H$  is a cyclic group of prime order. Moreover, that  $H$  is a unique minimal normal subgroup of  $G$  implies on the one hand the centralizer  $C_G(H) = H$  because  $C_G(H)$  is a normal subgroup of  $G$ , and on the other hand, any two nonzero normal subgroups  $H_1, H_2$  of  $G$  satisfying the commutator  $[H_1, H_2] = 0$  must have  $H_1 = H_2 = H$ .

It is now not difficult to verify that the hypothesis for Theorem 3.5 in [4] holds for the set  $K \setminus \{0\}$ . By using this theorem, there exists an  $\eta \in I(G)$  such that, for any  $\alpha \in \text{End}(G)$ , we have  $k\alpha(1-p) = k\eta$  for all  $k \in K$ . Hence  $p\alpha(1-p) = p\eta \in I(G)$  and thus  $G$  is an I–E group by Theorem 2.1.  $\square$

As a quick application of the above result, we immediately have the symmetric group  $S_n$  with  $n \geq 5$  and the dihedral group  $D_n$  of order  $2n$  with  $n$  odd are all I–E groups.

## 3. DIRECT SUMS OF I-E GROUPS

**Proposition 3.1.** *Let  $G$  be direct sum of normal subgroups  $H$  and  $K$  of  $G$ . Then:*

(1) *If  $G$  is an I-E group, then  $H$  and  $K$  are both I-E groups. In particular, all direct components of an I-E group are I-E groups.*

(2) *Suppose that both  $H$  and  $K$  are fully invariant subgroups of  $G$ . If  $H$  and  $K$  are both I-E groups, then  $G$  is an I-E group.*

*Proof.* (1) Since endomorphisms of  $H$  and  $K$  extend to endomorphisms of  $G$  and elements of  $I(G)$  restricted to  $H$  and  $K$  yield elements of  $I(H)$  and  $I(K)$ , it follows that  $H$  and  $K$  will be I-E groups if  $G$  is an I-E group.

(2) Let  $\alpha \in \text{End}(G)$ . Since  $H$  is fully invariant,  $\alpha$  restricted to  $H$  gives an endomorphism of  $H$ . Assuming that  $H$  is an I-E group, we may represent  $\alpha|_H$  as a finite sum of inner automorphisms of  $H$ . Therefore  $\alpha|_H \in I(G)$ . Similarly,  $\alpha|_K \in I(G)$ . Let  $\mu \in I(H)$ ,  $\nu \in I(K)$  such that  $\alpha|_H = \mu|_H$  and  $\alpha|_K = \nu|_K$ . Note that here  $\mu|_K = 1|_K$  and  $\nu|_H = 1|_H$  where 1 is the identity map on  $G$ . It is then routine to verify that  $\alpha = \mu - 1 + \nu \in I(G)$ .  $\square$

From Proposition 3.1(1), we know that the direct summand of an I-E group is an I-E group. However, this property does not hold for a fully invariant subgroup. Observe that  $V_4$  (i.e., Klein 4-group) is a fully invariant subgroup of the I-E group  $S_4$ , but  $V_4$  is not an I-E group. In Proposition 3.1(2), the requirement that both  $H$  and  $K$  be fully invariant is not superfluous as we can see in the following examples. Recall that an abelian group is I-E if and only if it is cyclic. So a direct sum of cyclic groups is not I-E if it is not cyclic.

(1) In the infinite case, consider the group  $G$  as the direct sum of the integers  $\mathbb{Z}$  and the group  $\mathbb{Z}_2$  of order 2.  $G$  is not an I-E group, but both  $\mathbb{Z}$  and  $\mathbb{Z}_2$  are I-E groups. Here  $\mathbb{Z}$  is not a fully invariant subgroup of  $G$ , but  $\mathbb{Z}_2$  is an I-E group.

(2) In the finite case, consider  $V_4$  as the direct sum of two copies of  $\mathbb{Z}_2$ .  $V_4$  is not an I-E group but  $\mathbb{Z}_2$  is an I-E group. However  $\mathbb{Z}_2$  is not a fully invariant subgroup of  $V_4$ .

Making use of Proposition 3.1, we can reprove the following corollary by C. G. Lyons and G. L. Peterson [10] without using Corollary 2.5.

**Corollary 3.2** ([10, Corollary 2.2]). *If  $G$  is the direct sum of  $H$  and  $K$  where both  $H$  and  $K$  are I-E groups and if  $(|H|, |K|) = 1$ , then  $G$  is an I-E group.*

*Proof.* We first show that  $H$  and  $K$  will be fully invariant subgroups of  $G$ . Pick  $h \in H$  and  $\alpha \in \text{End}(G)$ . Then  $h\alpha = a+b$  for some  $a \in H$  and  $b \in K$ . Suppose the order of  $H$  is  $m$  and the order of  $K$  is  $n$ . Then  $0 = (mh)\alpha = m(h\alpha) = ma + mb = mb$ , which then implies that the order of  $b$ , say  $|b|$ , is a factor of  $m$ . But  $|b|$  must divide  $n$  and so  $|b|$  is a common factor of  $m$  and  $n$ . Therefore  $|b| = 1$  and  $h\alpha = a \in H$ . Hence  $H$  is a fully invariant subgroup of  $G$ . Similarly,  $K$  is a fully invariant subgroup of  $G$ . The result then follows from Proposition 3.1.  $\square$

**Corollary 3.3.** *A finite nilpotent group is an I-E group if and only if all its Sylow subgroups are I-E groups.*

*Proof.* Recall that a finite group is nilpotent if and only if it is the direct sum of its Sylow subgroups. By Proposition 3.1 and Corollary 3.2, a finite nilpotent group is an I-E group if and only if all its Sylow subgroups are I-E groups.  $\square$

**Theorem 3.4.** *Let  $G$  be a finite group with an abelian normal subgroup  $H$  such that the order of  $H$  is coprime to its index. If  $G/H$  and the centralizer  $C_G(H)$  are I-E groups, then  $G$  is an I-E group.*

*Proof.* By the Schur-Zassenhaus theorem, the complement  $K$  of  $H$  in  $G$  exists. Since  $H$  is a normal subgroup of  $G$ , the centralizer  $C_G(H)$  is normal in  $G$ . Note that  $C_G(H)$  is the direct sum of  $H$  and  $C_K(H)$ . By Proposition 3.1(1),  $H$  is an I-E group. Since the order of  $H$  and  $K$  are coprime, the projection map  $p : G \rightarrow K$  is in  $I(G)$  by [25, Theorem 2]. Also  $K \cong G/H$  is an I-E group; therefore  $G$  is an I-E group by Corollary 2.5.  $\square$

As a corollary, we obtain Theorem 3.2 [10] of C. G. Lyons and G. L. Peterson [10, Theorem 3.2].

**Corollary 3.5** ([10, Theorem 3.2]). *If  $G$  is the semidirect sum of a cyclic normal subgroup  $H$  and a cyclic subgroup  $K$  where  $(|H|, |K|) = 1$ , then  $G$  is an I-E group.*

*Proof.* Note that  $C_K(H)$ , as a subgroup of a cyclic group  $K$ , is cyclic.  $C_K(H)$  is an I-E group. Because the order of  $H$  and  $C_K(H)$  are relatively prime,  $C_G(H) = H \oplus C_K(H)$  is an I-E group by Corollary 3.2. Hence  $G$  is an I-E group by Theorem 3.4.  $\square$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LOUISIANA AT LAFAYETTE, LAFAYETTE,  
LOUISIANA 70504–1010

*E-mail address:* fxb2858@us1.edu