

## THE DIRICHLET-JORDAN TEST AND MULTIDIMENSIONAL EXTENSIONS

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ABSTRACT. If  $\mathcal{F}$  is a foliation of an open set  $\Omega \subset \mathbb{R}^n$  by smooth  $(n-1)$ -dimensional surfaces, we define a class of functions  $\mathcal{B}(\Omega, \mathcal{F})$ , supported in  $\Omega$ , that are, roughly speaking, smooth along  $\mathcal{F}$  and of bounded variation transverse to  $\mathcal{F}$ . We investigate geometrical conditions on  $\mathcal{F}$  that imply results on pointwise Fourier inversion for these functions. We also note similar results for functions on spheres, on compact 2-dimensional manifolds, and on the 3-dimensional torus. These results are multidimensional analogues of the classical Dirichlet-Jordan test of pointwise convergence of Fourier series in one variable.

Suppose  $f \in L^1(\mathbb{R}^n)$ , with Fourier transform

$$(1) \quad \hat{f}(\xi) = (2\pi)^{-n/2} \int f(x) e^{-ix \cdot \xi} dx.$$

We set

$$(2) \quad S_R f(x) = (2\pi)^{-n/2} \int_{|\xi| \leq R} \hat{f}(\xi) e^{ix \cdot \xi} d\xi.$$

When  $n = 1$ , the Dirichlet-Jordan test for pointwise convergence of  $S_R f(x)$  as  $R \rightarrow \infty$  states that, if  $f$  has bounded variation, then for each  $x \in \mathbb{R}$ ,

$$(3) \quad \lim_{R \rightarrow \infty} S_R f(x) = \frac{1}{2} \lim_{\varepsilon \searrow 0} [f(x + \varepsilon) + f(x - \varepsilon)].$$

This can be established as follows. Pick a function  $h(t)$ , equal to 0 for  $t < 0$ , 1 for  $0 < t \leq 1$ , smooth on  $(0, \infty)$ , and equal to 0 for  $t \geq 2$ . Set  $h(0) = 1/2$ . By Riemann's localization principle there is no loss of generality in assuming  $f$  has compact support. If  $f$  has bounded variation, its distributional derivative  $f' = \mu$  is a (signed) measure, and we have

$$(4) \quad f(x) = \int h(x - y) d\mu(y) + g(x),$$

with  $g \in C_0^\infty(\mathbb{R})$ . If  $f(x)$  is adjusted to equal the right side of (3) at each point of discontinuity, then (4) holds for all  $x \in \mathbb{R}$ . Then we have

$$(5) \quad S_R f(x) = \int S_R h(x - y) d\mu(y) + S_R g(x).$$

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Obviously  $S_R g(x) \rightarrow g(x)$  for all  $x$ . The Dirichlet-Jordan result can then be proven using the following two properties of  $S_R h$ :

$$(6) \quad S_R h(x) \rightarrow h(x), \quad \text{for every } x \in \mathbb{R}$$

(including  $x = 0$ ), and, for some  $C < \infty$ , independent of  $x, R$ ,

$$(7) \quad |S_R h(x)| \leq C.$$

To establish (6), one can appeal to the Dini test, or use localization and smoothness for  $x \neq 0$ , plus a symmetrization argument to cover the case  $x = 0$ . The bound (7) is a consequence of the analysis of the Gibbs phenomenon for  $S_R h$ . From this, the Dirichlet-Jordan result can be deduced via Lebesgue's dominated convergence theorem. Let us state an abstract version of this last segment of the argument.

Let  $(Y, \mathfrak{B})$  be a set with sigma algebra, let  $\mu$  be a finite signed measure on  $\mathfrak{B}$ , and let  $X$  be a set. Let  $h_R : X \times Y \rightarrow \mathbb{C}$  be given, for each  $R \in (0, \infty)$ . Assume that  $h_R(x, \cdot)$  is  $\mathfrak{B}$ -measurable, for each  $x \in X, R \in (0, \infty)$ , that

$$(8) \quad |h_R(x, y)| \leq C, \quad \forall x \in X, y \in Y, R \in (0, \infty),$$

and that

$$(9) \quad \lim_{R \rightarrow \infty} h_R(x, y) = h(x, y), \quad \forall x \in X, y \in Y.$$

Then

$$(10) \quad \lim_{R \rightarrow \infty} \int_Y h_R(x, y) d\mu(y) = \int_Y h(x, y) d\mu(y), \quad \forall x \in X.$$

As mentioned, this is simply a consequence of the dominated convergence theorem. The role played by  $X$  here is, in essence, trivial, except for the fact that it arises in nontrivial contexts.

Multidimensional analogues of functions for which (6)–(7) hold arise as follows. Let  $\Sigma$  be a smooth  $(n - 1)$ -dimensional surface in  $\mathbb{R}^n$ . Let  $\mathcal{C}_1(\Sigma)$  denote the set of caustic points of order  $\geq 1$ , in the terminology used in §10 of [PT]. (This follows Definition 5.2.3 of [Dui], in the case where  $\Lambda$  is the Lagrangian flow-out of the unit normal bundle of  $\Sigma$ .) Let  $\mathcal{O}_\Sigma$  be an open neighborhood of  $\mathcal{C}_1(\Sigma)$ . Let  $h(x)$  be a piecewise smooth function, with compact support, with simple jump across  $\Sigma$ . For  $x \in \Sigma$ , set  $h(x)$  equal to the mean value of its limits from each side. The fact that

$$(11) \quad S_R h(x) \rightarrow h(x), \quad \forall x \in \mathbb{R}^n \setminus \mathcal{O}_\Sigma,$$

follows from Proposition 26 in §10 of [PT] (the result for  $x \in \Sigma$  holding by the analysis in §11). The fact that, for any compact  $K \subset \mathbb{R}^n \setminus \mathcal{O}_\Sigma$ ,

$$(12) \quad |S_R h(x)| \leq C_K, \quad \forall R \in (0, \infty), x \in K,$$

follows from the analysis of the Gibbs phenomenon in §11 of [PT] (cf. also [CV]). We note that  $\mathcal{C}_1(\Sigma)$  is empty when  $n = 2$ . Also, when  $n = 3$ ,  $\mathcal{C}_1(\Sigma)$  is empty if  $\Sigma$  is real analytic and not part of a sphere (as noted by [K]).

Now suppose we have a foliation of an open set  $\Omega \subset \mathbb{R}^n$  by such surfaces. More precisely, suppose we have smooth functions  $u_1, \dots, u_{n-1}, v$  on  $\Omega$ , producing a diffeomorphism

$$(13) \quad (u_1, \dots, u_{n-1}, v) : \Omega \rightarrow Q \subset \mathbb{R}^n,$$

where  $Q$  is the open cube  $(-\pi, \pi) \times \cdots \times (-\pi, \pi)$ . We consider the family of surfaces  $\Sigma_c = \{x \in \Omega : v(x) = c\}$ . Assume that  $\mathcal{O}$  is an open neighborhood of the union of the sets  $\mathcal{C}_1(\Sigma_c)$ . Fix  $\varphi \in C_0^\infty(\Omega)$ . Let  $h_t : \Omega \rightarrow \mathbb{R}$  be given by

$$h_t(x) = \begin{cases} 1 & \text{if } v(x) > t, \\ \frac{1}{2} & \text{if } v(x) = t, \\ 0 & \text{if } v(x) < t. \end{cases}$$

Let  $K$  be any compact set in  $\mathbb{R}^n \setminus \mathcal{O}$ . Then, for each  $g \in C^\infty(\Omega)$ , we have

$$(14) \quad |S_R(g\varphi h_t)(x)| \leq C_K(g), \quad \forall R \in (0, \infty), x \in K, t \in I = (-\pi, \pi).$$

Hence, if we set  $\Phi(g)(R, x, t) = S_R(g\varphi h_t)(x)$ , we have

$$(15) \quad \Phi : C^\infty(\Omega) \rightarrow L^\infty((0, \infty) \times K \times (-\pi, \pi)).$$

Now, if we compose this with the inclusion  $\iota : L^\infty((0, \infty) \times K \times (-\pi, \pi)) \rightarrow L_{\text{loc}}^\infty((0, \infty) \times K \times (-\pi, \pi))$ , it is easy to see that the map

$$\iota \circ \Phi : C^\infty(\Omega) \rightarrow L_{\text{loc}}^\infty((0, \infty) \times K \times (-\pi, \pi))$$

is continuous. It follows that the map  $\Phi$  in (15) has closed graph. Hence, we can apply the closed graph theorem and deduce that

$$(16) \quad \sup_{x \in K, t \in I, R \in (0, \infty)} |S_R(g\varphi h_t)(x)| \leq C_K \|g\|_{H^\ell(\Omega)},$$

for some finite  $\ell$ . This estimate can also be demonstrated by a recollection of what makes geometrical optics constructions work, up to any given finite order, and its implementation for the analysis of the Gibbs phenomenon in [PT]. (It would be of interest to study the optimal value of  $\ell$ , but we will not pursue this here. We will stipulate that  $\ell > n/2$ .)

Now, if  $\mu$  is a finite (signed) measure on  $I$  we can say that, for each  $g \in H^\ell(\Omega)$ ,

$$(17) \quad f(x) = \int_I g(x)\varphi(x)h_t(x) d\mu(t) \Rightarrow S_R f(x) \rightarrow f(x), \quad \forall x \in \mathbb{R}^n \setminus \mathcal{O}.$$

This class of synthesized functions is somewhat constrained, but it will serve as a starting point for an analysis of a much more natural class of functions, which we will now introduce.

Let  $\Omega \subset \mathbb{R}^n$  be open and let  $\mathcal{F} = \{\Sigma_c : c \in I\}$  be a foliation of  $\Omega$  by smooth  $(n - 1)$ -dimensional surfaces. Let  $\mathcal{M}(\Omega)$  denote the space of finite (signed) Borel measures on  $\Omega$ . We say

$$(18) \quad f \in \mathcal{B}(\Omega, \mathcal{F})$$

if  $f$  is a compactly supported element of  $L^\infty(\Omega)$  with the property that

$$(19) \quad X_1 \cdots X_k f \in \mathcal{M}(\Omega),$$

for any  $k$ , and any smooth vector fields  $X_1, \dots, X_k$  on  $\Omega$ , provided that *at most one* of them is not tangent to  $\mathcal{F}$ . One would have the same class of functions if one insisted the one exceptional vector field be  $X_1$  (or that it be  $X_k$ ). The following is our main result.

**Theorem 1.** *Given  $f \in \mathcal{B}(\Omega, \mathcal{F})$ , there exists a Borel measurable  $\tilde{f}$ , equal to  $f$  a.e., such that, as  $R \rightarrow \infty$ ,*

$$(20) \quad S_R f(x) \rightarrow \tilde{f}(x), \quad \forall x \in \mathbb{R}^n \setminus \mathcal{O},$$

where  $\mathcal{O}$  is a neighborhood of the union of  $\mathcal{C}_1(\Sigma_c)$ ,  $c \in I$ .

To begin the proof, we note that  $\mathcal{B}(\Omega, \mathcal{F})$  is clearly a module over  $C_0^\infty(\Omega)$ . Hence, using a partition of unity, we can assume that  $\Omega$  is as in (13), and  $\Sigma_c = \{v = c\}$ . Use the inverse of the diffeomorphism in (13) to pull  $f$  back to a compactly supported element  $g \in L^\infty(Q)$ , with the property on  $\omega = \partial g / \partial x_n$  that

$$(21) \quad \Delta_T^M \omega \in \mathcal{M}(Q), \quad M = 0, 1, 2, \dots,$$

where

$$(22) \quad \Delta_T = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_{n-1}^2}.$$

For the first  $n - 1$  factors of  $(-\pi, \pi)$  in  $Q$ , throw in the endpoints and identify them, to regard  $\omega$  as a compactly supported measure on  $\mathbb{T}^{n-1} \times (-\pi, \pi)$ . We have, for  $\varphi$  continuous on  $[-\pi, \pi]$ ,

$$(23) \quad |\langle \varphi(t) e^{-ik \cdot x'}, \Delta_T^M \omega \rangle| \leq C_M \|\varphi\|_{L^\infty},$$

with  $x' = (x_1, \dots, x_{n-1}) \in \mathbb{T}^{n-1}$ ,  $k \in \mathbb{Z}^{n-1}$ , so

$$(24) \quad |\langle \varphi(t) e^{-ik \cdot x'}, \omega \rangle| \leq C_M \langle k \rangle^{-M} \|\varphi\|_{L^\infty}.$$

Hence, we have measures  $\mu_k$  on  $(-\pi, \pi)$ , supported on  $[-a, a]$  for some  $a < \pi$ , such that

$$(25) \quad \|\mu_k\|_{\mathcal{M}(I)} \leq C_M \langle k \rangle^{-2M}, \quad \omega = \sum_k e^{ik \cdot x'} \mu_k,$$

where the norm denotes the total variation of  $\mu_k$ . Hence

$$(26) \quad g(x', y) = \int_{-\pi}^y \sum_k e^{ik \cdot x'} d\mu_k(t),$$

so

$$(27) \quad \begin{aligned} f(x) &= \sum_k e^{ik \cdot u(x)} \int_{-\pi}^{v(x)} d\mu_k(t) \\ &= \varphi(x) \sum_k e^{ik \cdot u(x)} \int_{-\pi}^{v(x)} d\mu_k(t) \\ &= \sum_k \varphi(x) g_k(x) \int_{-\pi}^{v(x)} d\mu_k(t), \end{aligned}$$

where we choose  $\varphi \in C_0^\infty(\Omega)$  equal to 1 on the support of  $f$ , and set  $g_k(x) = e^{ik \cdot u(x)}$ , with  $u(x) = (u_1(x), \dots, u_{n-1}(x))$ . The estimates done above imply convergence in sup-norm of the infinite series, to a function  $\tilde{f}(x)$  equal a.e. to  $f(x)$ . The analysis done above also shows that, for

$$(28) \quad f_k(x) = \varphi(x) g_k(x) \int_{-\pi}^{v(x)} d\mu_k(t),$$

we have

$$(29) \quad S_R f_k(x) \rightarrow f_k(x), \quad x \in \mathbb{R}^n \setminus \mathcal{O},$$

and, for each compact  $K \subset \mathbb{R}^n \setminus \mathcal{O}$ ,

$$(30) \quad \sup_{R \in (0, \infty), x \in K} |S_R f_k(x)| \leq C_K \|g_k\|_{H^\epsilon(\Omega)} \|\mu_k\|_{\mathcal{M}(I)}.$$

Now

$$(31) \quad \|g_k\|_{H^\ell(\Omega)} \leq C\langle k \rangle^\ell,$$

so, given  $N$ , we can produce  $M = M(\ell, N)$  and apply (25) to obtain

$$(32) \quad \sup_{R \in (0, \infty), x \in K} |S_R f_k(x)| \leq C_{KN} \langle k \rangle^{-N}.$$

Thus, from (27), we have, for  $x \in K$ ,

$$(33) \quad \lim_{R \rightarrow \infty} S_R f(x) = \sum_k f_k(x) = \tilde{f}(x),$$

and the theorem is proven.

It is clear what sort of representative of the class of  $f \in \mathcal{B}(\Omega, F)$  the function  $\tilde{f}(x)$  is. If  $x_0 \in \Sigma_c \subset \Omega$ , then  $\tilde{f}(x_0)$  is the mean of the limit of  $\tilde{f}(x)$  as  $x \rightarrow x_0$  from within  $\{v(x) > c\}$  and as  $x \rightarrow x_0$  from within  $\{v(x) < c\}$ . In particular, for each  $x_0 \in \Omega$ ,

$$(34) \quad \tilde{f}(x_0) = \lim_{r \searrow 0} \frac{1}{V_n r^n} \int_{|y| < r} f(x_0 + y) dy,$$

where  $V_n$  is the volume of the unit ball in  $\mathbb{R}^n$ .

There are other Riemannian manifolds  $M$  besides  $\mathbb{R}^n$  for which there are analogues of Theorem 1, with

$$(35) \quad S_R f(x) = \chi_R(\sqrt{-\Delta})f(x),$$

where  $\Delta$  is the Laplace-Beltrami operator on  $M$  and  $\chi_R(\lambda)$  is 1 for  $|\lambda| < R$ , 0 for  $|\lambda| > R$ , and  $1/2$  for  $|\lambda| = R$ . One class of examples is the class of “strongly scattering manifolds,” in the terminology of [PT], §10. Using the “compactification” trick from §6 of [PT], we can extend Theorem 1 to the case where  $M$  is a sphere  $S^n$ , or other compact rank-one symmetric space. Using results of [BC], we can extend Theorem 1 to compact 2-dimensional manifolds (and then  $\mathcal{O}$  is empty). Using Theorem 5.4 of [T], we can extend Theorem 1 to the case  $M = \mathbb{T}^3$ , as long as all the leaves  $\Sigma_c$  of  $\mathcal{F}$  have nonzero Gauss curvature in  $\Omega$ .

#### REFERENCES

- [BC] L. Brandolini and L. Colzani, *Localization and convergence of eigenfunction expansions*, J. Fourier Anal. **5** (1999), 431–447. CMP 2000:12
- [CV] L. Colzani and M. Vignati, *Gibbs phenomena for multiple Fourier integrals*, J. Approximation Theory **80** (1995), 119–131. MR **95k**:42021
- [Dui] J. J. Duistermaat, *Fourier Integral Operators*, Birkhäuser, Boston, 1996. MR **96m**:58245
- [K] J.-P. Kahane, *Le phénomène de Pinsky et la géométrie des surfaces*, CRAS Paris **321** (1995), 1027–1029. MR **96m**:42018
- [PT] M. Pinsky and M. Taylor, *Pointwise Fourier inversion: a wave equation approach*, J. Fourier Anal. **3** (1997), 647–703. MR **99d**:42019
- [T] M. Taylor, *Pointwise Fourier inversion on tori and other compact manifolds*, J. Fourier Anal. **5** (1999), 449–463. CMP 2000:12

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