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SPECTRUM OF INTERPOLATED OPERATORS

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Abstract. Let (X_0, X_1) be a compatible pair of Banach spaces and let T be an operator that acts boundedly on both X_0 and X_1. Let $T_{[\theta]} \quad (0 \le \theta \le 1)$ be the corresponding operator on the complex interpolation space $(X_0, X_1)_{[\theta]}$.

The aim of this paper is to study the spectral properties of $T_{[\theta]}$. We show that in general the set-valued function $\theta \mapsto \sigma(T_{[\theta]})$ is discontinuous even in inner points $\theta \in (0,1)$ and show that each operator satisfies the local uniqueness-of-resolvent condition of Ransford. Further we study connections with the real interpolation method.

I. Complex interpolation

Let $\bar{X} = (X_0, X_1)$ be a compatible pair of Banach spaces, i.e., $(X_0, \|\cdot\|_0)$ and $(X_1, \|\cdot\|_1)$ are Banach spaces continuously embedded in a Hausdorff topological vector space. Then $\bar{X}_\Delta = X_0 \cap X_1$ and $\bar{X}_\Sigma = X_0 + X_1$ endowed with norms $\|x\|_\Delta = \max\{\|x\|_0, \|x\|_1\}$ and $\|x\|_\Sigma = \inf\{\|a\|_0 + \|b\|_1 : a \in X_0, b \in X_1, a + b = x\}$ are Banach spaces.

Recall the construction of complex interpolation spaces $\bar{X}_{[\theta]} \quad (0 \le \theta \le 1)$ (see e.g. [C], [BL], [T]). Let $G = \{z \in \mathbf{C} : 0 < \operatorname{Re} z < 1\}$. Denote by $\mathcal{F}$ the set of all continuous functions $f : \bar{G} \to \bar{X}_\Sigma$ that are analytic on G such that $f(j + it) \in X_j \quad (j = 0, 1, t \in \mathbf{R})$ and $\lim_{|t|\to\infty} \|f(j + it)\|_j = 0 \quad (j = 0, 1)$. Then $\mathcal{F}$ with the norm

$$\|f\|_{\mathcal{F}} = \max\{\max_{t\in\mathbf{R}} \|f(it)\|_0, \max_{t\in\mathbf{R}} \|f(1 + it)\|_1\}$$

becomes a Banach space. The intermediate spaces $\bar{X}_{[\theta]}$ are defined by $\bar{X}_{[\theta]} = \{f(\theta) : f \in \mathcal{F}\}$ with the norm $\|x\|_{[\theta]} = \inf\{\|f\|_{\mathcal{F}} : f(\theta) = x\}$. Then $\bar{X}_\Delta \subset \bar{X}_{[\theta]} \subset \bar{X}_\Sigma$ and $\bar{X}_\Delta$ is dense in $\bar{X}_{[\theta]} \quad (0 \le \theta \le 1)$. Further $\bar{X}_{[0]}$ and $\bar{X}_{[1]}$ are closed subspaces of X_0 and X_1, respectively.

Clearly $\bar{X}_{[\theta]}$ can be identified with the quotient space $\mathcal{F}/N_\theta$ where $N_\theta = \{f \in \mathcal{F} : f(\theta) = 0\}$.

Let $T : \bar{X}_\Sigma \to \bar{X}_\Sigma$ be a linear mapping such that $TX_0 \subset X_0$, $TX_1 \subset X_1$ and the restrictions $T|X_j$ are bounded with respect to the norms on $X_j \quad (j = 0, 1)$ (we denote this situation by $T : \bar{X} \to \bar{X}$). Then $T\bar{X}_{[\theta]} \subset \bar{X}_{[\theta]}$ for all θ; the restriction $T|\bar{X}_{[\theta]}$ is denoted by $T_{[\theta]}$.

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It is well known that $T_{[\theta]}$ is bounded with respect to the norm on $\bar{X}_{[\theta]}$ and $\|T_{[\theta]}\| \leq \|T_0\|^{1-\theta} \cdot \|T_1\|^{\theta}$. By the spectral radius formula we also have

$$r(T_{[\theta]}) \leq r(T_0)^{1-\theta} \cdot r(T_1)^{\theta}.$$

The continuity properties of the set-valued function $\theta \mapsto \sigma(T_{[\theta]})$ were studied by a number of authors. By [Sv] (see also [A]), the function $\theta \mapsto \sigma(T_{[\theta]})$ is upper semi-continuous on $(0,1)$; at the points $0, 1$ it is in general neither upper nor lower semi-continuous (see [R]).

The spectral radius $r(T_{[\theta]})$ is continuous on $(0,1)$ and upper semi-continuous on $\langle 0,1\rangle$.

By [S], the polynomial convex hull of the spectrum $\hat{\sigma}(T_{[\theta]})$ $\bigl($= the union of $\sigma(T_{[\theta]})$ with the bounded components of $\mathbf{C}\setminus\sigma(T_{[\theta]})\bigr)$ is upper semi-continuous on $\langle 0,1\rangle$ and the outer boundary $\partial\hat{\sigma}(T_{[\theta]})$ is lower semi-continuous on $(0,1)$.

Recall that the capacity of a compact subset K of $\mathbf{C}$ is defined by

$$\operatorname{cap} K = \inf \|p\|_K^{1/\deg p},$$

where the infimum is taken over all polynomials with leading coefficient equal to 1 and $\|p\|_K = \sup\{|p(z)| : z \in K\}$. Clearly $\operatorname{cap} K = \operatorname{cap} \hat{K} = \operatorname{cap} \partial\hat{K}$. Consequently the capacity $\operatorname{cap} \sigma(T_{[\theta]})$ is continuous on $(0,1)$ and upper semi-continuous on $\langle 0,1\rangle$.

Our first example shows that the spectrum $\sigma(T_{[\theta]})$ is in general discontinuous even in inner points $\theta \in (0,1)$. This gives a negative answer to a question posed in [S].

Example 1. Let $(X_0, \|\cdot\|_0)$ and $(X_1, \|\cdot\|_1)$ be Hilbert spaces, each with orthogonal basis $\{e_j : j \in \mathbf{Z}\}$ such that $\|e_j\|_0 = 2^{-j}$ and $\|e_j\|_1 = 2^j \quad (j \in \mathbf{Z})$. For $0 \leq \theta \leq 1$ set $r_\theta = \left(\frac{1}{2}\right)^{1-\theta} \cdot 2^\theta = 2^{2\theta-1}$. Then (cf. [T], 1.18.5) the intermediate spaces $\bar{X}_{[\theta]} = (X_0, X_1)_{[\theta]}$ are Hilbert spaces of all sums $\sum_{j\in\mathbf{Z}} \alpha_j e_j$ with complex coefficients α_j such that

$$\Bigl\|\sum_{j\in\mathbf{Z}} \alpha_j e_j\Bigr\|_{[\theta]} = \Bigl(\sum_{j\in\mathbf{Z}} |\alpha_j|^2 r_\theta^{2j}\Bigr)^{1/2} < \infty.$$

Let $S : \bar{X} \to \bar{X}$ be defined by $Se_j = e_{j+1} \quad (j \in \mathbf{Z})$. Clearly $S_{[\theta]}$ is the r_θ-multiple of the unitary bilateral shift of multiplicity 1.

Let $H_0 = \bigoplus_{j\in\mathbf{Z}} X_0$, $H_1 = \bigoplus_{j\in\mathbf{Z}} X_1$ and $\bar{H} = (H_0, H_1)$. Then the intermediate spaces $H_{[\theta]}$ are orthogonal sums of infinitely many copies of $\bar{X}_{[\theta]}$. Define $T : \bar{H} \to \bar{H}$ by

$$T(\dots, x_{-1}, \boxed{x_0}, x_1, \dots) = (\dots, Sx_{-2}, \boxed{Sx_{-1}}, (S-I)x_0, Sx_1, \dots)$$

where the box denotes the zero position and I is the identity operator; the same formula also defines $T_{[\theta]}$ for all θ. For each $n \geq 1$ we have

$$\begin{aligned} &T^n(\dots, x_{-1}, \boxed{x_0}, x_1, \dots) \\ =&(\dots, S^n x_{-n-1}, \boxed{S^n x_{-n}}, (S^n - S^{n-1})x_{-n+1}, \dots, (S^n - S^{n-1})x_0, S^n x_1, \dots). \end{aligned}$$

Thus $\|T_{[\theta]}^n\| \leq r_\theta^n + r_\theta^{n-1}$ and $r(T_{[\theta]}) = \lim_{n\to\infty} \|T_{[\theta]}^n\|^{1/n} \leq r_\theta$. Hence $\sigma(T_{[\theta]}) \subset \{z \in \mathbf{C} : |z| \leq r_\theta\}$.

Let $\theta \neq 1/2$. Then $r_\theta \neq 1$ so that $S_{[\theta]} - I_{[\theta]}$ is invertible. Let m denote the injectivity modulus, $m(T_{[\theta]}) = \inf\{\|T_{[\theta]}x\| : x \in \bar{H}_{[\theta]}, \|x\|_{[\theta]} = 1\}$. Clearly

$$m(T^n_{[\theta]}) \geq \min\{r^n_\theta, |r^n_\theta - r^{n-1}_\theta|\} = |r^n_\theta - r^{n-1}_\theta|$$

(since $1/2 \leq r_\theta \leq 2$). Hence

$$r(T^{-1}_{[\theta]}) = \lim_{n\to\infty} \|T^{-n}_{[\theta]}\|^{1/n} = \lim_{n\to\infty} m(T^n_{[\theta]})^{-1/n} \geq r^{-1}_\theta \qquad (\theta \neq 1/2)$$

and thus $\sigma(T_{[\theta]}) \subset \{z \in \mathbf{C} : |z| = r_\theta\}$.

On the other hand, for $\theta = 1/2$ we have $m(S_{[1/2]} - I_{[1/2]}) = 0$ so that $m(T_{[1/2]}) = 0$ and $0 \in \sigma(T_{[1/2]})$. Thus the function $\theta \mapsto \sigma(T_{[\theta]})$ is discontinuous at $\theta = 1/2$.

In fact it is easy to check that $\sigma(T_{[\theta]}) = \{z : |z| = r_\theta\}$ for $\theta \neq 1/2$ and $\sigma(T_{[1/2]}) = \{z : |z| \leq r_{1/2} = 1\}$. □

It is not difficult to construct an example of an operator $T : \bar{X} \to \bar{X}$ with a much greater set of discontinuity points of the function $\theta \mapsto \sigma(T_{[\theta]})$ in $(0,1)$.

Since this function is upper semi-continuous, the set of all continuity points of $\sigma(T_{[\theta]})$ is a dense G_δ set (see e.g. [F]). In fact this is the only condition.

Theorem 2. *Let M be a dense G_δ subset of $(0,1)$. Then there exists a compatible pair $\bar{Z} = (Z_0, Z_1)$ of separable Hilbert spaces and an operator $W : \bar{Z} \to \bar{Z}$ such that M is the set of all continuity points of the function $\theta \mapsto \sigma(W_{[\theta]})$ in $(0,1)$.*

Proof. As in Example 1 set $r_\theta = 2^{2\theta-1}$ $\quad(0 \leq \theta \leq 1)$.

We give an outline of the proof in three steps:

(a) For each $\alpha \in (0,1)$ there is a compatible pair of separable Hilbert spaces $\bar{H} = (H_0, H_1)$ and $T : \bar{H} \to \bar{H}$ such that

$$\begin{aligned} &\sigma(T_{[\theta]}) = \{z : |z| = r_\theta\} \qquad (0 \leq \theta \leq 1, \theta \neq \alpha), \\ &\sigma(T_{[\alpha]}) = \{z : |z| \leq r_\alpha\}. \end{aligned}$$

Indeed, define $\bar{H}$ as in Example 1 and let

$$T(\dots, x_{-1}, \boxed{x_0}, x_1, \dots) = (\dots, Sx_{-2}, \boxed{Sx_{-1}}, (S - r_\alpha I)x_0, Sx_1, \dots).$$

It is easy to check that T satisfies the conditions of (a).

(b) Let F be a closed rare subset of $(0,1)$ $\quad$ (i.e., the interior of F is empty). Then there exists a compatible pair $\bar{Y} = (Y_0, Y_1)$ of separable Hilbert spaces and $V : \bar{Y} \to \bar{Y}$ such that

$$\begin{aligned} &\sigma(V_{[\theta]}) = \{z : |z| = r_\theta\} \qquad (\theta \notin F), \\ &\sigma(V_{[\theta]}) = \{z : |z| \leq r_\theta\} \qquad (\theta \in F). \end{aligned}$$

Indeed, let F_0 be a countable dense subset of F. For each $\alpha \in F_0$ let $T^{(\alpha)}$ be as in (a). Set $V = \bigoplus_{\alpha \in F_0} T^{(\alpha)}$. As in Example 1 it is easy to check that V satisfies the conditions of (b).

(c) Let M be a dense G_δ subset of $(0,1)$, i.e., $(0,1) \setminus M = \bigcup_{n=1}^\infty F^{(n)}$ where $F^{(n)}$ are closed rare sets. Set $s_n = 1 - 2^{-n}$ $\quad(n = 1, 2, \dots)$ and $r^{(n)}_\theta = 2^{2\theta-1} \cdot 2^{-(n+2)}$. As in (b) we can construct operators $V^{(n)}$ such that

$$\begin{aligned} &\sigma(V^{(n)}_{[\theta]}) = \{z : |z - s_n| = r^{(n)}_\theta\} \qquad (\theta \notin F^{(n)}), \\ &\sigma(V_{[\theta]}) = \{z : |z - s_n| \leq r^{(n)}_\theta\} \qquad (\theta \in F^{(n)}). \end{aligned}$$

Set $W = \bigoplus_{n=1}^{\infty} V^{(n)}$. It is easy to check that

$$\sigma(W_{[\theta]}) = \bigcup_{\{n:\theta \notin F^{(n)}\}} \{z : |z - s_n| = r_\theta^{(n)}\} \cup \bigcup_{\{n:\theta \in F^{(n)}\}} \{z : |z - s_n| \le r_\theta^{(n)}\} \cup \{1\}.$$

Clearly, M is the set of all continuity points of the function $\theta \mapsto \sigma(W_{[\theta]})$ in $(0,1)$. □

Let $\bar{X} = (X_0, X_1)$ be a compatible pair of Banach spaces and $T : \bar{X} \to \bar{X}$. Let $0 \le \alpha < \beta \le 1$. In general it is possible that both $T_{[\alpha]}$ and $T_{[\beta]}$ are invertible but the inverses $T_{[\alpha]}^{-1}$, $T_{[\beta]}^{-1}$ do not coincide on $\bar{X}_\Delta$. An operator $T : \bar{X} \to \bar{X}$ is said to have the uniqueness-of-resolvent (U.R.) property if the restrictions $(T_{[\alpha]} - z)^{-1}|\bar{X}_\Delta$ and $(T_{[\beta]} - z)^{-1}|\bar{X}_\Delta$ coincide for all $\alpha, \beta \in \langle 0, 1\rangle$ and $z \notin \sigma(T_{[\alpha]}) \cup \sigma(T_{[\beta]})$ (see [Z]).

In [R] a weaker property was suggested:

Definition 3. Let $\bar{X} = (X_0, X_1)$ be a compatible pair of Banach spaces. An operator $T : \bar{X} \to \bar{X}$ satisfies the local uniqueness-of-resolvent (local U.R.) condition if, for all $\alpha \in (0,1)$ and $w \notin \sigma(T_{[\alpha]})$, there exists a neighbourhood U of α such that $(T_{[\theta]} - w)^{-1}$ exists and $(T_{[\theta]} - w)^{-1}|\bar{X}_\Delta$ coincides with $(T_{[\alpha]} - w)^{-1}|\bar{X}_\Delta$.

We show that in fact each operator $T : \bar{X} \to \bar{X}$ satisfies the local U.R. condition.

The proof uses the argument outlined in [Sl2]; for the convenience of the reader we give a detailed proof.

Recall that the reduced minimum modulus $\gamma(S)$ of an operator $S : X \to Y$ is defined by $\gamma(S) = \inf\bigl\{\|Tx\| / \operatorname{dist}\{x, \ker S\} : x \in X \setminus \ker S\bigr\}$. Clearly $\gamma(S) > 0$ if and only if S has closed range.

Theorem 4. *Let $\bar{X} = (X_0, X_1)$ be a compatible pair of Banach spaces, $T : \bar{X} \to \bar{X}$ and $\alpha \in (0,1)$. Suppose that $T_{[\alpha]}$ is invertible. Then there exists a neighbourhood U of α such that $T_{[\theta]}$ is invertible for all $\theta \in U$ and $T_{[\theta]}^{-1}x = T_{[\alpha]}^{-1}x \quad (\theta \in U, x \in \bar{X}_\Delta)$.*

Proof. Let $\mathcal{F}$ be the space defined in the introduction. For $w \in G$ define the operator $V(w) : \mathcal{F} \to \mathcal{F}$ by

$$\bigl(V(w)f\bigr)(z) = (w - z)f(z) \qquad (f \in \mathcal{F}, z \in \bar{G}).$$

Clearly $\operatorname{Im} V(w) = \{f \in \mathcal{F} : f(w) = 0\}$ so that $\operatorname{Im} V(w)$ is closed and $\gamma(V(w)) > 0$ for all $w \in G$. Further $w \mapsto V(w)$ is an analytic (in fact linear affine) function,

$$V(w) = (w - \alpha)I_{\mathcal{F}} + V_0 \qquad (w \in G),$$

where $I_{\mathcal{F}}$ is the identity operator on $\mathcal{F}$ and $V_0 = V(\alpha) : \mathcal{F} \to \mathcal{F}$ is defined by $(V_0 f)(z) = (\alpha - z)f(z) \quad (f \in \mathcal{F}, z \in \bar{G})$.

For $w \in G$ write $N_w = \{f \in \mathcal{F} : f(w) = 0\}$ and define the operator $\tilde{T} : \mathcal{F} \to \mathcal{F}$ by $\bigl(\tilde{T}f\bigr)(z) = Tf(z) \quad (f \in \mathcal{F}, z \in \bar{G})$. Clearly $\|\tilde{T}\| \le \max\{\|T_0\|, \|T_1\|\}$. The operator induced by $\tilde{T}$ on the quotient space $\mathcal{F}/N_w$ is isometrically equivalent to $T_{[\operatorname{Re} w]}$.

Let c_1, c be positive constants satisfying $c_1 > \|T_{[\alpha]}^{-1}\|$ and $c > (1 + c_1\|\tilde{T}\|)\gamma(V_0)^{-1}$. Let $U_0 = \{w \in G : |w - \alpha| < c^{-1}$ and $T_{[\operatorname{Re} w]}$ is invertible$\}$. Clearly U_0 is an open neighbourhood of α.

Let $x \in \bar{X}_\Delta$. Then the function $k(z) = x\exp(z^2)$ belongs to $\mathcal{F}$. We show that there exist analytic functions $g, h : U_0 \to \mathcal{F}$ such that

$$\tilde{T}g(w) + V(w)h(w) = k \qquad (w \in U_0). \tag{1}$$

If $g(w)=\sum_{j=0}^{\infty} g_j(w-\alpha)^j$ and $h(w)=\sum_{j=0}^{\infty} h_j(w-\alpha)^j$ are the Taylor expansions of g and h about α, (1) reduces to the construction of coefficients $g_j, h_j \in \mathcal{F}$ satisfying

$$\begin{aligned} \tilde{T}g_0 + V_0 h_0 &= k,\\ \tilde{T}g_1 + V_0 h_1 &= -h_0,\\ &\vdots\\ \tilde{T}g_j + V_0 h_j &= -h_{j-1},\\ &\vdots \end{aligned} \tag{2}$$

such that the series defining g and h converge in U_0.

Since $T_{[\alpha]}$ is invertible, there exists a class $y+N_\alpha \in \mathcal{F}/N_\alpha$ such that $\tilde{T}y+N_\alpha = k+N_\alpha$ and $\|y+N_\alpha\|_{\mathcal{F}/N_\alpha} \le \|T_{[\alpha]}^{-1}\|\cdot\|k\|_{\mathcal{F}}$. Thus there exists $g_0\in\mathcal{F}$ such that $g_0+N_\alpha = y+N_\alpha$ and $\|g_0\|_{\mathcal{F}} \le c_1\|k\|_{\mathcal{F}}$. Hence $\tilde{T}g_0 - k \in N_\alpha$ and $\|\tilde{T}g_0-k\|_{\mathcal{F}} \le \|k\|_{\mathcal{F}}\cdot(1+c_1\|\tilde{T}\|)$. Therefore there exists $h_0\in\mathcal{F}$ such that $V_0h_0=\tilde{T}g_0-k$ and $\|h_0\|_{\mathcal{F}}\le c\cdot\|k\|_{\mathcal{F}}$.

In the same way we can find g_1 and $h_1 \in \mathcal{F}$ such that $\tilde{T}g_1+V_0h_1=-h_0$, $\|g_1\|_{\mathcal{F}}\le c_1c\|k\|_{\mathcal{F}}$ and $\|h_1\|_{\mathcal{F}}\le c^2\|k\|_{\mathcal{F}}$.

If we continue in this way, we construct functions $g_j,h_j\in\mathcal{F}\quad(j=0,1,\dots)$ satisfying (2) such that $\|g_j\|_{\mathcal{F}}\le c_1c^{j-1}\|k\|_{\mathcal{F}}$ and $\|h_j\|\le c^j\|k\|_{\mathcal{F}}$. Thus $g(w)=\sum_{j=0}^{\infty} g_j(w-\alpha)^j$ and $h(w)=\sum_{j=0}^{\infty} h_j(w-\alpha)^j$ are functions analytic in U_0 satisfying (1).

Define now a function $\bar{g}:U_0\to\bar{X}_\Sigma$ by

$$\bar{g}(w)=\big(g(w)\big)(w)\cdot\exp(-w^2)\qquad(w\in U_0).$$

Clearly, $\bar{g}$ is analytic in U_0 and $T\bar{g}(w)=x\quad(w\in U_0)$. Further $\bar{g}(w)\in\bar{X}_{[\operatorname{Re} w]}$ so that $\bar{g}(w)=T_{[\operatorname{Re} w]}^{-1}x$ and the function $\bar{g}$ is constant in the imaginary direction. Thus $\bar{g}$ is constant in U_0. In particular, $T_{[\theta]}^{-1}x=T_{[\alpha]}^{-1}x$ for $\theta\in U_0\cap\mathbf{R}$. ☐

By a standard argument it is possible to easily obtain the uniqueness of the inverse on any open interval:

Corollary 5. *Let $\bar{X}=(X_0,X_1)$ be a compatible pair of Banach spaces, $T:\bar{X}\to\bar{X}$, $0\le\alpha<\beta\le1$. Suppose that $T_{[\theta]}$ is invertible for all $\theta\in(\alpha,\beta)$. Then the restriction $T_{[\theta]}^{-1}|\bar{X}_\Delta$ is constant on (α,β).*

Corollary 6. *The set-valued function $w\mapsto\sigma(T_{[\operatorname{Re} w]})$ is analytic on G (for the definition and properties of analytic set-valued functions (see* [Sl1]*).*

Corollary 6 follows immediately from [R], Theorem 2.7; in fact it also follows from the general theory in [Sl2].

Corollary 7 (cf. [A], 3.7 and 3.9)**.** *Let $\bar{X}=(X_0,X_1)$ be a compatible pair of Banach spaces, $T:\bar{X}\to\bar{X}$. Then*

$$\operatorname{cap}\sigma(T_{[\theta]})\le\operatorname{cap}\sigma(T_0)^{1-\theta}\cdot\operatorname{cap}\sigma(T_1)^{\theta}$$

for all θ, $0<\theta<1$.

Proof. The previous corollary and [Au], Theorem 7.1.3, imply that the function $w\mapsto\log\operatorname{cap}\sigma(T_{[\operatorname{Re} w]})$ is subharmonic on G. Since it is constant in the imaginary direction, the following lemma shows that the function $\theta \mapsto \log \operatorname{cap} \sigma(T_{[\theta]})$ is convex in $(0,1)$. In particular, for $0 < \theta_0 < \theta < \theta_1 < 1$ we have

$$\operatorname{cap} \sigma(T_{[\theta]}) \leq \operatorname{cap} \sigma(T_{[\theta_0]})^{\frac{\theta_1 - \theta}{\theta_1 - \theta_0}} \cdot \operatorname{cap} \sigma(T_{[\theta_1]})^{\frac{\theta - \theta_0}{\theta_1 - \theta_0}}.$$

If $\theta_0 \to 0$ and $\theta_1 \to 1$, the upper semi-continuity of $\operatorname{cap} \sigma(T_{[\theta]})$ gives the statement of Corollary 7.

Lemma 8. *Let $f : G \to \mathbf{R}$ be a subharmonic function such that, for all $x \in (0,1)$, the function $y \mapsto f(x+iy)$ is constant on $\mathbf{R}$. Then the function $x \mapsto f(x)$ is convex on $(0,1)$. In particular, f is continuous on G.*

Proof. For $0 < s < t < 1$ the function

$$z \mapsto g(z) = f(z) - \frac{\operatorname{Re} z - s}{t - s} \cdot f(t) - \frac{t - \operatorname{Re} z}{t - s} \cdot f(s)$$

is subharmonic and hence attains a maximum on the compact set $\langle s, t\rangle \times \langle -1, 1\rangle$. Because of $g(s) = g(t) = 0$ and the fact that $y \mapsto g(x + iy)$ is constant on $\mathbf{R}$ for all $x \in (0,1)$, we conclude from the maximum principle for subharmonic functions that $g(z) \leq 0$ on $\langle s, t\rangle \times \langle -1, 1\rangle$. In particular, for all $\tau \in \langle s, t\rangle$ we have

$$f(\tau) - \frac{\tau - s}{t - s} \cdot f(t) - \frac{t - \tau}{t - s} \cdot f(s) \leq 0.$$

□

II. Applications to the real interpolation method

Let $\bar{X} = (X_0, X_1)$ be a compatible pair of Banach spaces. For $0 < \theta < 1$ and $1 \leq p \leq \infty$ let $\bar{X}_{\theta,p}$ be the real interpolation spaces; for definitions and basic properties see e.g. [BL], [T].

For an operator $T : \bar{X} \to \bar{X}$ denote by $T_{\theta,p}$ the corresponding operator acting on $\bar{X}_{\theta,p}$.

By [AS], $\sigma(T_{\theta,p})$ does not depend on $p, 1 \leq p \leq \infty$, and it is upper semi-continuous as a function of θ (see also [BKS]).

Clearly Example 1 shows that in general the function $\theta \mapsto \sigma(T_{\theta,p})$ is not continuous. Indeed, by [T], 1.18.5, in this case $\bar{H}_{\theta,2} = \bar{H}_{[\theta]}$.

Using the connections between the real and complex interpolation spaces it is possible to obtain other results from Section I for the real interpolation spaces.

Theorem 9 (local U.R. property for real interpolation spaces; cf. [K])**.** *Let $\bar{X} = (X_0, X_1)$ be a compatible pair of Banach spaces, let $T : \bar{X} \to \bar{X}$ and $0 \leq \alpha < \beta \leq 1$. Suppose that $T_{\theta,1}$ is invertible for all θ, $\alpha < \theta < \beta$. Then $T^{-1}_{\theta,p}|\bar{X}_\Delta$ is constant for all θ, p with $\alpha < \theta < \beta$, $1 \leq p \leq \infty$.*

Proof. By [AS], the existence of $T^{-1}_{\theta,p}$ is independent of p, $1 \leq p \leq \infty$. Since $\bar{X}_\Delta \subset \bar{X}_{\theta,1} \subset \bar{X}_{\theta,p}$, the restriction $T^{-1}_{\theta,p}|\bar{X}_\Delta$ is constant for $1 \leq p \leq \infty$. Therefore we may consider only the operators $T_{\theta,2}$ $(\alpha < \theta < \beta)$. Clearly we can assume that $0 < \alpha < \beta < 1$. Using the formula $(\bar{X}_{\alpha,2}, \bar{X}_{\beta,2})_{[\eta]} = \bar{X}_{\theta,2}$ where $0 < \eta < 1$, $\theta = (1 - \eta)\alpha + \eta\beta$ (see [BL], Theorem 4.7.2), Corollary 5 implies that $T^{-1}_{\theta,2}|\bar{X}_\Delta$ is constant for $\alpha < \theta < \beta$. □

Theorem 10. *Let $\bar{X} = (X_0, X_1)$ be a compatible pair of Banach spaces, let $T : \bar{X} \to \bar{X}$, $0 < \theta < 1$, $1 \leq p \leq \infty$. Then $\sigma(T_{\theta,p}) \subset \sigma(T_{[\theta]})$.*

Proof. Suppose that $T_{[\theta]}$ is invertible. By Theorem 4 there exist θ_0, θ_1, $0 < \theta_0 < \theta < \theta_1 < 1$ such that $T_{[\theta_0]}$ and $T_{[\theta_1]}$ are invertible and $T_{[\theta_0]}^{-1}|\bar{X}_\Delta = T_{[\theta_1]}^{-1}|\bar{X}_\Delta$. By [BL], Theorem 4.7.2, $\bar{X}_{\theta,2} = (\bar{X}_{[\theta_0]}, \bar{X}_{[\theta_1]})_{\eta,2}$ where $\eta = \frac{\theta-\theta_0}{\theta_1-\theta_0}$, so that $T_{\theta,2}$ is invertible. Since $\sigma(T_{\theta,p})$ is independent of p, we have $\sigma(T_{\theta,p}) \subset \sigma(T_{[\theta]})$. □

Corollary 11.

$$\operatorname{cap}\sigma(T_{\theta,p}) \leq \operatorname{cap}\sigma(T_0)^{1-\theta} \cdot \operatorname{cap}\sigma(T_1)^{\theta}$$

for all θ, p *with* $0 < \theta < 1$, $1 \leq p \leq \infty$.

The next example shows that the real and complex methods yield in general different spectra.

Example 12. Choose $1 < p_0 < p_1 < \infty$, $0 < \theta < 1$. Let p satisfy $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

For the definition and basic properties of the Lorentz spaces L_{rs} (with respect to the Lebesgue measure on the real line), see [BL] or [T]. In particular, we have

$$\begin{aligned}(L_{p_0}, L_{p_1})_{[\theta]} &= L_p,\\ (L_{p_0}, L_{p_1})_{\theta,1} &= L_{p,1},\\ (L_{p_0,1}, L_{p_1,1})_{[\theta]} &= L_{p,1},\\ (L_{p_0,1}, L_{p_1,1})_{\theta,1} &= L_{p,1}.\end{aligned}$$

Further $L_{r1} \subset L_r$ $(= L_{rr})$ for all $r > 1$ and these two spaces are not isomorphic. Set

$$\begin{aligned}X_0 &= \cdots \oplus L_{p_0,1} \oplus L_{p_0,1} \oplus L_{p_0} \oplus L_{p_0} \oplus \cdots,\\ X_1 &= \cdots \oplus L_{p_1,1} \oplus L_{p_1,1} \oplus L_{p_1} \oplus L_{p_1} \oplus \cdots\end{aligned}$$

(ℓ_1 direct sums). By [T], 1.18.1,

$$\begin{aligned}(X_0, X_1)_{[\theta]} &= \cdots \oplus L_{p,1} \oplus L_p \oplus \cdots,\\ (X_0, X_1)_{\theta,1} &= \cdots \oplus L_{p,1} \oplus L_{p,1} \oplus \cdots.\end{aligned}$$

Let T be the right shift operator. Then $T_{\theta,1}$ is invertible but $T_{[\theta]}$ is not invertible.

Problem. In the previous example the polynomial convex hulls of $\sigma(T_{[\theta]})$ and $\sigma(T_{\theta,p})$ coincide $\big($and consequently, $r(T_{[\theta]}) = r(T_{\theta,p})\big)$. Is this always the case?

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