

## A TWO-DIMENSIONAL HAHN-BANACH THEOREM

B. L. CHALMERS AND B. SHEKHTMAN

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ABSTRACT. Let  $\tilde{T} = \sum_{i=1}^n \tilde{u}_i \otimes v_i : V \rightarrow V = [v_1, \dots, v_n] \subset X$ , where  $\tilde{u}_i \in V^*$  and  $X$  is a Banach space. Let  $T = \sum_{i=1}^n u_i \otimes v_i : X \rightarrow V$  be an extension of  $\tilde{T}$  to all of  $X$  (i.e.,  $u_i \in X^*$ ) such that  $T$  has minimal (operator) norm. In this paper we show in particular that, in the case  $n = 2$  and the field is  $\mathbf{R}$ , there exists a rank- $n$   $\tilde{T}$  such that  $\|T\| = \|\tilde{T}\|$  for all  $X$  if and only if the unit ball of  $V$  is either not smooth or not strictly convex. In this case we show, furthermore, that, for some  $\|T\| = \|\tilde{T}\|$ , there exists a choice of basis  $v = v_1, v_2$  such that  $\|u_i\| = \|\tilde{u}_i\|$ ,  $i = 1, 2$ ; i.e., each  $u_i$  is a Hahn-Banach extension of  $\tilde{u}_i$ .

### 1. INTRODUCTION

Let  $X$  be a Banach space and  $V$  be a subspace of  $X$ . If  $T$  is an operator from  $X$  into  $V$ , then  $T|_V$  is the restriction of  $T$  onto  $V$ . Let  $\tilde{T} : V \rightarrow V$ .

Define

$$e(\tilde{T}; X) = \inf_T \{\|T\| : T|_V = \tilde{T}\}$$

and

$$e(\tilde{T}) = \sup_X \{e(\tilde{T}; X) : X \supset V\}.$$

The classical Hahn-Banach Theorem states that, for every rank-one operator  $\tilde{T}$ ,

$$e(\tilde{T}) = \|\tilde{T}\|.$$

The Nachbin Theorem shows that, if  $V$  is  $n$ -dimensional and, for every  $\tilde{T} : V \rightarrow V$ ,

$$e(\tilde{T}) = \|\tilde{T}\|,$$

then  $V \cong \ell_\infty^{(n)}$ .

**Definition 1.1.** An operator  $\tilde{T} : V \rightarrow V$  is called a Hahn-Banach operator if  $e(\tilde{T}) = \|\tilde{T}\|$ .

In this paper we are concerned with the existence of non-trivial Hahn-Banach operators on a Banach space  $V$ . In particular we show (Theorem 2.2) that the two-dimensional real Banach space  $V$  possesses a rank-two Hahn-Banach operator if and only if  $V$  is either non-smooth or not strictly convex.

For the proof of the theorem we will need a number of definitions.

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Let  $X$  be a Banach space and  $V$  be a finite-dimensional subspace. Let  $T$  be a bounded linear operator from  $X$  into  $V$  and let  $\tilde{T}$  be the restriction of  $T$  onto  $V$ . ( $\tilde{T} = T|_V$ .)

**Definition 1.2.**  $T$  is a minimal norm extension of  $\tilde{T}$  means

$$\|T\| = \inf\{\|S\| : S : X \rightarrow V : S|_V = \tilde{T}\}.$$

**Definition 1.3.** The extremal set of  $T : X \rightarrow V$  is

$$\mathcal{E}(T) = \{(x^*, x^{**}) \in B(X^*) \times B(X^{**}) : x^*(T^{**}x^{**}) = \|T\|\},$$

where  $B$  denotes the unit ball and  $T^{**}$  denotes the second adjoint extension of  $T$  ( $T^{**}x^{**} = \sum_{i=1}^n \langle u_i, x^{**} \rangle v_i$  and  $\|T^{**}\| = \|T\|$ ).

*Note 1.1.* In the following we will designate by  $x^* \otimes x^{**} : X \rightarrow X^{**}$  the usual dyad operator given by  $x^* \otimes x^{**}(x) = \langle x, x^* \rangle x^{**}$ .

**Theorem 1.1** ([2]).  $T$  is a minimal norm extension of  $\tilde{T}$  if and only if there exists a probability measure  $\mu$  on  $\mathcal{E}(T)$  such that the operator

$$E_T = \int x^* \otimes x^{**} d\mu$$

maps  $V$  into  $V$ .

**Definition 1.4.** An  $n$ -dimensional subspace  $V \subset L_1[\mathcal{T}, \nu]$  is said to be smooth if no non-zero element  $v$  of  $V$  vanishes on a set of positive  $\nu$ -measure.

*Note 1.2.* It is well known that the  $n$ -dimensional real space  $V$  is smooth if and only if the unit sphere  $S(V)$  is differentiable (has a unique  $(n-1)$ -dimensional tangent plane) at every point. This fact follows immediately from  $\|\vec{a}\| := \|\vec{a} \cdot \vec{v}\|$  and the formula (see e.g. [1])

$$\frac{\partial}{\partial a_i} \|\vec{a}\| = \int_T v_i(s) \operatorname{sgn}[\vec{a} \cdot \vec{v}(s)] d\nu(s) \text{ for } \vec{a} \neq 0$$

and the Implicit Function Theorem. A point where  $S(V)$  is non-differentiable is called a point of non-smoothness.

**Theorem 1.2** ([1]). Let  $V = [\vec{v}]$  be a smooth real subspace of  $L_1[\mathcal{T}, \nu]$ . Then  $T = \sum_{i=1}^n u_i \otimes v_i$  is a minimal extension of  $\tilde{T}$  (to  $L_1[\mathcal{T}, \nu]$ ) implies that, for almost all  $t$ ,

$$\vec{u}(t) = \|T\| \vec{z}(t)$$

with  $\vec{z}(t)$  being a point of intersection of  $S(V)$  and its tangent plane perpendicular to  $M\vec{v}(t)$ , where  $M$  is the  $n \times n$  matrix  $M$  of  $(E_T)|_V$  with respect to the basis  $\vec{v}$ . (See Figure 1.)

**Corollary 1.1.** Let  $V = [\vec{v}]$  be a smooth real subspace of  $L_1[\mathcal{T}, \nu]$ . Then without loss  $\mathcal{T}$  can be enlarged so that there exists  $T = \sum_{i=1}^n u_i \otimes v_i$  which is a minimal extension of  $\tilde{T}$  and the operator  $T^*$  is an extension of  $\tilde{T}^* : V^* \rightarrow V^*$ .

*Proof.* Let  $T = \sum_{i=1}^n u_i \otimes v_i$  be a minimal extension of  $\tilde{T}$ . Theorem 1.2 provides that  $\vec{u}/\|T\|$  lies on  $S(V)$  a.e.  $[\nu]$ . Next replace  $\mathcal{T}$  by  $\mathcal{T} \cup \Delta\mathcal{T}$ , where  $\Delta\mathcal{T} = S(V)$ -range  $(\vec{u}/\|T\|)$ , enlarge  $\nu$  and  $\vec{v}$  to be zero on  $\Delta\mathcal{T}$ , and enlarge  $\vec{u}/\|T\|$  to be the coordinate functions on  $\Delta\mathcal{T}$ . Then let  $T$  and  $\tilde{T}$  be replaced by their corresponding enlargements, recall that  $V^*$  is isometric to the space of coordinate functions of  $S(V)$  regarded as a subspace of  $L_\infty(S(V))$ , and note that  $T^*_{|_{V^*}} = \tilde{T}^*$ .  $\square$

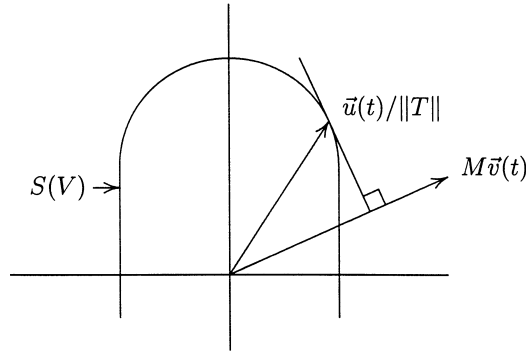


FIGURE 1.

2. MAIN THEOREM

**Lemma 2.1.** *Let  $T$  be an extension of  $\tilde{T}$ . Then  $\|T\| = \|\tilde{T}\|$  if and only if there exists a pair*

$$(x^*, v) \in \mathcal{E}(T), v \in V.$$

Hence the dyad operator  $E_T = x^* \otimes v$  satisfies the conclusion of Theorem 1.1.

*Proof.* If  $(x^*, v) \in \mathcal{E}(T)$ , then

$$\|T\| = x^*(Tv) = \|Tv\|.$$

Thus  $T$  attains its norm on  $V$  and

$$\|T\| = \|Tv\| = \|\tilde{T}v\| \leq \|\tilde{T}\|.$$

Conversely, if  $\|T\| = \|\tilde{T}\|$ , let  $v \in V$  be such that  $\|\tilde{T}v\| = \|Tv\| = \|\tilde{T}\| = \|T\|$ . Let  $x^* \in X^*$  be such that  $x^*(Tv) = \|Tv\|$ . Then  $(x^*, v) \in \mathcal{E}(T)$ . □

The following theorem shows that, if  $n > 1$  and the field is  $\mathbf{R}$ , a smooth, strictly convex,  $n$ -dimensional  $L_1$ -subspace  $V$  possesses no rank- $n$  Hahn-Banach operator. In fact it shows more, namely, that  $V$  possesses no rank- $k$  operator which extends to  $L_1$  of the same norm,  $k > 1$ .

**Theorem 2.1.** *Let  $V$  be a smooth, strictly convex,  $n$ -dimensional real subspace of  $L^1[-1, 1]$ . Let  $\tilde{T} : V \rightarrow V$  be such that there exists an extension  $T : L^1[-1, 1] \rightarrow V$  with  $\|T\| = \|\tilde{T}\|$ . Then  $\tilde{T}$  is a rank-one operator.*

*Proof.* By Lemma 2.1, corresponding to  $T = \sum_{i=1}^n u_i \otimes v_i$ , there exists a rank-one operator  $E_T$  satisfying the conclusion of the lemma. Hence the matrix  $M$  defined as the matrix of  $(E_T)|_V$ , with respect to the basis  $\vec{v}$ , has rank one.

Since  $V$  is smooth there exists a minimal extension  $T$  of  $\tilde{T}$  satisfying the conditions of Theorem 1.2. But  $M$  has rank one, whence the vectors  $M\vec{v}(t)$  lie on the same one-dimensional subspace of  $\mathbf{R}^n$  for almost all  $t \in [-1, 1]$ .

Since  $V$  is strictly convex, the vectors  $\vec{u}(t)$  also lie in the same one-dimensional subspace of  $\mathbf{R}^n$  for almost all  $t \in [-1, 1]$ . Hence the operator  $T$  given by Theorem 1.2 has rank one and thus so does  $\tilde{T}$ . □

**Corollary 2.1.**  *$n$ -dimensional Hilbert space  $(\ell_2^{(n)})$  possesses no Hahn-Banach operator with rank  $> 1$ .*

*Proof.* It is well known (see [5]) that  $\ell_2^{(n)}$  can be realized as a subspace of  $L_1$ .  $\square$

**Theorem 2.2.** *Let  $V$  be a two-dimensional real Banach space. Then the following are equivalent:*

- (1)  $V$  is smooth and strictly convex.
- (2) For every rank-two operator

$$\tilde{T} : V \rightarrow V, \quad e(\tilde{T}) > \|\tilde{T}\|.$$

*Proof.* (1) implies (2) by Theorem 2.1 since every two-dimensional space  $V$  can be isometrically imbedded into  $L_1[-1, 1]$  (see e.g. [6]).

We now show (2) implies (1).

Suppose that  $V$  with unit sphere  $S(V) \subset \mathbf{R}^2$  is not strictly convex. Then (via a linear transformation) we can assume that  $S(V)$  lies inside the unit square  $C$ , touches  $C$  on all four sides, and has a flat spot running from  $(1, d)$  to  $(1, -d)$  (and of course a corresponding flat spot running from  $(-1, -d)$  to  $(-1, d)$ ), where “flat spot” is defined in the usual obvious way. Let  $P$  be the rectangle with corners  $(\pm 1, \pm d)$ , whence  $P$  lies inside  $S(V)$ . (See Figure A.)

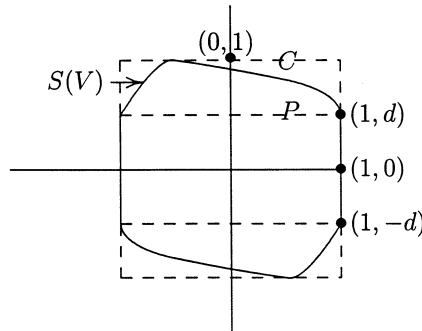


FIGURE A.

Now consider the linear transformation  $\tilde{T}(a_1, a_2) := (a_1, da_2)$  taking  $C$  into  $P$ . Thus  $\tilde{T}(S(V))$  lies inside  $P$ , which lies inside  $S(V)$ , and, since  $\tilde{T}(1, 0) = (1, 0) \in S(V) \cap \tilde{T}(S(V))$ , we see that  $\tilde{T} : V \rightarrow V$  has norm one. Note that we can also write  $\tilde{T}$  in the form

$$\tilde{T} = \tilde{\phi}_1 \otimes (1, 0) + d\tilde{\phi}_2 \otimes (0, 1),$$

where  $\tilde{\phi}_i$  are the norm-1 coordinate functionals on  $V$  defined by

$$\tilde{\phi}_i(a_1, a_2) = a_i, \quad i = 1, 2.$$

Next let  $\phi_i$  be a Hahn-Banach extension to  $X$  of the functional  $\tilde{\phi}_i$  on  $V$ ,  $i = 1, 2$ . Then the extension operator  $T : X \rightarrow V$  has the form

$$T = \phi_1 \otimes (1, 0) + d\phi_2 \otimes (0, 1).$$

Finally, since  $T(S(X)) \subset \tilde{T}(\overline{c\partial}C) = \overline{c\partial}P \subset \overline{c\partial}S(V) = B(V)$ , we see that  $\|T\| \leq 1$  and hence, since  $\|T\| \geq \|\tilde{T}\| = 1$ , it follows that  $\|T\| = \|\tilde{T}\|$ .

Furthermore, suppose that  $V$  is strictly convex but not smooth. Then, as is well known,  $V^*$  is smooth but not strictly convex, and the above argument shows that  $V^*$  possesses a rank-two Hahn-Banach operator. But then, by Corollary 1.1

( $1 = \|T^*\| = \|\tilde{T}^*\| = \|\tilde{T}\| = \|T\|$  and  $e(\tilde{T}) = e(\tilde{T}; L_\infty)$ ),  $V$  also possesses a rank-two Hahn-Banach operator.  $\square$

**Corollary 2.2.** *If the unit ball of the 2-dimensional real space  $V$  is either not smooth or not strictly convex, then  $V$  possesses a basis  $v_1, v_2$  and a rank-2 Hahn-Banach operator  $\tilde{T} = \sum_{i=1}^2 \tilde{u}_i \otimes v_i$  such that, for the extension  $T = \sum_{i=1}^2 u_i \otimes v_i$ , each  $\|u_i\| = \|\tilde{u}_i\|$ ,  $i = 1, 2$ , i.e., each  $u_i$  is a Hahn-Banach extension of  $\tilde{u}_i$ . A reasonable conjecture is that this phenomenon holds for every rank-2 Hahn-Banach operator on  $V$ .*

*Note 2.1.* In [3] it is shown in particular that, for each real non-singular  $2 \times 2$  real matrix  $A$  not equal to a scalar multiple of the identity  $I$ , there exists a real space  $V = [\vec{v}] \not\cong \ell_\infty^{(2)}$  which possesses a Hahn-Banach operator  $\tilde{T}$  such that  $\tilde{T}\vec{v} = A\vec{v}$ . Theorem 2.2 explains why any such space must be either non-smooth or non-strictly convex. Indeed all the spaces provided in [3] are hexagonal spaces.

Note, furthermore, that the rectangle  $P$  in the proof of Theorem 2.2 can be replaced by any parallelogram (symmetric with respect to the origin) contained inside  $S(V)$  and intersecting the flat sides of  $S(V)$ . In this way we see that  $V$  possesses an entire family of two-dimensional Hahn-Banach operators of norm-1 ( $\tilde{T}p = p$  for some  $p \in S(V) \cap \tilde{T}(S(V))$ ) of the form

$$\tilde{T} = (\alpha\tilde{\phi}_1 + \beta\tilde{\phi}_2) \otimes (1, 0) + (\gamma\tilde{\phi}_1 + \delta\tilde{\phi}_2) \otimes (0, 1),$$

all of which have two real eigenvalues (1 and  $e$ , where  $|e| \leq 1$ ). In [4] we show that, if the spectrum of  $\tilde{T} \neq I$  lies on the unit circle, then the Hahn-Banach operator  $\tilde{T}$  has norm  $> 1$ .

**Corollary 2.3.** *If the two-dimensional real Banach space  $V$  possesses a rank-two Hahn-Banach operator  $\tilde{T} \neq I$  whose spectrum lies on the unit circle, then  $V$  also possesses a rank-two Hahn-Banach operator with eigenvalues 1,  $d$ ,  $|d| < 1$ .*

**Corollary 2.4.** *The two-dimensional real Banach space  $V$  possesses a rank-two Hahn-Banach operator if and only if its dual space  $V^*$  possesses a rank-two Hahn-Banach operator.*

**Corollary 2.5.** *For every two-dimensional subspace  $V$  of  $\ell_1$  or  $\ell_\infty$  there exists a rank-two operator  $\tilde{T} : V \rightarrow V$  such that  $e(\tilde{T}) = \|\tilde{T}\|$ .*

*For every two-dimensional subspace  $V$  of  $\ell_p$  ( $p \in (1, \infty)$ ) and for every rank-two operator  $\tilde{T} : V \rightarrow V$  we have  $e(\tilde{T}) > \|\tilde{T}\|$ .*

**Example 2.1.** Let  $V = [\vec{v}] \subset L^1[-1, 1]$ , where  $(v_1, v_2) = (\frac{1}{2}, t)$  and let  $\tilde{T}v_1 = v_1$  and  $\tilde{T}v_2 = av_2$ ,  $a \geq \frac{1}{2}$ . Then Theorem 1.2 yields that the minimal extension operator  $T = \sum_{i=1}^2 u_i \otimes v_i : L^1[-1, 1] \rightarrow V$  is given by

$$\vec{u} = \kappa \left[ \frac{(1, 2mt)}{\sqrt{1 + 4m^2t^2}} + \left(0, \frac{\text{sgn } t}{2}\right) \right],$$

where  $\kappa = \|T\|$ ,  $m$  is a non-negative constant, and  $M = \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}$ . (In fact  $a = (\frac{1}{2} + \int_0^1 \frac{4mt^2 dt}{\sqrt{1+4m^2t^2}}) / \int_0^1 \frac{dt}{\sqrt{1+4m^2t^2}}$ .) Since  $V$  is smooth, Theorem 2.2 provides that  $\|T\| = \|\tilde{T}\| = 1$  precisely when  $m = 0$ . Note that then  $u_1 = 1$  and  $u_2 = \frac{1}{2}\text{sgn } t$  shows that  $u_i \in L_\infty[-1, 1]$  is the Hahn-Banach extension of  $\tilde{u}_i$ ,  $i = 1, 2$ . (See Figure 2.)

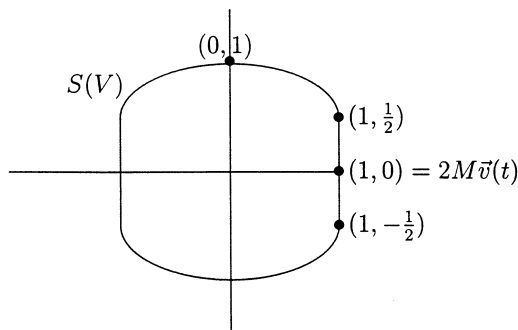


FIGURE 2.

As a further example, following Corollary 1.1, note that the space  $V$  is also realized as  $V = [\frac{1}{2}, t] \subset L^1[\overline{\mathbf{R}}, \nu]$ , where  $\nu$  is Lebesgue measure on  $[-1, 1]$  and zero off  $[-1, 1]$ . Thus replacing  $[-1, 1]$  by  $\overline{\mathbf{R}}$  in the above, we see that, for  $m > 0$ ,  $\vec{u}/\|T\|$  now covers all of  $S(V)$  and, for  $m = 0$ , let  $\vec{u}(t)/\|T\| = \vec{u}(t)$ ,  $|t| > 1$ , be any convenient parameterization of the curved portion of  $S(V)$ . Thus for  $m = 0$  the adjoint operator  $\tilde{T}^*$  (with same-norm extension  $T^*$ ) provides a Hahn-Banach operator for the non-smooth, strictly convex space  $V^* = [t, 1 - t^2] \subset L_\infty[-1, 1]$  with a “corner” at  $(1, 0)$ . ( $V^* \cong [\vec{u}/\|T\|] \subset L_\infty[\overline{\mathbf{R}}]$ .)

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, RIVERSIDE, CALIFORNIA 92507  
*E-mail address*: [blc@math.ucr.edu](mailto:blc@math.ucr.edu)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH FLORIDA, TAMPA, FLORIDA 33620  
*E-mail address*: [boris@math.usf.edu](mailto:boris@math.usf.edu)