ON THE GRUSHIN OPERATOR 
AND HYPERBOLIC SYMMETRY

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Abstract. Complexity of geometric symmetry for differential operators with 
mixed homogeneity is examined here. Sharp Sobolev estimates are calculated 
for the Grushin operator in low dimensions using hyperbolic symmetry and 
conformal geometry.

Considerable interest exists in understanding differential operators with 
mixed homogeneity. A simple example is the Grushin operator on $\mathbb{R}^2$

$$\Delta_G = \frac{\partial^2}{\partial t^2} + 4t^2 \frac{\partial^2}{\partial x^2}.$$ 

The purpose of this note is to demonstrate the complexity of geometric symmetry 
that may exist for operators defined on Lie groups. Here the existence of an un-
derlying $SL(2, \mathbb{R})$ symmetry for $\Delta_G$ is used to compute the sharp constant for the 
associated $L^2$ Sobolev inequality.

**Theorem 1.** For $f \in C^1(\mathbb{R}^2)$

$$\left[ \|f\|_{L^6(\mathbb{R}^2)} \right]^2 \leq \pi^{-2/3} \int_{\mathbb{R}^2} \left[ \left( \frac{\partial f}{\partial t} \right)^2 + 4t^2 \left( \frac{\partial f}{\partial x} \right)^2 \right] \, dx \, dt.$$ 

This inequality is sharp, and an extremal is given by $\left( (1 + |t|^2)^{3/2} + |x|^2 \right)^{-1/4}$.

This result follows from the analysis of a Sobolev inequality on $SL(2, R)/SO(2)$. But the hyperbolic embedding estimate requires some interpretation to take into account cancellation effects. It will be essential to include contributions to the hyperbolic Dirichlet form from non-$L^2$ functions. Let $z = x + iy$ denote a point in the upper half-plane $\mathbb{H}^2 \approx \mathbb{M} \approx SL(2, R)/SO(2)$. Here the invariant 
distance is given by the Poincaré metric

$$d(z, z') = \frac{|z - z'|}{2\sqrt{yy}}$$

with the corresponding invariant gradient $D = y\nabla$ and left-invariant Haar measure $d\nu = y^{-2} dy \, dx$. 

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Theorem 2. For $F \in C^1_c(M)$

\[
\left( \|F\|_{L^6(M)} \right)^2 \leq 4\pi^{-2/3} \left[ \int_M |DF|^2 \, d\nu - \frac{3}{16} \int_M |F|^2 \, d\nu \right].
\]

Both estimates are sharp as limiting forms.

These two estimates would seem to be contradictory, but it must be understood that the right-hand sides are to be evaluated as limiting forms for functions that may not be in $L^2(M)$. So the issue of which is the sharper Sobolev inequality must be studied carefully. On the hyperbolic manifold the Dirichlet form can be represented as a weighted Sobolev form so that for $\alpha > 0$

\[
\int_M y^{2\alpha} |\nabla (y^{-\alpha} f)|^2 \, dx \, dy = \int_M |Df|^2 \, d\nu + \alpha(\alpha - 1) \int_M |f|^2 \, d\nu.
\]

On the right-hand side of equation (3) observe the appearance of the spectral limit $\frac{1}{4}$ for the hyperbolic Laplacian $-y^2 \Delta$:

\[
\frac{1}{4} \int_M |F|^2 \, d\nu \leq \int_M |DF|^2 \, d\nu.
\]

Proof of Theorem 1. Let $\hat{f}$ denote the Fourier transform of $f$ in the first variable. That is, for integrable functions

\[
\hat{f}(\xi, t) = \int_R e^{2\pi i \xi x} f(x, t) \, dx
\]

so that by using the Plancherel identity, inequality (1) for some constant $A_0$ is equivalent to

\[
\left[ \int_{\mathbb{R}^2} |\hat{f} * \hat{f} * \hat{f}|^2 \, d\xi \, dt \right]^{1/3} \leq A_0 \int_{\mathbb{R}^2} \left[ \left( \frac{\partial \hat{f}}{\partial t} \right)^2 + 16\pi^2 |t|^2 |\xi|^2 |\hat{f}|^2 \right] \, d\xi \, dt
\]

where here convolution is only with respect to the first variable. Now one can apply standard rearrangement arguments of Riesz-Sobolev type to see that it suffices to consider this inequality only for non-negative functions $\hat{f}(\xi, t)$ that are symmetric decreasing in each of the two variables separately. Hence, the function $f(x, t)$ in (1) can be taken to be symmetric in $t$. Moreover, $f(x, t)$ can be taken to be symmetric decreasing in $x$ because the Dirichlet form in (1) taken only with respect to integration in $x$ is diminished by a symmetric decreasing equimeasurable rearrangement in the first variable.

Since $f(x, t)$ is even in $t$, set $y = t^2$ and let $f(x, |t|) = y^{-1/4} F(x, y)$. Then

\[
\|f\|_{L^6(R^2)} = \|F\|_{L^6(M)}
\]

and inequality (1) is now equivalent to

\[
\left[ \|F\|_{L^6(M)} \right]^2 \leq 4A_0 \left[ \int_M |DF|^2 \, d\nu - \frac{3}{16} \int_M |F|^2 \, d\nu \right].
\]

This is an a priori inequality where the function $F$ can be taken to be smooth but still the form will extend to functions that are not in $L^2(M)$. One can also restrict this result to consideration of functions that are radial decreasing in the Poincaré distance from the origin $\tilde{0} = (0, 1) = i$. Now Theorem 1 will follow from the first
part of Theorem 2 with $A_0 = \pi^{-2/3}$. To make this argument clearer, observe that estimates for the Grushin operator are maximized on the class of even functions. Every even function of compact support in $t$ can be smoothly approximated by a Schwartz class function that is analytic in $t^2$ through convolution with a Gaussian function. This class of functions suffices to determine the Grushin bound (1). But by using the functional transformation $F(x, y) = y^{1/4}f(x, |t|)$ to transfer the estimate to hyperbolic space, one finds that both the $L^6$ norm and the Dirichlet form for the elliptic operator $L = -y^2\Delta - (3/16)I$ are finite. Every smooth function on $M$ with suitable decay at infinity which vanishes to order one-quarter at the boundary $y = 0$ has finite Dirichlet form for inequality (2).

**Proof of equation (2) in Theorem 2.** By using equimeasurable radial decreasing rearrangement corresponding to the metric on hyperbolic space, it suffices to consider this inequality for radial decreasing functions of the distance from the origin. Let $u = [d(z, i)]^2$. Then for functions depending on distance the gradient is given by

$$|DF| = \sqrt{u + u^2} \frac{dF}{du}$$

and the volume form restricted to integration for radial function is given by $d\nu = 4\pi du$. Then (2) is equivalent to

$$\left[ \int_0^\infty |F|^6 du \right]^{1/3} \leq 2^{10/3} \left[ \int_0^\infty (u + u^2)|F'|^2 du + \frac{3}{16} \int_0^\infty |F|^2 du \right].$$

Let $G \in C^2_c([0, \infty))$ and set $F(u) = (1 + u)^{-1/4}G(u)$. Then inequality (2) is equivalent to

$$\left[ \int_0^\infty |G|^6 (1 + u)^{-3/2} du \right]^{1/3} \leq 2^{10/3} \left[ \int_0^\infty u\sqrt{1 + u}|G'|^2 du + \frac{1}{16} \int_0^\infty |G|^2 (1 + u)^{-3/2} du \right].$$

But now this estimate will be considered for all Lipschitz functions $G$ such that the right-hand side is finite. Make the change of variables $u \to 1/u$ with $H(u) = G(1/u)$;

$$(5) \left[ \int_0^\infty |H|^6 (1 + u)^{-3/2} u^{-1/2} du \right]^{1/3} \leq 2^{10/3} \left[ \int_0^\infty \sqrt{u(1+u)}|H'|^2 du + \frac{1}{16} \int_0^\infty |H|^2 (1 + u)^{-3/2} u^{-1/2} du \right].$$

By evaluating this estimate for $H(u) = (1 + u)^{-\varepsilon}$ as $\varepsilon \to 0$, one sees that the constant cannot be smaller than $2^{10/3}$. This calculation also suggests that the inequality should be associated with sharp Sobolev embedding on $S^2$. Such intuition is realized by the following argument.

Define a new variable $w$ by setting

$$(1 + w)^{-2} dw = \frac{1}{2} u^{-1/2}(1 + u)^{-3/2} du$$

so that

$$\sqrt{\frac{u}{1 + u}} = \frac{w}{1 + w}.$$
and \( w = u + \sqrt{u(1+u)} \). With this change of variables (5) becomes
\[
\left[ \int_0^\infty |G|^6 (1+w)^{-2} \, dw \right]^{1/3} \leq 4 \int_0^\infty (2w+1)|G'|^2 \, dw + \int_0^\infty |G|^2 (1+w)^{-2} \, dw .
\]
This inequality is controlled by sharp Sobolev embedding on \( S^2 \); more precisely, the family of sharp Sobolev inequalities on \( S^2 \) that are determined by the Hardy-Littlewood-Sobolev inequality (see Theorem 4 in [2])
\[
\left( \int_{S^2} |F|^p \, d\xi \right)^{2/p} \leq \frac{p-2}{2} \int_{S^2} |\nabla F|^2 \, d\xi + \int_{S^2} |F|^2 \, d\xi \quad \text{for } 2 \leq p < \infty \text{ and } d\xi \text{ is a normalized surface measure on } S^2 .
\]
Inequality (6) follows from the case \( p = 6 \). Observe that the change of variables defined by stereographic projection between \( \mathbb{R}^2 \) and \( S^2 \{ \text{pole} \} \) can be realized for the polar angle on \( S^2 \) by \( \cos \theta = (1-|x|^2)/(1+|x|^2) \) and \( w = |x|^2 \) in (6). Since inequality (6) then corresponds to functions of the polar angle, it suffices simply to match up the “radial coordinates” in each domain. Then
\[
w \left( \frac{dG}{dw} \right)^2 \, dw = \frac{1}{2} \left( \frac{dG}{d\theta} \right)^2 \sin \theta \, d\theta
\]
so that (7) for \( p = 6 \) and radial variables give a stronger inequality than (6):
\[
\left[ \int_0^\infty |G|^6 (1+w)^{-2} \, dw \right]^{1/3} \leq 2 \int_0^\infty w|G'|^2 \, dw + \int_0^\infty |G|^2 (1+w)^{-2} \, dw .
\]
This shows that equality is achieved in (6) only for constants.

**Proof of equation (3) in Theorem 2.** This result is a special case of an argument in [4] that uses axial symmetry and \( SL(2, \mathbb{R}) \) to derive the sharp Sobolev embedding constant on \( \mathbb{R}^n \) and characterize the extremals for that problem. The motivation for this approach came from problems in fluid mechanics and vortex dynamics (see the discussion concerning the Stokes stream function on page 829 in [4] and [10]). For \( n > 2 \) and \( 1/p = 1/2 - 1/n \)
\[
\|f\|_{L^p(\mathbb{R}^n)} \leq A_p \|\nabla f\|_{L^2(\mathbb{R}^n)}, \quad A_p = [\pi n(n-2)]^{-1/2}[\Gamma(n)/\Gamma(n/2)]^{1/n}
\]
and up to the action of the conformal group, the sharp constant is only attained for functions of the form \( A(1+|x|^2)^{-n/p} \). By using the technique of symmetrization (equimeasurable radial decreasing rearrangement), it suffices to consider this inequality for non-negative radial decreasing functions. For radial functions use the product structure for Euclidean space \( \mathbb{R}^n \cong \mathbb{R} \times \mathbb{R}^{n-1} \) with \( x = (t, x') \) and set \( y = |x'| \). Being radial in \( x \) means that the function is also radial in \( x' \). Let \( g(t, y) = y^{n/p} f(t, x') \) and inequality (9) becomes
\[
\left[ \int_M |g|^p \, d\nu \right]^{2/p} \leq B_p \left[ \int_M |Dg|^2 \, d\nu + \frac{n}{p} \left( \frac{n}{p} - 1 \right) \int_M |g|^2 \, d\nu \right]
\]
where
\[
B_p = \frac{4}{n(n-2)} \left[ \frac{n-1}{2\pi} \right]^{2/n} .
\]
Theorem 4. For one simply calculates that equality is attained for the indicated extremal. The fact that the Dirichlet form in (11) taken only with respect to integration in $x$ and symmetric decreasing in $t$ variables separately. Hence, the function $f$ for non-negative functions ~Riesz-Sobolev rearrangement arguments, it suffices to consider this inequality only.

For higher dimensions this problem has corresponding behavior. Consider $(x, t) \in \mathbb{R} \times \mathbb{R}^2 \simeq \mathbb{R}^3$ with

$$\Delta_G = \Delta_t + 4|t|^2 \frac{\partial^2}{\partial x^2}.$$ 

The homogeneous dimension of this operator is 4. Here one can also use the underlying $SL(2, R)$ symmetry to compute the sharp constant for the associated $L^2$ Sobolev inequality with a similar analysis.

Theorem 3. For $f \in C^1(\mathbb{R}^3)$

$$\left(\|f\|_{L^4(\mathbb{R}^3)}\right)^2 \leq \frac{1}{2\pi} \int_{\mathbb{R} \times \mathbb{R}^2} \left[|\nabla_t f|^2 + 4|t|^2 \left(\frac{\partial f}{\partial x} \right)^2\right] \, dx \, dt.$$ 

This inequality is sharp, and an extremal is given by $[(1 + |t|^2)^2 + |x|^2]^{-1/2}$. 

Theorem 4. For $F \in C^1(\mathbb{R}^3)$

$$\left(\|F\|_{L^4(M)}\right)^2 \leq \frac{2}{\sqrt{\pi}} \left[ \int_{M} |DF|^2 \, dv - \frac{1}{4} \int_{M} |F|^2 \, dv \right].$$ 

Proof of Theorem 3. Let $\hat{f}$ denote the Fourier transform of $f$ in the first variable $x$. Using the Plancherel identity, inequality (11) for some constant $A_0$ is equivalent to

$$\left[ \int_{\mathbb{R}^3} |\hat{f} * \hat{f}|^2 \, d\xi \, dt \right]^{1/2} \leq A_0 \int_{\mathbb{R}^3} \left[ |\nabla_t \hat{f}|^2 + 16\pi^2 |t|^2 |\xi|^2 |\hat{f}|^2 \right] \, d\xi \, dt$$

where here convolution is only with respect to the first variable. By applying Riesz-Sobolev rearrangement arguments, it suffices to consider this inequality only for non-negative functions $\tilde{f}(x, t)$ that are radial decreasing in each of the two variables separately. Hence, the function $f(x, t)$ in (11) can be taken to be radial in $t$ and symmetric decreasing in $x$. The second part of this remark follows from the fact that the Dirichlet form in (11) taken only with respect to integration in $x$ is diminished by a symmetric decreasing equimeasurable rearrangement with respect to the first variable.

Since $f(x, t)$ is radial in $t$, set $y = |t|^2$ and let $f(x, |t|) = y^{-1/2}F(x, y)$. Then

$$\|f\|_{L^4(\mathbb{R}^3)} = \pi^{1/4} \|F\|_{L^4(M)}$$

and inequality (11) is now equivalent to

$$\left(\|F\|_{L^4(M)}\right)^2 \leq 4\sqrt{\pi} A_0 \left[ \int_{M} |DF|^2 \, dv - \frac{1}{4} \int_{M} |F|^2 \, dv \right].$$

This is an a priori inequality where the function $F$ can be taken to be smooth with compact support. Now Theorem 3 will follow from Theorem 4 with $A_0 = 1/(2\pi)$. One simply calculates that equality is attained for the indicated extremal. The
details of the argument to show equivalence follow the method outlined in the proof of Theorem 1. Estimates for the Grushin operator are maximized on the class of functions that are radial in the variable \( t \). Every radial function of compact support in \( t \) can be smoothly approximated by a Schwartz class function that is analytic in \(|t|^2\) through convolution with a Gaussian function. This class of functions suffices to determine the Grushin bound (11). By using the functional transformation \( F(x, y) = y^{1/2} f(x, |t|) \) to transfer the estimate to hyperbolic space, one finds that both the \( L^4 \) norm and the Dirichlet form for the elliptic operator \( L = -y^2 \Delta - (1/4) I \) are finite. Every smooth function on \( M \) with suitable decay at infinity which vanishes to order one-half at the boundary \( y = 0 \) has finite Dirichlet form for inequality (12).

**Proof of Theorem 4.** Using equimeasurable radial decreasing rearrangement corresponding to the metric on hyperbolic space, it suffices to consider this inequality for radial decreasing functions of the distance from the origin. Set \( u = d^2(z, i) \). Then the volume form restricted to integration for radial functions is given by \( du = 4\pi du \) and inequality (12) becomes (see [3])

\[
\left( \int_0^\infty |F|^4 \, du \right)^{1/2} \leq \frac{4}{3} \left( \int_0^\infty \left( u^2 + u \right) |\frac{dF}{du}|^2 \, du - \frac{1}{4} \int_0^\infty |F|^2 \, du \right).
\]

If one can show that this is a good upper bound, then the sequence of functions \( F^*_\varepsilon(u) = (1 + u)^{-\varepsilon} \) for \( \varepsilon > \frac{1}{2} \) shows that the estimate is sharp. Let \( G \in C^2_c([0, \infty)) \) and set \( F(u) = (1 + u)^{-1/2} G(u) \). Then inequality (13) takes the form

\[
\left( \int_0^\infty |G|^4 \frac{1}{(1 + u)^2} \, du \right)^{1/2} \leq \frac{4}{3} \int_0^\infty u(G')^2 \, du + \int_0^\infty |G|^2 \frac{1}{(1 + u)^2} \, du.
\]

This inequality is controlled by sharp Sobolev embedding on \( S^2 \); more precisely, the family of sharp Sobolev inequalities on \( S^2 \) that are determined by the Hardy-Littlewood-Sobolev inequality (see Theorem 4 in [3])

\[
\left( \int_{S^2} |F|^p \, d\xi \right)^{2/p} \leq \frac{p - 2}{2} \int_{S^2} |\nabla F|^2 \, d\xi + \int_{S^2} |F|^2 \, d\xi
\]

for \( 2 \leq p < \infty \) and \( d\xi \) is a normalized surface measure on \( S^2 \). Inequality (14) follows from the case \( p = 4 \). Observe that change of variables defined by stereographic projection between \( \mathbb{R}^2 \) and \( S^2 \{ \text{pole} \} \) can be realized for the polar angle on \( S^2 \) by \( \cos \theta = (1 - |x|^2)/(1 + |x|^2) \) and \( u = |x|^2 \) in (14). Since inequality (14) corresponds to functions of the polar angle, it suffices simply to match up the “radial coordinates” in each domain. Then

\[
u \left( \frac{dG}{d\theta} \right)^2 \, d\theta = \frac{1}{2} \left( \frac{dG}{d\theta} \sin \theta \, d\theta\right)^2
\]

so that (15) for \( p = 4 \) and radial variables gives a stronger inequality than (14):

\[
\left( \int_0^\infty |G|^4 \frac{1}{(1 + u)^2} \, du \right)^{1/2} \leq \frac{4}{3} \int_0^\infty u(G')^2 \, du + \int_0^\infty |G|^2 \frac{1}{(1 + u)^2} \, du.
\]

This shows that Theorem 4 is sharp as a limiting form. However, the limit “extremal”

\[
F(x, y) = [1 + d^2(z, i)]^{-1/2}
\]
is not in \( L^2(M) \). This observation emphasizes that the appropriate Dirichlet form for Sobolev embedding on hyperbolic space \( \mathbb{H}^2 \) should correspond to the intrinsic positive elliptic differential operator

\[
L_s = -y^2 \Delta + s(s - 1)1.
\]

These two results illustrate the complexity and interdependence of Sobolev estimates on Lie groups and symmetric spaces, and demonstrate that there is still much to understand about the geometry of Grushin operators. The elementary nature of these calculations was facilitated by the capability to use rearrangement arguments which here depended on the Sobolev index being an even integer. An interesting aspect of the analysis is that the intermediate estimate on hyperbolic space must be defined as a limiting form using the positive elliptic operator \( L_s \) at the critical exponent for the Grushin embedding estimate.

**Appendix**

The argument used here to relate sharp Sobolev embedding on \( S^2 \) to embedding estimates on hyperbolic space determines a more general family of such estimates.

**Theorem 5.** For \( F \in C^1_c(M) \), \( 0 < s \leq \frac{1}{2} \) and \( p = 2 + \frac{1}{s} \geq 4 \)

\[
\left[ \left\| F \right\|_{L^p(M)} \right]^2 \leq A_p \left[ \int_M |DF|^2 \, dv + s(s - 1) \int_M |F|^2 \, dv \right] ,
\]

\[
A_p = (2\pi)^{\frac{1}{2}} s^{1 - \frac{s}{2}}.
\]

**Proof.** By using equimeasurable radial decreasing rearrangement corresponding to the metric on hyperbolic space, it suffices to consider this inequality for radial decreasing functions of the distance from the origin. Let \( u = [d(z,i)]^2 \). Then (17) is equivalent to

\[
\left[ \int_0^\infty |g|^{2/p} \, du \right]^{2/p} \leq (4\pi)^{1 - \frac{s}{2}} A_p \left[ \int_0^\infty (u + u^2)|g'|^2 \, du + s(s - 1) \int_0^\infty |g|^2 \, du \right] .
\]

Set \( g = (1 + u)^{-\alpha} h \) and \( C_p = (4\pi)^{(p-2)/p} A_p \). Then

\[
\left[ \int_0^\infty |h|^p (1 + u)^{-p\alpha} \, du \right]^{2/p} \leq C_p \left[ \int_0^\infty u(1 + u)^{1 - 2\alpha}|h'|^2 \, du \\
+ \alpha^2 \int_0^\infty |h|^2 (1 + u)^{-1 - 2\alpha} \, du \\
+ (s^2 - s + \alpha - \alpha^2) \int_0^\infty |h|^2 (1 + u)^{-2\alpha} \, du \right] .
\]

Set \( \alpha = s \), \( p\alpha = 2\alpha + 1 \) and \( \beta = 2\alpha > 0 \). Then

\[
\left[ \int_0^\infty |h|^p (1 + u)^{-\beta - 1} \, du \right]^{2/p} \leq C_p \left[ \int_0^\infty u(1 + u)^{1 - \beta}|h'|^2 \, du \\
+ \beta^2 \int_0^\infty |h|^2 (1 + u)^{-1 - \beta} \, du \right] .
\]
Make the change of variables $u \to 1/u$ with $H(u) = h(1/u)$ so that $|h'(1/u)| = u^2|H'(u)|$ and
\[
\left[ \int_0^\infty |H'|^p (1 + u)^{-\beta - 1} u^{\beta - 1} du \right]^{2/p} \leq C_p \left[ \int_0^\infty u^\beta (1 + u)^{-\beta} |H'|^2 du + \beta^2 \int_0^\infty |H|^2 (1 + u)^{-\beta - 1} u^{\beta - 1} du \right].
\]

Now set $(1 + w)^{-2} dw = \beta(1 + u)^{-\beta - 1} u^{\beta - 1} du$ so that
\[
\frac{w}{1 + w} = \left( \frac{u}{1 + u} \right)^\beta
\]
which gives for $G(w) = H(u)$, $2/p = \beta/(1 + \beta)$ and $B_p = \frac{1}{4}\beta(1+2\beta)/(1+\beta)C_p$
\begin{equation}
\left[ \int_0^\infty |G|^p (1 + w)^{-2} dw \right]^{\beta/(1+\beta)} \leq B_p \left[ 4\int_0^\infty \left( (1 + w)^{1/\beta} - w^{1/\beta} \right) |G'|^2 dw + \int_0^\infty |G|^2 (1 + w)^{-2} dw \right].
\end{equation}

This equation can be simplified using the change of variables $w \to 1/w$ and setting $\tilde{G}(w) = G(1/w)$:
\begin{equation}
\left[ \int_0^\infty |\tilde{G}|^p (1 + w)^{-2} dw \right]^{\beta/(1+\beta)} \leq B_p \left[ 4\int_0^\infty \left( (1 + w)^{1/\beta} - 1 \right) |\tilde{G}'|^2 dw + \int_0^\infty |\tilde{G}|^2 (1 + w)^{-2} dw \right].
\end{equation}

Now this estimate should be compared with the sharp Sobolev embedding on $S^2$ that is determined by the Hardy-Littlewood-Sobolev inequality:
\[
\left[ \int_{S^2} |F|^p d\xi \right]^{2/p} \leq \frac{p-2}{2} \int_{S^2} |\nabla F|^2 d\xi + \int_{S^2} |F|^2 d\xi
\]
where $d\xi$ denotes normalized surface measure and $p > 2$, and in turn gives for radial functions and $p = 2(1 + 1/\beta)$
\begin{equation}
\left[ \int_0^\infty |F|^p (1 + w)^{-2} dw \right]^{\beta/(1+\beta)} \leq \frac{1}{\beta} \int_0^\infty w|F|^2 dw + \int_0^\infty |F|^2 (1 + w)^{-2} dw.
\end{equation}

Now set $B_p = 1$ in (19) which corresponds to the value of $A_p$ in (17) and observe that for $r = 1/\beta$ and $w \geq 0$, then $(1 + w)^r \geq 1 + rw$ for $r \geq 1$. Hence the estimate (20) derived from a Sobolev embedding on $S^2$ implies that (19) holds for $B_p = 1$ and $\beta \geq 1$. The proof of Theorem 5 is then complete for $0 < s \leq 1/2$.

A discrete set of hyperbolic embedding estimates can be obtained in the analysis of a Sobolev embedding on the Heisenberg group $\mathcal{H}_n$ realized as the manifold $\mathbb{C}^n \times \mathbb{R}$ and restricted to radial symmetry in the complex variables (see Theorem 18 in [3]).
Theorem 6. For $F \in C^1(M) \cap L^2(M)$, $s = n/2$ for $n \in \mathbb{N}$ and $p = 2 + \frac{1}{s} \leq 4$
\[
\left[ \|F\|_{L^p(M)} \right]^2 \leq A_p \left[ \int_M |DF|^2 \, dv + s(s-1) \int_M |F|^2 \, dv, \right.
\]
\[\left. A_p = (2\pi)^{\frac{n}{2}} \Gamma(-1)^{\frac{n}{2}} \Gamma(-\frac{n}{2}). \right\]
For $n > 1$ and up to the “conformal structure” of $M$, an extremal is given by
\[F(z) = \left[ 1 + d^2(z,i) \right]^{-s}.\]

This family of hyperbolic embedding estimates can be extended to include values of $s \geq 1$ by using duality and the fundamental solution corresponding to the differential operator $L_s$. Note that in this case an $L^2$ extremal function will exist. The fundamental solution for $L_s = -y^2\Delta + s(s-1)I$ for $s \geq 1$ is given by
\[\psi_s(u) = \frac{1}{4\pi} \int_0^1 \left[ f(1-t) \right]^{s-1}(t+u)^{-s} \, dt = \frac{\Gamma(s)\Gamma(s)}{4\pi\Gamma(2s)}(1+u)^{-s}F\left(s, s, 2s; \frac{1}{1+u}\right)\]
where $u = [d(z,i)]^2$ and $F$ is the hypergeometric function. The transition from Sobolev embedding estimates to a Hardy-Littlewood-Sobolev convolution inequality is made using the following lemma.

Lemma. Let $K$ and $\Lambda$ be densely defined, positive-definite, self-adjoint operators acting on functions defined on a $\sigma$-finite measure space $M$ and satisfying the relation
\[\Lambda K = K \Lambda = 1.\]

Then the following two inequalities are equivalent:

(\ast) \quad \|Kf\|_{L^p(M)} \leq C_p \|f\|_{L^p(M)},

(\ast\ast) \quad \|g\|_{L^p(M)} \leq \sqrt{C_p} \|\Lambda^{1/2}g\|_{L^2(M)}.

Here $1 < p < 2$ and $1/p + 1/p' = 1$. Extremal functions for one inequality will determine extremal functions for the other inequality if the operator forms are well-defined.

Proof. In (\ast\ast) substitute $g = Kf$ so that
\[
\left[ \|Kf\|_{L^p(M)} \right]^2 \leq C_p \int_M (Kf)\Lambda(Kf) \, dm = C_p \int_M (Kf)f \, dm
\]
\[\leq C_p \|Kf\|^2_{L^p(M)} \|f\|_{L^p(M)}, \]
which is now (\ast). For equivalence in the reverse direction, $K$ is a positive-definite self-adjoint operator and notice that (\ast) implies
\[\|K^{1/2}f\|^2_{L^2(M)} \leq \sqrt{C_p} \|f\|^2_{L^p(M)}\]
which by duality implies
\[\|K^{1/2}h\|_{L^p(M)} \leq \sqrt{C_p} \|h\|_{L^2(M)} .\]
Now substitute $h = K^{1/2}(Ag)$ which results in (\ast\ast). The full equivalence is obtained by taking limits on dense domains.
For \( s > 0 \) define the fractional integral operator
\[
(I_s G)(z) = \int_M \psi_s[\alpha^2(z, w)] G(w) \ d\nu.
\]

The symmetric space \( SL(2, R)/SO(2) \cong \mathbb{H}^2 \) can be identified with the subgroup of \( SL(2, R) \) given by all matrices of the form
\[
\begin{pmatrix}
\sqrt{y} & x/\sqrt{y} \\
0 & 1/\sqrt{y}
\end{pmatrix}
\]
with \( y > 0 \) and \( x \in \mathbb{R} \) which act via fractional linear transformations on \( \mathbb{R}^2_+ \cong \mathbb{H}^2 \).

\[ z = x + iy \in \mathbb{R}^2_+ \rightarrow \frac{az+b}{cz+d} \]

for \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, R) \). The modular function is \( \Delta(x, y) = 1/y \) and \( d\nu = y^{-2} \ d\sigma \ dy \)
is a left-invariant Haar measure on the group. Observe that the group action here corresponds to the multiplication rule
\[
(x, y)(u, v) = (xuy, yv)
\]
for \( x, u \in \mathbb{R} \) and \( y, v > 0 \). This \( SL(2, R) \) subgroup is the “\( ax+b \) group”, namely the group of all linear transformations of the real line to itself that preserve orientation.

With this framework, the operator \( I_s G \) can be represented as a convolution operator
\[
I_s G = G * \psi_s
\]
where convolution for the left-invariant Haar measure on a locally compact group is defined by
\[
(f * g)(x) = \int_G f(y)g(y^{-1}x) \ d\nu.
\]
Observe that \( L_s(I_s G) = G \) for \( s \geq 1 \). The Riesz-Sobolev inequality and an extension of Young’s inequality to non-unimodular groups provide good estimates for the fractional integral operator \( I_s \).

**Riesz-Sobolev Inequality on \( SL(2, R)/SO(2) \).**
\[
\int_M (f * g)(w)h(w) \ d\nu \leq \int_M (f^* * g^*)(w)h^*(w) \ d\nu
\]
where \( f, g \) and \( h \) are non-negative measurable functions with \( f^*, g^* \) and \( h^* \) denoting their respective equimeasurable, geodesically decreasing rearrangements on \( M \cong SL(2, R)/SO(2) \cong \mathbb{H}^2 \) and \( d\nu \) is a left-invariant Haar measure on \( M \).

**Young’s inequality.** Let \( G \) be a locally compact group with the left-invariant Haar measure denoted by \( m \). For \( 1 \leq p \leq \infty \)
\[
\|f * g\|_{L^p(G)} \leq \|f\|_{L^p(G)}\|\Delta^{-1/p'}g\|_{L^{1/p'}(G)},
\]
\[
\|f * g\|_{L^p(G)} \leq \|f\|_{L^1(G)}\|g\|_{L^p(G)},
\]
\[
\|f * g\|_{L^p(G)} \leq \|f\|_{L^{1}(G)}\|\Delta^{-1/p'}g\|_{L^{q}(G)}
\]
where \( \Delta \) denotes the modular function defined by \( m(Ey) = \Delta(y)m(E) \), \( 1/p + 1/p' = 1 \) and \( 1/r = 1/p + 1/q - 1 \).
The critical estimate is now reduced to the fact that
\[ y(24) \]
where 1
\[ C \]
on \[ 0 \]
For a constant for this estimate will be obtained using duality. The plan of the argument is to use the Riesz-Sobolev inequality to show that
\[ \text{Proof.} \]
\[ (27) \]
show the existence of an extremal for (25), it suffices to consider the functional
\[ \text{for (26) which can be calculated using the Euler-Lagrange variational equation. To show an extremal function exists for (25) and hence by duality an extremal function exists for (26) the necessary estimates to show that I_s is a bounded map from } L^q(M) \text{ to } L^p(M) \text{ where } 1/q + 1/p = 1, \ p = 2 + 1/s \text{ and } q = 2 - 1/(1 + s):
\]
\[ \psi_s(u) \simeq \frac{\Gamma(s)\Gamma(s)}{4\pi\Gamma(2s)} u^{-s} \quad \text{as } u \to \infty \]
\[ \simeq -\frac{1}{4\pi} \ln u \quad \text{as } u \to 0. \]
\[ \text{Hence, any power of } \psi_s \text{ is locally integrable and using Young's inequality} \]
\[ (24) \quad \|f * \psi_s\|_{L^p(M)} \leq \|f\|_{L^q(M)} \|\Delta^{-1/p}\psi_s\|_{L^{p/2}(M)} = \|f\|_{L^q(M)} \|y^{1/p}\psi_s\|_{L^{p/2}(M)}. \]
The critical estimate is now reduced to the fact that \( y^{s-1}(y + 1)^{-2s} \) is integrable on \([0, \infty)\) for \( s > 0 \). So the map \( I_s \) is bounded from \( L^q(M) \) to \( L^p(M) \). The sharp constant for this estimate will be obtained using duality.

**Theorem 7.** For \( s \geq 1, \ p = 2 + 1/s, \ q = 2 - 1/(1 + s) \)
\[ (25) \quad \|I_s G\|_{L^p(M)} \leq A_p \|G\|_{L^q(M)}, \]
\[ A_p = (2\pi)^{\frac{s}{2} - 1} \frac{s - 1 - \frac{2}{p}}{s}. \]
This inequality is sharp and an extremal is given by \([1 + d^2(z, i)]^{-1-s}\). For \( F \in C^2(M) \)
\[ (26) \quad \left[\|F\|_{L^p(M)}^p\right] \leq A_p \int_M F(L_s F) \, d\nu. \]
Here the extremal is \([1 + d^2(z, i)]^{-s}\). Because \( s \geq 1 \), this latter result can be represented for \( F \in C^2(M) \cap L^2(M) \) as
\[ (27) \quad \left[\|F\|_{L^p(M)}^p\right] \leq A_p \left[\int_M |DF|^2 \, d\nu + s(s - 1) \int_M |F|^2 \, d\nu\right]. \]

**Proof.** The plan of the argument is to use the Riesz-Sobolev inequality to show that an extremal function exists for (25) and hence by duality an extremal function exists for (26) which can be calculated using the Euler-Lagrange variational equation. To show the existence of an extremal for (25), it suffices to consider the functional
\[ \int_{M \times M} F(a)\psi_s[d^2(z, w)]G(w) \, d\nu \, d\nu \]
for \( F, G \geq 0 \) and \( \|F\|_q = \|G\|_q = 1 \). By (24) this form is bounded above and by applying the Riesz-Sobolev inequality one can restrict attention to the case where
$F$ and $G$ are geodesically radial decreasing functions. Then consider sequences of functions $\{F_n, G_n\}$ with $\|F_n\|_q = \|G_n\|_q = 1$ so that

$$\int_{M \times M} F_n(z) \psi_s(d^2(z, w)) G_n(w) \, dv \, dw$$

converges to its maximum value. Since these functions are decreasing, one can use the Helly selection principle to choose subsequences that converge almost everywhere to functions $F, G \in L^q(M)$. By Fatou’s lemma $\|F\|_q \leq 1$, $\|G\|_q \leq 1$. Notice that $F_n(z) \leq (4\pi u)^{-1/q}$, $G_n(z) \leq (4\pi u)^{-1/q}$ using the radial variable $u = d^2(z, i)$ since Haar measure restricted to the radial variable is $dv = 4\pi du$. Observe that

$$[d(z, i)]^{-2/q} \psi_s[d^2(z, w)][d(w, i)]^{-2/q} \in L^1(M \times M).$$

Re-label the subsequences to have index $n$. By the dominated convergence theorem

$$\int_{M \times M} F_n(z) \psi_s[d^2(z, w)] G_n(w) \, dv \, dw \longrightarrow \int_{M \times M} F(z) \psi_s[d^2(z, w)] G(w) \, dv \, dw,$$

and so $\|F\|_{L^q(M)} = \|G\|_{L^q(M)} = 1$ and $F, G$ must be extremal functions for (25).

A somewhat similar argument is given in [3], page 40.

From the lemma above, one sees that if $G$ is an extremal for (25), then $F = I_s G$ is an extremal for (26). Moreover, if $G$ is radial decreasing, then $F$ will be radial decreasing since the convolution of two radial decreasing functions is radial decreasing. Hence, such an extremal $F$ must satisfy the Euler-Lagrange variational equation for (26):

$$L_s F = \gamma F^{p-1}, \quad \|F\|_{L^p(M)} = 1.$$

For $F$ a decreasing function of the radial variable $u = d^2(z, i)$, one looks for solutions of the differential equation

$$(28) \quad -\frac{d}{du} \left[ u(u + 1) \frac{dF}{du} \right] + s(s - 1)F = cF^{p-1}, \quad p = 2 + 1/s.$$  

Note that if $F = I_s G$ for $G \in L^q(M)$ with $q = 2 - 1/(1 + s)$, then $F$ is bounded. Hence, there will be a unique solution to (28) that is bounded and monotonically decreasing on $[0, \infty)$. This solution is

$$F(u) = B(1 + u)^{-s},$$

where the constant $B$ is determined by the condition that $\|F\|_q = 1$. Now one can calculate the value of the sharp constant $A_p$. An extremal for (25) is obtained by

$$L_s F = L_s(I_s G) = G.$$

This calculation completes the proof of Theorem 7. The argument developed here complements the result of Theorem 5. Similar methods can also be applied for the case $0 < s < 1$ and will be discussed in a more comprehensive treatment of Riesz potentials and Sobolev embedding on hyperbolic space [6].

Postscript. After this work was completed, I received in April 1999 three interesting papers (see [12], [20] and [21]) that treat some issues discussed here using complementary methods. Mugelli and Talenti also recognize the relation of the Fraenkel inequality to Sobolev inequalities on hyperbolic space and the usefulness of geodesic rearrangement in the hyperbolic setting. Their approach emphasizes geometric aspects, and they do not use the Hardy-Littlewood-Sobolev framework.
of duality. The Garofalo-Vassilev paper considers a broad class of nonlinear subelliptic problems that arise from geometric function theory on nilpotent Lie groups.

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