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ADJOINTS AND THE IMAGE OF THE BALL

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Abstract. A bounded operator between Hilbert spaces has an adjoint if and only if the image of the unit ball is located.

1. Introduction

Let $A : H \to K$ be a bounded operator between Hilbert spaces, and H_1 the unit ball of H. If AH_1 is totally bounded, then A is compact. What happens if we relax this condition and require only that AH_1 be *located*, that is, given $v \in K$ and $\varepsilon > 0$ we can find $h \in H_1$ so that $\|v - Ah\| < \rho(v, AH_1) + \varepsilon$? This is a purely constructive notion, having to do with what you can calculate. In the case when AH_1 is totally bounded, we are able to construct, for each $\delta > 0$, a finite set that approximates AH_1 up to δ. It is easy to show that totally bounded subsets are located.

We will show that A has an adjoint if and only if AH_1 is located. What is the significance of that? After all, from a classical point of view, AH_1 is always located, and A always has an adjoint. It means that if you can calculate A^*u for each vector u in K, then you can calculate approximate nearest points in AH_1 to each vector v in K, and vice versa.

You need not be able to calculate A^*u just because you can calculate Ah for each $h \in H$. The simplest example of this is built on a binary sequence (a_n) that contains at most one 1. On a Hilbert space with orthonormal basis (e_n) define A by $Ae_n = a_n e_1$. If $h = \sum h_n e_n$, then $\sum a_n h_n$ converges to a real number r because $h_n \to 0$ so you can compute $Ah = re_1$. But you can't calculate A^*e_1 unless you know n such that $a_n = 1$, or you know that $a_n = 0$ for all n.

The proof that A has an adjoint if and only if AH_1 is located divides into two parts. One is an easy characterization, given the Riesz representation theorem, of when an operator has an adjoint (Theorem 2.1). The other deals with the geometry of nonempty bounded balanced convex subsets C of a real inner product space. The main fact there is that C is located if and only if every linear programming problem on C admits arbitrarily good solutions.

2. Reduction to real geometry

For a separable Hilbert space, Ishihara [4, Theorem 2] showed that a bounded operator $A : H \to H$ has an adjoint if and only if AH_1 is totally bounded with respect to the double norm, $\sum_{n=1}^{\infty} |\langle x, y_n \rangle| / 2^n$, where (y_n) is a dense sequence in

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H_1 (see [1, page 350]). The following is a characterization, in the same spirit, for arbitrary Hilbert spaces.

Theorem 2.1. *Let $A : H \to K$ be a bounded operator between Hilbert spaces, and H_1 the unit ball of H. Then A has an adjoint if and only if $\pi A H_1$ is located for each one-dimensional projection π of K.*

Proof. Suppose A has an adjoint A^* and π is a one-dimensional projection of K. Then $\pi x = \langle x, u\rangle u$ for some unit vector u in K, so $\pi A H_1$ is located if and only if $\langle \pi A H_1, u\rangle = \langle H_1, A^* u\rangle$ is located. If $A^* u \neq 0$, then $\langle H_1, A^* u\rangle = \{z : |z| \leq \|A^* u\|\}$, which is located. In any case, $\langle H_1, A^* u\rangle$ is dense in $\{z : |z| \leq \|A^* u\|\}$, hence is located.

Conversely, let u be a unit vector and $\pi x = \langle x, u\rangle u$. If $\pi A H_1$ is located, then

$$f(x) = \langle Ax, u\rangle$$

defines a linear functional with norm $\sup\{\|y\| : y \in \pi A H_1\}$. So $f(x) = \langle x, A^* u\rangle$ for a unique vector $A^* u$ (the Riesz representation theorem; see [1, 2.3, page 419] for a proof of the separable case, using countable choice, and [2] for a general proof). □

We want to show that A has an adjoint if and only if AH_1 is located. Recall that a subset C is **balanced** if $rc \in C$ whenever $c \in C$ and $|r| \leq 1$. In light of Theorem 2.1, it suffices to show the following.

- *Let C be a nonempty, bounded, convex, balanced subset of an inner product space. Then C is located if and only if πC is located for each one-dimensional projection π.*

It suffices to prove this for *real* inner product spaces. A complex inner product space H can be considered as a real inner product space H_r by restricting the scalars to the real numbers and using the inner product $\langle x, y\rangle_r = \mathrm{Re}\langle x, y\rangle$. Distances remain the same, so C is located in H if and only if it is located in H_r. As C is balanced, πC is a union of concentric disks, hence is located if and only if its projection onto a one-dimensional real subspace of πH is located.

Note that to say that πC is located for each one-dimensional projection π is to say that any linear programming problem on C can be solved to arbitrary precision, while to say that C is located is to say that certain quadratic programming problems on C can be solved to arbitrary precision.

3. Real geometry

First a few lemmas.

Lemma 3.1. *Let a, b, c be points in a real inner product space with $a \neq b$ and $a \neq c$. Let c' be the projection of c onto the line through a and b. If c' is on the line segment joining a to b, then the distance from b to the line segment joining a to c is at most*

$$\|b - a\| - \frac{\|c' - a\|^3}{2\,\|c - a\|^2}.$$

Proof. The squared distance from c' to the line segment joining a to c is

$$\|c' - a\|^2 - \frac{\langle c' - a, c - a\rangle^2}{\|c - a\|^2}$$

which is

$$\|c' - a\|^2 \left(1 - \frac{\|c' - a\|^2}{\|c - a\|^2}\right),$$

so the distance is at most

$$\|c' - a\| \left(1 - \frac{\|c' - a\|^2}{2\|c - a\|^2}\right).$$

Therefore the distance from b to the line segment joining a to c is at most

$$\|b - c'\| + \|c' - a\| \left(1 - \frac{\|c' - a\|^2}{2\|c - a\|^2}\right) = \|b - a\| - \frac{\|c' - a\|^3}{2\|c - a\|^2}.$$

□

Lemma 3.2. *Let C be a convex balanced subset of a real inner product space H. Let $v \in H$ and $c \in C$. Then for each $\delta > 0$ there exists c' in C such that $\|v - c'\| \leq \|v - c\|$ and either $\|c'\| < \delta$, or $\langle v, c'\rangle > 0$ and $\langle v, c'\rangle + \delta > \|c'\|^2$.*

Proof. If $\|c\| < \delta$, then set $c' = c$. Otherwise $\|c\| > 0$. If $|\langle v, c\rangle| \leq \|c\|^2$, set

$$c' = \frac{\langle v, c\rangle}{\|c\|^2} c.$$

Then $c' \in C$ and

$$v - c = (v - c') + (c' - c).$$

But $v - c'$ is orthogonal to c, hence to $c' - c$. So $\|v - c'\| \leq \|v - c\|$ and $\langle v - c', c'\rangle = 0$. Otherwise $|\langle v, c\rangle| > \sup(0, \|c\|^2 - \delta)$. As C is balanced and

$$\|v - c\|^2 = \|v\|^2 + \|c\|^2 - 2\operatorname{Re}\langle v, c\rangle,$$

we can choose c' a unit multiple of c so that $\langle v, c'\rangle = |\langle v, c\rangle| > 0$ whence $\|v - c'\| \leq \|v - c\|$. □

We will use the notation $\rho(v, C)$ to indicate the distance from the vector v to the set C, even if the indicated infimum does not exist (in a constructive sense). That is, we treat $\rho(v, C)$ as a generalized real number in the sense of [3] and [5]. For example, it makes sense to write $\rho(v, C) \leq r$ or to write $r \leq \rho(v, C)$ for any real number r.

Lemma 3.3. *Let H be a real inner product space and C a convex balanced subset whose diameter is bounded by β. Suppose πC is located for each one-dimensional projection π. Let $v \in H$ and $c_0 \in C$. Then for each $\varepsilon > 0$, there exists $c_1 \in C$ such that $\|v - c_1\| \leq \|v - c_0\|$ and either*

$$\rho(v, C) \geq \|v - c_1\| - 2\varepsilon$$

or

$$\rho(v, C) \leq \|v - c_1\| - \frac{\varepsilon^3}{3\beta^2}.$$

Proof. Let $\delta < \varepsilon$ be a positive number to be determined later. From Lemma 3.2 there exists $c_1 \in C$ such that $\|v - c_1\| \leq \|v - c_0\|$ and either

1. $\|c_1\| < \delta$, or
2. $\langle v, c_1\rangle > 0$ and $\langle v - c_1, c_1\rangle + \delta > 0$.

If $\|v - c_1\| < 2\varepsilon$, then $\rho(v, C) \geq \|v - c_1\| - 2\varepsilon$, so we may assume that $\|v - c_1\| > \varepsilon$. Let $c_2 = 0$ in Case 1, and $c_2 = c_1$ in Case 2. Note that $v \neq c_2$ in either case.

Let π be the projection on the span of $v - c_2$. So

$$\pi x = \frac{\langle x, v - c_2\rangle}{\|v - c_2\|^2}(v - c_2).$$

In either case, both πc_2 and πv are on the span of $v - c_2$.

Let L be the line through c_2 and v, and λ the projection on L. (In Case 1, $\lambda = \pi$.) Note that λ is the composition of the projection on the span of $v - c_2$ with an isometry (the inverse of π restricted to L) of that span with L. As C is balanced and bounded, and λC is located in L, there exists a real number r so that

$$\{x \in L : \|x - \lambda 0\| < r\} \subset \lambda C \subset \{x \in L : \|x - \lambda 0\| \leq r\},$$

so the closure of λC is the interval of radius r around $\lambda 0$.

Now we branch. As $\rho(v, C) \geq \rho(v, \lambda C)$, if $\rho(v, \lambda C) \geq \|v - c_2\| - 2\varepsilon + \delta$, then we're done. So we may assume that $\rho(v, \lambda C) < \|v - c_2\| - \varepsilon$. Therefore there is $c \in C$ such that $\|v - \lambda c\| < \|v - c_2\| - \varepsilon$. As $c_2 \in \lambda C$, we may assume that λc is on the line segment joining c_2 and v (rather than on the other side of v). Thus

$$\|\lambda c - c_2\| + \|v - \lambda c\| = \|v - c_2\|,$$

so $\|\lambda c - c_2\| > \varepsilon$.

As $c_2 \neq v$ and $c_2 \neq c$, and λc is on the line segment joining c_2 and v, Lemma 3.1 applies, so

$$\rho(v, C) < \|v - c_2\| - \frac{\varepsilon^3}{2\beta^2}$$

and, for $\delta < \varepsilon^3/6\beta^2$,

$$\rho(v, C) < \|v - c_1\| - \frac{\varepsilon^3}{3\beta^2}.$$

□

Theorem 3.4. *Let H be a real inner product space and C a nonempty, bounded, convex, balanced subset such that πC is located for each one-dimensional projection π. Then C is located.*

Proof. We want to compute $\rho(v, C)$ up to $2\varepsilon > 0$. Let β be a bound on the diameter of C. Choose $c_0 \in C$ and iterate Lemma 3.3. Eventually we construct c so that $\rho(v, C) \geq \|v - c\| - 2\varepsilon$ □

Theorem 3.5. *Let H be a real inner product space, and C a nonempty, bounded, located, convex subset. Then πC is located for each one-dimensional projection π.*

Proof. Let $\pi x = \langle x, u\rangle u$ where u is a unit vector. Given $\varepsilon > 0$, we will construct $c \in C$ such that $\sup\langle C, u\rangle \leq \langle c, u\rangle + \varepsilon$. That is, we will show that $\sup\langle C, u\rangle$ exists. Replacing u by $-u$ shows that $\inf\langle C, u\rangle$ also exists. As C is convex, that will do it. We may assume that $\varepsilon < 1$.

Let d be a bound on the diameter of C, and c_0 any element of C. If $\langle c_1, u\rangle > \langle c_0, u\rangle + \varepsilon$, for some $c_1 \in C$, then there is c in C such that $\langle c, u\rangle = \langle c_0, u\rangle + \varepsilon$. In that case, $\rho(c_0 + u, C) \leq 1 - \varepsilon^3/2d^2$ (Lemma 3.1 with $a = c_0$ and $b = c_0 + u$). So if $\rho(c_0 + u, C) > 1 - \varepsilon^3/2d^2$, then $\sup\langle C, u\rangle \leq \langle c_0, u\rangle + \varepsilon$, as desired.

Otherwise $\rho(c_0 + u, C) < 1 - \varepsilon^3/3d^2$ so $\|c_0 + u - c_2\| < 1 - \varepsilon^3/3d^2$ for some $c_2 \in C$. But

$$1 - \langle c_2 - c_0, u\rangle = \langle c_0 + u - c_2, u\rangle \leq \|c_0 + u - c_2\|,$$

so $\langle c_2, u\rangle > \langle c_0, u\rangle + \varepsilon^3/3d^2$. Iterating this construction, we eventually construct c so that $\sup\langle C, u\rangle \leq \langle c, u\rangle + \varepsilon$. □

4. A Brouwerian example

We end with a Brouwerian example showing that boundedness is necessary in Theorem 3.5.

Theorem 4.1. *Let (e_n) be a basis for a Hilbert space, (a_n) a binary sequence with at most one 1, and $C = \{a_n t(e_1 + ne_n) : n \in \mathbf{N},\ |t| \leq 1\}$. Then C is located, but $\pi_1 C$ is located only if $a_n = 1$ for some n or $a_n = 0$ for all n.*

Proof. The second claim is clear: compute the distance from e_1 to $\pi_1 C$. To see that C is located, let y be an element of H. The squared distance between y and $t(e_1 + ne_n)$ is

$$\begin{aligned}
&\|y\|^2 - |y_1|^2 - |y_n|^2 + |y_1 - t|^2 + |y_n - tn|^2 \\
&\quad = \|y\|^2 - 2\operatorname{Re}(y_1 t + n y_n t) + |t|^2 (1 + n^2) \\
&\quad \geq \|y\|^2 - 2|t|(|y_1| + n|y_n|) + |t|^2 (1 + n^2).
\end{aligned}$$

The latter has minimum value when $|t| = (|y_1| + n|y_n|)/(1 + n^2)$ and the minimum value is

$$\|y\|^2 - \frac{(|y_1| + n|y_n|)^2}{1 + n^2}.$$

This is very near $\|y\|^2$ for large n, because y_n is small, so C is located. □

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