

ANALYTIC NORMS IN ORLICZ SPACES

P. HÁJEK AND S. TROYANSKI

(Communicated by Dale Alspach)

ABSTRACT. It is shown that an Orlicz sequence space h_M admits an equivalent analytic renorming if and only if it is either isomorphic to l_{2n} or isomorphically polyhedral. As a consequence, we show that there exists a separable Banach space admitting an equivalent C^∞ -Fréchet norm, but no equivalent analytic norm.

In this note, we denote by h_M as usual the subspace of an Orlicz sequence space l_M generated by the unit vector basis.

More terminology and notation concerning Orlicz spaces can be found in [LT].

Let us also point out that by C^k -smoothness (or analyticity) of a norm we always mean away from the origin (as is usual in renorming theory).

The characterization of the best order of C^k -Fréchet smoothness of some renorming, $k \in \mathbb{N} \cup \{+\infty\}$, for h_M was obtained in [M], [MT1], [MT2]. In our present note, we complete the characterization also for analytic renormings. We show that an Orlicz sequence space h_M has an analytic renorming if and only if $h_M \cong l_{2n}$, $n \in \mathbb{N}$, or h_M is isomorphically polyhedral. Let us recall that a separable Banach space X is isomorphically polyhedral if it has an equivalent polyhedral norm. By a theorem of Fonf [F], this is the case if and only if X admits an equivalent norm with a countable boundary. More precisely, there exists a sequence $\{f_i\}_{i \in \mathbb{N}}$ in X^* such that

$$\|x\| = \max\{|f_i(x)|, \quad i \in \mathbb{N}\}.$$

According to one of the results from [DFH], we have the following:

Theorem 1. *Every separable isomorphically polyhedral Banach space X admits an equivalent analytic form.*

We prove that the converse is also true if we impose additional conditions on the space X . In connection with our result it should be noted that by recent work of Gonzalo and Jaramillo ([GJ]) every separable Banach space with a symmetric basis and C^∞ -Fréchet smooth norm is isomorphic to l_{2n} , provided it does not contain a copy of c_0 .

Our approach is entirely different from that in [MT1] and relies on methods from [DFH] and [H1]–[H3]. As a corollary, relying on an example of Leung [L], we show that there exists a separable Banach space with C^∞ -Fréchet smooth norm which

Received by the editors March 30, 1998 and, in revised form, November 25, 1998.

2000 *Mathematics Subject Classification.* Primary 46B03, 46B45.

Key words and phrases. Analytic norm, Orlicz space, polyhedral space.

The second author was partially supported by NFSR of Bulgaria, Grant MM-808-98.

admits no analytic norm. A search of such an example was in fact a motivation of our work, since the previously known examples of such spaces (e.g. $c_0(\Gamma)$, Γ uncountable; see [P] and [BF] for a result of Kuiper) were nonseparable.

Let us recall that a Banach space X with an unconditional basis is said to satisfy an *upper p -estimate*, $p \geq 1$, if for some $C > 0$:

$$\left\| \sum_{i=1}^n u_i \right\| \leq C \left(\sum_{i=1}^n \|u_i\|^p \right)^{\frac{1}{p}}$$

whenever u_i are disjointly supported in X .

An important notion in our consideration is that of weak sequential continuity.

Definition 2. Let $\mathcal{U} \subseteq X$ be an open, convex and bounded subset of a Banach space X , f be a real function on \mathcal{U} . We say that f is *weakly sequentially continuous* (*wsc*-for short) if it maps weakly Cauchy sequences from \mathcal{U} into convergent ones. A function f defined on an open subset $\mathcal{O} \subseteq X$ is said to be *locally wsc* if there exists a covering of \mathcal{O} by a family of open sets \mathcal{U} as above such that f is wsc on \mathcal{U} for all \mathcal{U} .

In order to verify wsc-property for polynomials, it is sufficient to check the convergence only for weakly convergent sequences in \mathcal{U} ([AHV]).

Using this fact, the following lemma follows from results in [G].

Lemma 3. *Let X be a Banach space with an unconditional basis satisfying an upper p -estimate. Then all polynomials of degree $n < p$ on X are wsc (on B_X).*

The importance of the notion of wsc stems from the following lemma, which comes from [H3], and which was shown for polynomials in [AHV].

Lemma 4. *Let X be a Banach space $l_1 \not\hookrightarrow X$, f be a C^2 -Fréchet differentiable real function defined on some open set $\mathcal{O} \subseteq X$. TFAE:*

- (1) f is locally wsc,
- (2) f' is locally norm compact.

By f' being locally norm compact we mean that there exists a covering by a family of open sets \mathcal{U} of \mathcal{O} such that $f'(\mathcal{U})$ is relatively norm compact in X^* for all \mathcal{U} .

The following is a generalization of the main result in [H1].

Theorem 5. *Let $(X, \|\cdot\|)$ be a Banach space, where $\|\cdot\|$ is analytic. If all polynomials on X are wsc, then X is separable and isomorphically polyhedral.*

Proof. By ∂ we denote the duality map corresponding to $\|\cdot\|$, i.e.

$$\partial : X \setminus \{0\} \rightarrow S_{X^*} \quad \text{and} \quad \partial x(x) = \|x\| \quad \text{for all } x \in X \setminus \{0\}.$$

Since $\|\cdot\|$ is differentiable, ∂x is the derivative of $\|x\|$ at $x \in X \setminus \{0\}$.

Let us first show that $\|\cdot\|$ is locally wsc on $X \setminus \{0\}$.

Fix $x \neq 0$. Since $\|\cdot\|$ is analytic at x we can find $\delta > 0$ so that if $\|h\| < \delta$ we have

$$\|x + h\| = \sum_{n=1}^{\infty} p_n(h),$$

where p_n are homogeneous polynomials of degree n , and the convergence is uniform with respect to all $\|h\| < \delta$. Since all polynomials p_n are wsc, we see that $\|\cdot\|$ is wsc on $\{y : \|x - y\| < \delta\}$.

We proceed by showing that X is separable. Since X is an Asplund space, from Lemma 4 we get that there exist $0 < \eta < \delta$ so that the set $\{\partial y : \|x - y\| < \eta\}$ is norm relatively compact. Thus the subspace Y of X^* generated by $\{\partial y : \|y - x\| < \eta\}$ is separable. If $Y = X^*$ the proof is finished. Otherwise, assume $Y \neq X^*$. By the Hahn-Banach theorem, there exists $x^{**} \in S_{X^{**}}$ such that

$$x^{**}(\partial y) = 0 \quad \text{whenever} \quad \|y - x\| < \eta.$$

Since $\|\cdot\|$ is analytic on $X \setminus \{0\}$ we get that ∂ is analytic as well on $X \setminus \{0\}$. Hence $f = x^{**} \circ \partial$ is a real analytic function on $X \setminus \{0\}$.

Since $f(y) = 0$ for $\|y - x\| < \eta$, clearly $f \equiv 0$ on $X \setminus \{0\}$. On the other hand, by the Bishop-Phelps theorem, ∂S_X is dense in S_{X^*} , so there exists $y \in S_X$ such that $f(y) = x^{**}(\partial y) \neq 0$, a contradiction. So X is separable.

Since X is separable and $\|\cdot\|$ is locally wsc, by Lemma 4 and the Lindelöf property, $(S_X, \|\cdot\|)$ can be covered by a countable system $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$ of norm open convex bounded subsets of X such that $\partial \mathcal{U}_n$ is relatively compact. Thus the boundary of $(X, \|\cdot\|)$ can be covered by a countable system of compacts, and the result follows from [H2]. \square

Theorem 6. *Let M be an Orlicz function. Then h_M admits an equivalent analytic norm if and only if either $h_M \cong l_{2n}$, $n \in \mathbb{N}$, or h_M is isomorphically polyhedral. In particular if*

- (1) $\lim_{t \rightarrow 0} \frac{M(2t)}{M(t)} = +\infty$, then h_M has an equivalent analytic norm.
- (2) $a_M = +\infty$ and there exists a sequence $t_i \searrow 0$ such that $\sup_{i \in \mathbb{N}} \frac{M(at_i)}{M(t_i)} < +\infty$ for all $a \geq 1$, then h_M does not admit an equivalent analytic norm.

Proof. The “if” part follows from the well-known result that the canonical norm on l_{2n} , $n \in \mathbb{N}$, is analytic and from Theorem 1.

The “only if” part: By classical results ([LT]), the existence of an analytic norm on X implies $\alpha_M = \beta_M \in \{2n\}_{n \in \mathbb{N}} \cup \{+\infty\}$.

The case $\alpha_M = 2n$ implies that $X \cong l_{2n}$ by [MT1].

If $\alpha_M = \infty$, then ([LT]) X has an upper p -estimate for every $p > 1$.

Combination of Lemma 3 and Theorem 5 finishes the proof of the “only if” part.

Leung [L] showed that if M satisfies (1), then h_M is isomorphically polyhedral and if M satisfies (2) h_M is not isomorphically polyhedral. \square

Corollary 7. *There exists a c_0 -saturated separable Banach space which admits an equivalent C^∞ -Fréchet norm but no equivalent analytic norm.*

Proof. Leung [L] constructed an Orlicz function M satisfying (2). By a result of [MT2] the corresponding space h_M admits an equivalent C^∞ -Fréchet smooth norm. On the other hand, by Theorem 6, no equivalent analytic norm exists on this space. \square

Let us pass to some final remarks. A natural question is the following: Is there a separable c_0 -saturated non-polyhedral Banach space with an equivalent analytic norm?

By a careful analysis of [DFH], we obtain that on every separable polyhedral space there exists a dense set of equivalent analytic norms whose boundaries can be covered by countably many compacts. Such norms in turn immediately imply the polyhedrality of the space (using [H2]).

However, there are examples of polyhedral spaces (e.g. [S], [PS]) with analytic norms failing this property.

More precisely, the space S of Schreier has an unconditional basis $\{e_n\}$ such that the formal identity operator id from S into l_2 is bounded. It is easy to show that given an equivalent analytic norm $\|\cdot\|$ on S whose boundary is covered by countably many compacts, the equivalent analytic norm

$$\|x\| = (\|x\|^2 + \|\text{id}x\|_2^2)^{\frac{1}{2}}$$

fails the covering property.

The problem is therefore how to recognize the polyhedrality of S based on its norm $\|\cdot\|$.

ACKNOWLEDGEMENT

The authors would like to thank the Department of Analysis of Universidad Complutense in Madrid, for hospitality and excellent working conditions during the preparation of this note.

REFERENCES

- [AHV] R. Aron, C. Hervés and M. Valdivia, *Weakly continuous mappings on Banach spaces*, J. Funct. Anal. **52** (1983), 189–204. MR **84g**:46066
- [BF] R. Bonic and J. Frampton, *Smooth functions on Banach manifolds*, J. of Math. and Mech. **15** (1966), 877–898. MR **33**:6647
- [D] R. Deville, *Geometrical implications of the existence of very smooth bump functions in Banach spaces*, Israel J. Math. **67** (1989), 1–22. MR **90m**:46023
- [DFH] R. Deville, V. Fonf and P. Hájek, *Analytic and polyhedral approximation of convex bodies in separable polyhedral Banach spaces* Israel J. Math. **105** (1998), 139–154. MR **99h**:46006
- [F] V. Fonf, *One property of Lindenstrauss-Phelps spaces*, Functional Anal. Appl. **13** (1979), 66–67.
- [G] R. Gonzalo, *Upper and lower estimates in Banach sequence spaces*, Comment Math. Univ. Carolinae **36** (1995), 641–653. MR **97d**:46020
- [GJ] R. Gonzalo and J. A. Jaramillo, *Smoothness and estimates of sequences in Banach spaces*, Israel J. Math. **89** (1995), 321–341. MR **96h**:46017
- [H1] P. Hájek, *Analytic renormings of $C(K)$ spaces*, Serdica Math. J. **22** (1996), 25–28. MR **97h**:46014
- [H2] P. Hájek, *Smooth norms that depend locally on finitely many coordinates*, Proc. Amer. Math. Soc. **123** (1995), 3817–3821. MR **96e**:46017
- [H3] P. Hájek, *Smooth functions on $C(K)$* , Israel J. Math. **107** (1998), 237–252. MR **99k**:46072
- [L] D. H. Leung, *Some isomorphically polyhedral Orlicz sequence spaces*, Israel J. Math. **87** (1994), 117–128. MR **95f**:46033
- [LT] J. Lindenstrauss and R. R. Tzafriri, *Classical Banach Spaces I: Sequences Spaces*, Springer-Verlag, New York, 1977. MR **54**:3344
- [M] R. P. Maleev, *Norms of best smoothness of Orlicz spaces*, Z. Anal. Anwendungen **12** (1993), 123–135. MR **94m**:46049
- [MT1] R. P. Maleev and S. L. Troyanski, *Smooth functions in Orlicz spaces*, Contemp. Math. **85** (1989), 336–351. MR **90a**:46034
- [MT2] R. P. Maleev and S. L. Troyanski, *Smooth norms in Orlicz spaces*, Canad. Math. Bul. **34** (1) (1991), 74–82. MR **92j**:46052
- [P] A. Pelczynski, *A property of multilinear operations*, Studia Math. **16** (1957), 173–182. MR **20**:221

- [PS] A. Pelczynski and W. Szlenk, *An example of a non-shrinking basis*, Rev. Roumaine Math. Pures Appl. **10** (7) (1965), 961–966. MR **34**:3284
- [S] J. Schreier, *Ein Gegenbeispiel zur Theorie der schwachen Konvergenz*, Studia Math. **2** (1930), 58–62.

DEPARTAMENTO ANÁLISIS MATEMÁTICO, UNIVERSIDAD COMPLUTENSE DE MADRID, 28040 MADRID, SPAIN AND MATHEMATICAL INSTITUTE, CZECH ACADEMY OF SCIENCE, ŽITNÁ 25, PRAGUE, CZECH REPUBLIC

Current address: Department of Mathematics, Texas A&M University, College Station, Texas 77843

E-mail address: phajek@math.tamu.edu

DEPARTAMENTO ANÁLISIS MATEMÁTICO, UNIVERSIDAD COMPLUTENSE DE MADRID, 28040 MADRID, SPAIN AND DEPARTMENT OF MATHEMATICS AND INFORMATICS, SOFIA UNIVERSITY, 5, JAMES BOURCHIER BLVD., 1126 SOFIA, BULGARIA