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A BIFURCATION RESULT FOR HARMONIC MAPS FROM AN ANNULUS TO S^2 WITH NOT SYMMETRIC BOUNDARY DATA

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Abstract. We consider the problem of minimizing the energy of the maps $u(r,\theta)$ from the annulus $\Omega_\rho = B_1 \backslash \bar{B}_\rho$ to S^2 such that $u(r,\theta)$ is equal to $(\cos\theta, \sin\theta, 0)$ for $r=\rho$, and to $(\cos(\theta+\theta_0), \sin(\theta+\theta_0), 0)$ for $r=1$, where $\theta_0 \in [0,\pi]$ is a fixed angle.

We prove that the minimum is attained at a unique harmonic map u_ρ which is a planar map if $\log^2\rho + 3\theta_0^2 \le \pi^2$, while it is not planar in the case $\log^2\rho + \theta_0^2 > \pi^2$.

Moreover, we show that u_ρ tends to $\bar{v}$ as $\rho \to 0$, where $\bar{v}$ minimizes the energy of the maps $v(r,\theta)$ from B_1 to S^2, with the boundary condition $v(1,\theta) = (\cos(\theta+\theta_0), \sin(\theta+\theta_0), 0)$.

1. Introduction

Let $\Omega_\rho = \{(x,y) \in \mathbb{R}^2 \mid \rho^2 < x^2+y^2 < 1\}$ be an annulus of $\mathbb{R}^2$, let $\theta_0 \in [0,\pi]$ be fixed, and let

$$\gamma = \gamma(r,\theta) = \begin{cases} (\cos\theta, \sin,\theta, 0) & \text{if } r=\rho, \\ (\cos(\theta_0), \sin(\theta+\theta_0), 0) & \text{if } r=1. \end{cases}$$

We consider the problem of minimizing the energy functional

$$E_\rho(u) = \int_{\Omega_\rho} |\nabla u|^2 \, dxdy$$

in the class

$$\mathcal{E} = \{u \in H^1(\Omega_\rho, S^2) \mid u = \gamma \quad \text{on } \partial\Omega_\rho\}.$$

The critical points of the functional E_ρ, and, in particular, its minima, are harmonic maps which satisfy the Dirichlet boundary condition $u=\gamma$ on $\partial\Omega_\rho$.

It is well known that the minimum of E_ρ is always attained at some point u_p of $\mathcal{E}$, and in the following we shall prove that this minimum point is unique, and satisfies some kind of symmetry.

On the other hand, we consider the same problem for planar harmonic maps, namely for maps whose image $u(\Omega_\rho)$ lies in the equator S^1 of S^2.

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The latter problem has a simple solution, that is,

$$u_0(r,\theta) = (\cos(\theta + \chi_0(r)), \sin(\theta + \chi_0(r)), 0), \tag{1.1}$$

where

$$\chi_0(r) = \frac{\theta_0 \log(r/\rho)}{\log(1/\rho)}, \tag{1.2}$$

so it is a natural question to wonder when u_0 is also a solution of the first.

The main result of this paper shows that this is the case for $\log^2\rho + 3\theta_0^2 \le \pi^2$, namely for $\theta_0 \le \pi/\sqrt{3}$ and $\rho \ge e^{-\sqrt{\pi^2 - 3\theta_0^2}}$, while for $\log^2\rho + \theta_0^2 > \pi^2$, namely for $\rho < e^{-\sqrt{\pi^2-\theta_0^2}}$, we have a situation of "escape in the third dimension".

In the case $\log^2\rho + 3\theta_0^2 > \pi^2$ and $\log^2\rho + \theta_0^2 \le \pi^2$, the behavior of the energy functional is not clear; we have probably a planar minimum, but we have not been able to prove this.

Let us consider now the minimum $\bar{v}$ of the energy functional on the set

$$\mathcal{E}_B = \{v \in H^1(B_1, S^2) \mid v = \gamma \quad \text{on } \partial B_1\}.$$

Since the energy functional is invariant for rotations, from the results of [2] we know that the minimum is unique, and we have, in polar coordinates,

$$\bar{v}(r,\theta) = (\cos\bar{\Psi}(r)\cos(\theta+\theta_0), \cos\bar{\Psi}(r)\sin(\theta+\theta_0), \sin\bar{\Psi}(r)) \tag{1.3}$$

where

$$\bar{\Psi}(r) = 2\operatorname{arctg}\left(\frac{1}{r}\right) - \frac{\pi}{2}. \tag{1.4}$$

We prove that the minimum u_ρ of E_ρ converges to $\bar{v}$ as $\rho \to 0$, uniformly over the compact subsets of $B_1 \backslash \{(0,0)\}$, and the minimum $E_\rho(u_\rho)$ goes to 8π.

This kind of bifurcation phenomena has been studied in [1] in the case of radially symmetric boundary data, namely in the case $\theta_0 = 0$. In [1] the bifurcation point is $e^{-\pi}$, so that our result is a natural generalization of this situation.

Finally, we remark that, in the case $\theta_0 = \pi$, no bifurcation can occur, in the sense that for all values of $\rho \in]0,1[$, the map u_ρ is not contained in S^1.

2. Proof of the results

Let $\mathcal{E}_0$ be the class of the maps $u = u(r,\theta) \in \mathcal{E}$ such that

$$u(r,\theta) = (\cos\Psi(r)\cos(\theta+\chi(r)), \cos\Psi(r)\sin(\theta+\chi(r)), \sin\Psi(r)) \tag{2.1}$$

for some functions $\Psi : [\rho,1] \to [0,\pi/2], \chi : [\rho,1] \to \mathbb{R}$, with $\chi(\rho) = 0, \chi(1) = \theta_0$.

First of all, we shall prove that, if $E_\rho(u) = \min_{\mathcal{E}} E_\rho$, then $u \in \mathcal{E}_0$, and it is unique.

The proof is based on the symmetrization method of [3] (see also [4], [5]), and we sketch it for the reader's convenience.

Theorem 2.1. *If $E_\rho(u) = \min_{\mathcal{E}} E_\rho$, then $u \in \mathcal{E}_0$; moreover, if $E_\rho(u_1) = E_\rho(u_2) = \min_{u\in\mathcal{E}} E_\rho(u)$, then $u_1 = u_2$.*

Proof. Let $u \in \mathcal{E}$ be such that $E_\rho(u) = \min_{\mathcal{E}} E_\rho$. Then, in polar coordinates,

$$E_\rho(u) = \int_\rho^1 \left(\int_{S_r} \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 ds \right) d\rho,$$

where $S_r = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = r^2\}$.

We set

$$e^N(r) = \int_{S_r} \left(\frac{\partial u}{\partial r}\right)^2 ds, \qquad e^T(r) = \int_{S_r} \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 ds, \tag{2.2}$$

and consider the components of u in the radial, tangential and vertical directions:

$$\begin{aligned} u^N(r,\theta) &= \langle u(r,\theta) \mid (\cos\theta, \sin\theta, 0)\rangle, \\ u^T(r,\theta) &= \langle u(r,\theta) \mid (-\sin\theta, \cos\theta, 0)\rangle, \\ u^Z(r,\theta) &= \langle u(r,\theta) \mid (0,0,1)\rangle. \end{aligned} \tag{2.3}$$

Then, a simple calculation shows that

$$e^N(r) = \int_{S_r} \left(\left(\frac{\partial u^N}{\partial r}\right)^2 + \left(\frac{\partial u^T}{\partial r}\right)^2 + \left(\frac{\partial u^Z}{\partial r}\right)^2\right) ds$$

and

$$\begin{aligned} e^T(r) = \frac{1}{r^2}\int_{S_r} &\left(\frac{\partial u^N}{\partial \theta}\right)^2 + \left(\frac{\partial u^T}{\partial \theta}\right)^2 + \left(\frac{\partial u^Z}{\partial \theta}\right)^2 \\ &+ (u^N)^2 + (u^T)^2 - 2\frac{\partial u^N}{\partial \theta}u^T + 2\frac{\partial u^T}{\partial \theta}u^N\, ds. \end{aligned}$$

We consider now the function $\tilde{u}$ with components

$$\begin{aligned} \tilde{u}^N(r,\theta) &\equiv \tilde{u}^N(r) = \frac{1}{2\pi r}\int_{S_r} u^N(r,\theta) ds, \\ \tilde{u}^T(r,\theta) &\equiv \tilde{u}^T(r) = \frac{1}{2\pi r}\int_{S_r} u^T(r,\theta) ds, \\ \tilde{u}^Z(r,\theta) &\equiv \tilde{u}^Z(r) = \sqrt{1 - \tilde{u}^N(r)^2 - \tilde{u}^T(r)^2}. \end{aligned}$$

Clearly $\tilde{u} \in \mathcal{E}_0$, and we prove now that

$$\tilde{e}^N(r) \le e^N(r), \tag{2.4}$$

$$\tilde{e}^T(r) \le e^T(r), \tag{2.5}$$

where $\tilde{e}^N(r)$ and $\tilde{e}^T(r)$ are the analogues of (2.2) for the function $\tilde{u}$.

In fact, since

$$\frac{\partial u^Z}{\partial r} = \frac{\partial}{\partial r}\left(\sqrt{1 - u^N(r)^2 - u^T(r)^2}\right),$$

we have

$$e^N(r) = \int_{S_r} \left(\left(\frac{\partial u^N}{\partial r}\right)^2 + \left(\frac{\partial u^T}{\partial r}\right)^2 + \frac{\left\langle \left(\frac{\partial u^N}{\partial r}, \frac{\partial u^T}{\partial r}\right) \mid (u^N, u^T)\right\rangle^2}{1 - (N^2)^2 - (u^T)^2}\right) ds$$

so (2.4) follows from the convexity of the function $G(v,w) = |w|^2 + \frac{\langle w|v\rangle^2}{1-|v|^2}$ (see [5], Lemma 1).

Moreover, since the components of $\tilde{u}$ do not depend on θ, we have:

$$\frac{1}{r^2}\int_{S_r}\left(\frac{\partial(u-\tilde{u})}{\partial\theta}\right)^2 ds = \frac{1}{r^2}\int_{S_r}\left(\frac{\partial u}{\partial\theta}\right)^2 ds - \frac{1}{r^2}\int_{S_r}\left(\frac{\partial\tilde{u}}{\partial u}\right)^2 ds$$
$$= e^T(r) - \tilde{e}^T(r),$$

so that (2.5) follows.

Since u is a minimum of E_ρ, we have $e^T(r) = \tilde{e}^T(r)$, so that $u - \tilde{u}$ is constant on S_r, thus $u = \tilde{u}$.

In order to prove uniqueness, let us suppose that the minimum of E_ρ is attained at u_1 and u_2, with $u_1 \neq u_2$, and set $u_3 = (u_1 + u_2)/2$.

Using the fact that the components of u_i, $i = 1, 2, 3$, do not depend on θ, we get

$$\frac{1}{2}(e_1^T(r) + e_2^T(r)) - e_3^T(r) = \frac{1}{4r^2}\int_{S_r}(u_1^N - u_2^N)^2 + (u_1^T - u_2^T)^2\, ds,$$

where $e_i^T(r)$ is the analogue of (2.2) for u_i, so

$$e_3^T(r) < \frac{1}{2}(e_1^T(r) + e_2^T(r)).$$

On the other hand, we have, because of the convexity of $G(v, w)$,

$$e_3^N(r) \leq \frac{1}{2}(e_1^N(r) + e_2^N(r)).$$

Then

$$E_\rho(u_3) < \frac{1}{2}(E_\rho(u_1) + E_\rho(u_2)) = \min_{u\in\mathcal{E}} E_\rho(u),$$

a contradiction. □

Now, let us consider the energy E_ρ for the functions (2.1), with the substitution $r = e^t$:

$$E_\rho(u) = 2\pi\int_{\log\rho}^{0}(\varphi'^2 + (1+\sigma'^2)\cos^2\varphi)dt,$$

where $\varphi(t) = \Psi(e^t)$ and $\sigma(t) = \chi(e^t)$.

It is easy to see that the critical points (φ, σ) of E_ρ satisfy the following system of ordinary differential equations:

$$\begin{cases}\varphi'' + (1+\sigma'^2)\sin\varphi\cos\varphi = 0,\\ -\sigma''\cos\varphi + 2\varphi'\sigma'\sin\varphi = 0\end{cases} \tag{2.6}$$

on $[\log\rho, 0]$.

From (2.6) we have that φ is concave on $[\log\rho, 0]$ and symmetric with respect to $t = (\log\rho)/2$, with a maximum point at $t = (\log\rho)/2$, whereas $\sigma(t) = \theta_0 - \sigma(\log\rho - t)$, namely σ is symmetric with respect to $\theta_0/2$, and it is convex on $[\log\rho, \log\rho/2]$, and concave on $[\log\rho/2, 0]$.

Moreover, from the second equation of the system, we get $d(\sigma'\cos^2\varphi)/dt = 0$, so that

$$\sigma'(t)\cos^2\varphi(t) = \lambda \tag{2.7}$$

on $[\log\rho, 0]$, and, by inserting this in the first equation, $d(\varphi'^2 + \sin^2(\varphi) + \lambda^2\mathrm{tg}^2(\varphi))/dt = 0$.

Finally, we notice that

$$E_\rho(u) - E_\rho(u_0) = 2\pi \int_{\log\rho}^{0} \left(\varphi'^2 - \sin^2\varphi + \sigma'^2 \cos^2\varphi - \frac{\theta_0^2}{\log^2\rho} \right) dt,$$

where u_0 is the function (1.1).

Then we can prove the following theorem.

Theorem 2.2. *Let $\rho \in]0,1[$; we have:*
1⁰) *if $\log^2\rho + \theta_0^2 > \pi^2$, then $\min_{u\in\mathcal{E}} E_\rho(u) < E_\rho(u_0)$;*
2⁰) *if $\log^2\rho + 3\theta_0^2 \le \pi^2$, then $\min_{u\in\mathcal{E}} E_\rho(u) = E_\rho(u_0)$.*

Proof. Let us suppose first that $\log^2\rho + \theta_0^2 > \pi^2$, that is, $\rho < e^{-\sqrt{\pi^2-\theta_0^2}}$, and set

$$u(r,\theta) = (\cos\Psi(r)\cos(\theta+\chi_0(r)), \cos\Psi(r)\sin(\theta+\chi_0(r)), \sin\Psi(r)),$$

where χ_0 is the function defined in (1.2). Then we have

$$\begin{aligned}
\min_{u\in\mathcal{E}} E_\rho(u) - E_\rho(u_0) &\le E_\rho(u) - E_\rho(u_0) \\
&= 2\pi \int_{\log\rho}^{0} \left(\varphi'^2 - \sin^2\varphi + \frac{\theta_0^2}{\log^2\rho}\cos^2\varphi - \frac{\theta_0^2}{\log^2\rho} \right) dt \\
&= 2\pi \int_{\log\rho}^{0} \left(\varphi'^2 - \left(1 + \frac{\theta_0^2}{\log^2\rho}\right)\sin^2\varphi \right) dt.
\end{aligned}$$

Setting $\varphi(t) = \beta\sin(\pi t/|\log\rho|)$ as in [1], and using the fact that $\rho < e^{-\sqrt{\pi^2-\theta_0^2}}$ implies $\pi^2/\log^2\rho < (1+\theta_0^2/\log^2\rho)$, we have

$$\begin{aligned}
&2\pi \int_{\log\rho}^{0} \left(\varphi'^2 - \left(1 + \frac{\theta_0^2}{\log^2\rho}\right)\sin^2\varphi \right) dt \\
&\quad < 2\pi \left(1 + \frac{\theta_0^2}{\log^2\rho}\right) \int_{\log\rho}^{0} \left(\beta^2\cos^2\left(\frac{\pi t}{|\log\rho|}\right) - \sin^2\left(\beta\sin\left(\frac{\pi t}{|\log\rho|}\right)\right)\right) dt,
\end{aligned}$$

so that $E_\rho(u) - E_\rho(u_0) < 0$ for some value of β, and this proves the first claim of the theorem.

Let us suppose now that $\log^2\rho + 3\theta_0^2 \le \pi^2$, and let $u \in \mathcal{E}_0$ be such that

$$E_\rho(u) = \min_{u\in\mathcal{E}} E_\rho(u).$$

We want to show that $\varphi \equiv 0$, so that $u = u_0$. Arguing by contradiction, we suppose $\varphi \ne 0$; then we have $M \equiv \max_{t\in[\log\rho,0]} \varphi(t) > 0$.

Because of the concavity of φ on $[\log\rho, 0]$, we get

$$\varphi(t) \ge \frac{2M}{\log\rho}(\log\rho - t)$$

on $[\log\rho, \log\rho/2]$, so that

$$\int_{\log\rho}^{0} \varphi^2 dt \ge \frac{|\log\rho|}{3} M^2.$$

Since

$$\int_{\log\rho}^{0} \varphi'^2 dt \ge \frac{\pi^2}{\log^2\rho} \int_{\log\rho}^{0} \varphi^2 dt$$

and $\pi^2/\log^2\rho > 1$, we have

$$\int_{\log\rho}^{0} (\varphi'^2 - \varphi^2)dt \geq \int_{\log\rho}^{0} \left(\frac{\pi^2}{\log^2\rho} - 1\right)\varphi^2\,dt \geq \left(\frac{\pi^2}{\log^2\rho} - 1\right)\frac{|\log\rho|}{3}M^2. \tag{2.8}$$

From (2.7) we get

$$\lambda = \sigma'(t)\cos^2(\varphi(t)) \geq \sigma'(t)\cos^2 M$$

and, integrating over $[\log\rho, 0]$:

$$\lambda \geq \frac{\theta_0}{|\log\rho|}\cos^2 M.$$

Then

$$\sigma'^2(t)\cos^2(\varphi(t)) = \lambda\sigma'(t) \geq \sigma'(t)\frac{\theta_0}{|\log\rho|}\cos^2 M. \tag{2.9}$$

By using (2.8) and (2.9), we get

$$\begin{aligned}
\min_{u\in\mathcal{E}} E_\rho(u) - E_\rho(u_0) &= E_\rho(u) - E_\rho(u_0)\\
&= 2\pi\int_{\log\rho}^{0}\left(\varphi'^2 - \sin^2\varphi + \sigma'^2\cos^2\varphi - \frac{\theta_0^2}{\log^2\rho}\right)dt\\
&= 2\pi\int_{\log\rho}^{0}\left((\varphi'^2 - \varphi^2) + (\varphi^2 - \sin^2\varphi) + \sigma'^2\cos^2\varphi - \frac{\theta_0^2}{\log^2\rho}\right)dt\\
&\geq 2\pi\left(\left(\frac{\pi^2}{\log^2\rho} - 1\right)\frac{|\log\rho|}{3}M^2 + \frac{\theta_0^2}{|\log\rho|}\cos^2 M - \frac{\theta_0^2}{|\log|}\right)\\
&= 2\pi\left(\left(\frac{\pi^2}{\log^2\rho} - 1\right)\frac{|\log\rho|}{3}M^2 - \frac{\theta_0^2}{|\log\rho|}\sin^2 M\right) > 0,
\end{aligned}$$

since $\log^2\rho + 3\theta_0^2 \leq \pi^2$ implies

$$\left(\frac{\pi^2}{\log^2\rho} - 1\right)\frac{|\log\rho|}{3} \geq \frac{\theta_0^2}{|\log\rho|},$$

and M is strictly positive. We have obtained a contradiction, so that $M = 0$, and the theorem is proved. □

Let us denote now by u_ρ the minimum point of the functional E_ρ; we set

$$u_\rho(r,\theta) = (\cos\Psi_\rho(r)\cos(\theta + \chi_\rho(r)), \cos\Psi_\rho(r)\sin(\theta + \chi_\rho(r)), \sin\Psi_\rho(r)),$$

so that

$$E_\rho(u_\rho) \equiv E_\rho(\varphi_\rho, \sigma_\rho) = 2\pi\int_{\log\rho}^{0}(\varphi_\rho'^2 + (1 + \sigma_\rho'^2)\cos^2(\varphi_\rho))dt,$$

where $\varphi_\rho(t) = \Psi_\rho(e^t)$ and $\sigma_\rho(t) = \chi_\rho(e^t)$.

We recall that φ_ρ and σ_ρ are solutions of the system (2.6), namely

$$\begin{cases} \varphi_\rho'' + (1 + \sigma_\rho'^2)\sin\varphi_\rho\cos\varphi_\rho = 0,\\ -\sigma_\rho''\cos\varphi + 2\varphi_\rho'\sigma_\rho'\sin\varphi_\rho = 0, \end{cases} \tag{2.10}$$

and that

$$\sigma_\rho'(t)\cos^2\varphi_\rho(t) = \lambda_\rho \tag{2.11}$$

on $[\log\rho, 0]$.

We want to study the behaviour of u_ρ as $\rho \to 0$. For this purpose, it is convenient to introduce some notation for the sets of the functions $u = u(r,\theta)$ from Ω_ρ and B_1 to S^2 with standard radially symmetric boundary conditions.

We set

$$\bar{\mathcal{E}}_\rho = \{u \in H^1(\Omega_\rho, S^2) \mid u(\rho,\theta) = u(1,\theta) = (\cos\theta, \sin\theta, 0)\},$$
$$\bar{\mathcal{E}}_B = \{u \in H^1(B_1, S^2) \mid u(1,\theta) = (\cos\theta, \sin\theta, 0)\},$$

and denote by $\bar{E}_\rho$ and $\bar{E}$ the corresponding energy functionals.

From [1], [4] we know that the minima of the functionals $\bar{E}_\rho$ and $\bar{E}$ are attained at a unique map $\bar{u}_\rho$ and $\bar{u}$ respectively (modulo reflections), which is radially symmetric, namely

$$\bar{u}_\rho(r,\theta) = (\cos(\bar{\Psi}_\rho(r))\cos\theta, \cos(\bar{\Psi}_\rho(r))\sin\theta, \sin(\bar{\Psi}_\rho(r)))$$

and

$$\bar{u}(r,\theta) = (\cos(\bar{\Psi}(r))\cos\theta, \cos(\bar{\Psi}(r))\sin\theta, \sin(\bar{\Psi}(r))).$$

The map $\bar{u}$ corresponds to the stereographic projection of B_1 onto the northern hemisphere, so that

$$\bar{\Psi}(r) = 2\operatorname{arctg}\left(\frac{1}{r}\right) - \frac{\pi}{2}.$$

Moreover, after the substitution $r = e^t$, we have

$$\bar{E}_\rho(\bar{u}_\rho) = 2\pi \int_{\log\rho}^{0} (\bar{\varphi}_\rho'^2 + \cos^2(\bar{\varphi}_\rho))dt,$$

where $\bar{\varphi}_\rho(t) = \bar{\Psi}_\rho(e^t)$.

As $\rho \to 0$, the map $\bar{u}_\rho$ tends to $\bar{u}$ uniformly over the compact sets contained in $B_1 \backslash \{(0,0)\}$, and the minimum $\bar{E}_\rho(\bar{u}_\rho)$ tends to $8\pi = \bar{E}(\bar{u}) + 4\pi$ (see [1], Theorem 2).

We now prove the following theorem.

Theorem 2.3. *As $\rho \to 0$, the map u_ρ tends to $\bar{v}$ (see* (1.3)*) uniformly over the compact set of $B_1 \backslash \{(0,0)\}$, and $E_\rho(u_\rho)$ tends to 8π.*

Proof. Since u_ρ is a minimum point, replacing the functions φ_ρ and σ_ρ with $\bar{\varphi}_\rho$ and $\theta_0 - t\theta_0/\log\rho$ respectively, we have

$$\begin{aligned} E_\rho(u_\rho) &\le 2\pi \int_{\log\rho}^{0} \left(\bar{\varphi}_\rho'^2 + \left(1 + \frac{\theta_0^2}{\log^2\rho}\right)\cos^2(\bar{\varphi}_\rho)\right) dt \\ &= \bar{E}_\rho(\bar{u}_\rho) + \frac{2\pi\theta_0^2}{\log^2\rho} \int_{\log\rho}^{0} \cos^2(\bar{\varphi}_\rho)dt. \end{aligned}$$

On the other hand, since $\bar{u}_\rho$ is the minimum point of $\bar{E}_\rho$, we have

$$E_\rho(u_\rho) \ge 2\pi \int_{\log\rho}^{0} (\varphi_\rho'^2 + \cos^2(\varphi_\rho))dt \ge 2\pi \int_{\log\rho}^{0} (\bar{\varphi}_\rho'^2 + \cos^2(\bar{\varphi}_\rho))dt = \bar{E}_\rho(\bar{u}_\rho),$$

so that

$$\bar{E}_\rho(\bar{u}_\rho) \le E_\rho(u_\rho) \le \bar{E}_\rho(\bar{u}_\rho) + \frac{2\pi\theta_0^2}{\log^2\rho} \int_{\log\rho}^{0} \cos^2(\bar{\varphi}_\rho)dt.$$

Passing to the limit as $\rho \to 0$, we get

$$\lim_{\rho\to 0} E_\rho(u_\rho) = 8\pi. \tag{2.12}$$

Moreover, (2.12) implies that $\int_{\log\rho}^{0} \varphi_\rho'^2 dt$ is bounded as $\rho \to 0$, so $(\varphi_\rho)_\rho$ converges weakly in $H^1_{\mathrm{loc}}(]-\infty,0])$ to a function $\hat\varphi :]-\infty,0] \to [0,\pi/2]$. Notice that, modulo subsequences, $\varphi_\rho \to \hat\varphi$ uniformly on the compact subsets of $]-\infty,0]$, so $\hat\varphi$ is a continuous and concave function, and $\hat\varphi(0) = 0$.

We want to show that

$$\hat\varphi(t) = 2\,\mathrm{arctg}\left(\frac{1}{e^t}\right) - \frac{\pi}{2},$$

and that $\sigma_\rho \to \theta_0$ uniformly on the compact subsets of $]-\infty,0]$, which implies that u_ρ tends to $\bar v$ uniformly on the compact subsets of $B_1\backslash\{(0,0)\}$. □

For this purpose we now need the following simple lemma.

Lemma 2.1. *We have $\hat\varphi \in C^\infty(]-\infty,0])$, $\varphi_\rho \to \hat\varphi$, $\sigma_\rho \to \theta_0$ in $C^\infty_{\mathrm{loc}}(]-\infty,0])$ (modulo subsequences), and $\hat\varphi'' + \sin\hat\varphi\cos\hat\varphi = 0$ on $]-\infty,0]$.*

Proof. Let $L < 0$ be fixed, and let $\rho < e^{2L}$, so that $\log\rho/2 < L$. Since σ_ρ is concave on $[\log\rho/2, 0]$, we have $\sigma_\rho'(L) \le \sigma_\rho'(t)$ on $[\log\rho/2, L]$; then

$$\frac{\theta_0}{2} > \sigma_\rho(L) - \sigma_\rho\left(\frac{\log\rho}{2}\right) = \int_{\log\rho/2}^{L} \sigma_\rho'(t)dt \ge \sigma_\rho'(L)\left(L - \frac{\log\rho}{2}\right),$$

so that $\sigma_\rho'(L) \to 0$ as $\rho \to 0$.

Since $\sigma_\rho'(L) \ge \sigma_\rho'(t)$ on $[L,0]$, we have $\sigma_\rho'(t) \to 0$ uniformly on $[L,0]$. At this point the lemma follows by using standard arguments. □

Proof of Theorem 2.3 continued. From Lemma 2.1, we have $\hat\varphi'' + \sin\hat\varphi\cos\hat\varphi = 0$ on $]-\infty,0]$, so the map

$$\hat u(r,\theta) = (\cos(\hat\Psi(r))\cos\theta, \cos(\hat\Psi(r))\sin\theta, \sin(\hat\Psi(r))),$$

where $\hat\Psi(r) = \hat\varphi(\log r)$, is harmonic from B_1 to S^2. Because of [1], Theorem A.1 and Corollary, $\hat u = \bar u$, and this completes the proof. □

References

1. F. Bethuel, H. Brezis, B. D. Coleman and F. Hélein, *Bifurcation analysis of minimizing harmonic maps describing the equilibrium of nematic phases between cylinders*, Arch. Rational Mech. Anal. **118** (1992), 149–168. MR **93h:**49059
2. H. Brezis and J. M. Coron, *Large solutions for harmonic maps in two dimensions*, Comm. Math. Phys. **92** (1983), 203–215. MR **85a:**58022
3. S. Kaniel and I. Shafrir, *A new symmetrization method for vector valued maps*, C. R. Acad. Sci. Paris, t. **315**, Série I (1992), 413–416. MR **93g:**58037
4. E. Sandier and I. Shafrir, *On the symmetry of minimizing harmonic maps in N dimensions*, Differential and Integral Eq. **6** no. 6 (1993), 1531–1541. MR **94i:**58046
5. E. Sandier and I. Shafrir, *On the uniqueness of minimizing harmonic maps to a closed hemisphere*, Calc. Var **2** (1994), 113–122. MR **97b:**58041

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