THE DIRICHLET-JORDAN TEST
AND MULTIDIMENSIONAL EXTENSIONS

MICHAEL TAYLOR

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Abstract. If \( F \) is a foliation of an open set \( \Omega \subset \mathbb{R}^n \) by smooth \((n-1)\)-dimensional surfaces, we define a class of functions \( B(\Omega, F) \), supported in \( \Omega \), that are, roughly speaking, smooth along \( F \) and of bounded variation transverse to \( F \). We investigate geometrical conditions on \( F \) that imply results on pointwise Fourier inversion for these functions. We also note similar results for functions on spheres, on compact 2-dimensional manifolds, and on the 3-dimensional torus. These results are multidimensional analogues of the classical Dirichlet-Jordan test of pointwise convergence of Fourier series in one variable.

Suppose \( f \in L^1(\mathbb{R}^n) \), with Fourier transform

\[
\hat{f}(\xi) = (2\pi)^{-n/2} \int f(x) e^{-ix\cdot \xi} \, dx.
\]

We set

\[
S_R f(x) = (2\pi)^{-n/2} \int_{|\xi| \leq R} \hat{f}(\xi) e^{ix\cdot \xi} \, d\xi.
\]

When \( n = 1 \), the Dirichlet-Jordan test for pointwise convergence of \( S_R f(x) \) as \( R \to \infty \) states that, if \( f \) has bounded variation, then for each \( x \in \mathbb{R} \),

\[
\lim_{R \to \infty} S_R f(x) = \frac{1}{2} \lim_{\varepsilon \to 0} [f(x + \varepsilon) + f(x - \varepsilon)].
\]

This can be established as follows. Pick a function \( h(t) \), equal to 0 for \( t < 0 \), 1 for \( 0 < t \leq 1 \), smooth on \((0, \infty)\), and equal to 0 for \( t \geq 2 \). Set \( h(0) = 1/2 \). By Riemann’s localization principle there is no loss of generality in assuming \( f \) has compact support. If \( f \) has bounded variation, its distributional derivative \( f' = \mu \) is a (signed) measure, and we have

\[
f(x) = \int h(x - y) \, d\mu(y) + g(x),
\]

with \( g \in C^\infty_0(\mathbb{R}) \). If \( f(x) \) is adjusted to equal the right side of (3) at each point of discontinuity, then (3) holds for all \( x \in \mathbb{R} \). Then we have

\[
S_R f(x) = \int S_R h(x - y) \, d\mu(y) + S_R g(x).
\]

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Obviously $S_R g(x) \to g(x)$ for all $x$. The Dirichlet-Jordan result can then be proven using the following two properties of $S_R h$:

\begin{equation}
S_R h(x) \to h(x), \quad \text{for every } x \in \mathbb{R}
\end{equation}

(including $x = 0$), and, for some $C < \infty$, independent of $x, R$,

\begin{equation}
|S_R h(x)| \leq C.
\end{equation}

To establish (6), one can appeal to the Dini test, or use localization and smoothness for $x \neq 0$, plus a symmetrization argument to cover the case $x = 0$. The Dirichlet-Jordan result can then be deduced via Lebesgue’s dominated convergence theorem. Let us state an abstract version of this last segment of the argument.

Let $(Y, \mathcal{B})$ be a set with sigma algebra, let $\mu$ be a finite signed measure on $\mathcal{B}$, and let $X$ be a set. Let $h_R : X \times Y \to \mathbb{C}$ be given, for each $R \in (0, \infty)$. Assume that $h_R(x, \cdot)$ is $\mathcal{B}$-measurable, for each $x \in X, R \in (0, \infty)$, that

\begin{equation}
|h_R(x, y)| \leq C, \quad \forall x \in X, y \in Y, R \in (0, \infty),
\end{equation}

and that

\begin{equation}
\lim_{R \to \infty} h_R(x, y) = h(x, y), \quad \forall x \in X, y \in Y.
\end{equation}

Then

\begin{equation}
\lim_{R \to \infty} \int_Y h_R(x, y) \, d\mu(y) = \int_Y h(x, y) \, d\mu(y), \quad \forall x \in X.
\end{equation}

As mentioned, this is simply a consequence of the dominated convergence theorem. The role played by $X$ here is, in essence, trivial, except for the fact that it arises in nontrivial contexts.

Multidimensional analogues of functions for which (6)–(7) hold arise as follows. Let $\Sigma$ be a smooth $(n - 1)$-dimensional surface in $\mathbb{R}^n$. Let $C_1(\Sigma)$ denote the set of caustic points of order $\geq 1$, in the terminology used in §10 of [PT]. (This follows Definition 5.2.3 of [Dm], in the case where $\Lambda$ is the Lagrangian flow-out of the unit normal bundle of $\Sigma$.) Let $\mathcal{O}_\Sigma$ be an open neighborhood of $C_1(\Sigma)$. Let $h(x)$ be a piecewise smooth function, with compact support, with simple jump across $\Sigma$. For $x \in \Sigma$, set $h(x)$ equal to the mean value of its limits from each side. The fact that

\begin{equation}
S_R h(x) \to h(x), \quad \forall x \in \mathbb{R}^n \setminus \mathcal{O}_\Sigma,
\end{equation}

follows from Proposition 26 in §10 of [PT] (the result for $x \in \Sigma$ holding by the analysis in §11). The fact that, for any compact $K \subset \mathbb{R}^n \setminus \mathcal{O}_\Sigma$,

\begin{equation}
|S_R h(x)| \leq C_K, \quad \forall R \in (0, \infty), x \in K,
\end{equation}

follows from the analysis of the Gibbs phenomenon in §11 of [PT] (cf. also [CV]). We note that $C_1(\Sigma)$ is empty when $n = 2$. Also, when $n = 3$, $C_1(\Sigma)$ is empty if $\Sigma$ is real analytic and not part of a sphere (as noted by [K]).

Now suppose we have a foliation of an open set $\Omega \subset \mathbb{R}^n$ by such surfaces. More precisely, suppose we have smooth functions $u_1, \ldots, u_{n-1}, v$ on $\Omega$, producing a diffeomorphism

\begin{equation}
(u_1, \ldots, u_{n-1}, v) : \Omega \to Q \subset \mathbb{R}^n,
\end{equation}
where $Q$ is the open cube $(-\pi, \pi) \times \cdots \times (-\pi, \pi)$. We consider the family of surfaces $\Sigma_c = \{ x \in \Omega : v(x) = c \}$. Assume that $\mathcal{O}$ is an open neighborhood of the union of the sets $\mathcal{C}_1(\Sigma_c)$. Fix $\varphi \in C^0_0(\Omega)$. Let $h_t : \Omega \to \mathbb{R}$ be given by
\[
h_t(x) = \begin{cases} 
1 & \text{if } v(x) > t, \\
\frac{1}{2} & \text{if } v(x) = t, \\
0 & \text{if } v(x) < t.
\end{cases}
\]

Let $K$ be any compact set in $\mathbb{R}^n \setminus \mathcal{O}$. Then, for each $g \in C^\infty(\Omega)$, we have
\[
|S_R(g \varphi h_t)(x)| \leq C_K(g), \quad \forall R \in (0, \infty), x \in K, t \in I = (-\pi, \pi).
\]
Hence, if we set $\Phi(g)(R, x, t) = S_R(g \varphi h_t)(x)$, we have
\[
\Phi : C^\infty(\Omega) \to L^\infty((0, \infty) \times K \times (-\pi, \pi)).
\]
Now, if we compose this with the inclusion $\iota : L^\infty((0, \infty) \times K \times (-\pi, \pi)) \to L^\infty_{\text{loc}}((0, \infty) \times K \times (-\pi, \pi))$, it is easy to see that the map
\[
\iota \circ \Phi : C^\infty(\Omega) \to L^\infty_{\text{loc}}((0, \infty) \times K \times (-\pi, \pi))
\]
is continuous. It follows that the map $\Phi$ in (15) has closed graph. Hence, we can apply the closed graph theorem and deduce that
\[
\sup_{x \in K, t \in I, R \in (0, \infty)} |S_R(g \varphi h_t)(x)| \leq C_K \|g\|_{H^\ell(\Omega)},
\]
for some finite $\ell$. This estimate can also be demonstrated by a recollection of what makes geometrical optics constructions work, up to any given finite order, and its implementation for the analysis of the Gibbs phenomenon in [PT]. (It would be of interest to study the optimal value of $\ell$, but we will not pursue this here. We will stipulate that $\ell > n/2$.)

Now, if $\mu$ is a finite (signed) measure on $I$ we can say that, for each $g \in H^\ell(\Omega)$,
\[
f(x) = \int_I g(x) \varphi(x) h_t(x) \, d\mu(t) \to S_R f(x) \to f(x), \quad \forall x \in \mathbb{R}^n \setminus \mathcal{O}.
\]
This class of synthesized functions is somewhat constrained, but it will serve as a starting point for an analysis of a much more natural class of functions, which we will now introduce.

Let $\Omega \subset \mathbb{R}^n$ be open and let $\mathcal{F} = \{ \Sigma_c : c \in I \}$ be a foliation of $\Omega$ by smooth $(n-1)$-dimensional surfaces. Let $\mathcal{M}(\Omega)$ denote the space of finite (signed) Borel measures on $\Omega$. We say
\[
f \in \mathcal{B}(\Omega, \mathcal{F})
\]
if $f$ is a compactly supported element of $L^\infty(\Omega)$ with the property that
\[
X_1 \cdots X_k f \in \mathcal{M}(\Omega),
\]
for any $k$, and any smooth vector fields $X_1, \ldots, X_k$ on $\Omega$, provided that at most one of them is not tangent to $\mathcal{F}$. One would have the same class of functions if one insisted the one exceptional vector field be $X_1$ (or that it be $X_k$). The following is our main result.

**Theorem 1.** Given $f \in \mathcal{B}(\Omega, \mathcal{F})$, there exists a Borel measurable $\tilde{f}$, equal to $f$ a.e., such that, as $R \to \infty$,
\[
S_R f(x) \to \tilde{f}(x), \quad \forall x \in \mathbb{R}^n \setminus \mathcal{O},
\]
where $\mathcal{O}$ is a neighborhood of the union of $\mathcal{C}_1(\Sigma_c), c \in I$. 

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To begin the proof, we note that $B(\Omega, F)$ is clearly a module over $C_0^\infty(\Omega)$. Hence, using a partition of unity, we can assume that $\Omega$ is as in (13), and $\Sigma_c = \{v = c\}$. Use the inverse of the diffeomorphism in (13) to pull $f$ back to a compactly supported element $g \in L^\infty(Q)$, with the property on $\omega = \partial g/\partial x_n$ that
\begin{equation}
\Delta^M T \omega \in \mathcal{M}(Q), \quad M = 0, 1, 2, \ldots,
\end{equation}
where
\begin{equation}
\Delta T = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{n-1}^2}.
\end{equation}
For the first $n - 1$ factors of $(-\pi, \pi)$ in $Q$, throw in the endpoints and identify them, to regard $\omega$ as a compactly supported measure on $\mathbb{T}^{n-1} \times (-\pi, \pi)$. We have, for $\varphi$ continuous on $[-\pi, \pi]$,
\begin{equation}
|\langle \varphi(t)e^{-ik \cdot x'}, \Delta^M T \omega \rangle| \leq C_M \|\varphi\|_{L^\infty},
\end{equation}
with $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{T}^{n-1}$, $k \in \mathbb{Z}^{n-1}$, so
\begin{equation}
|\langle \varphi(t)e^{-ik \cdot x'}, \omega \rangle| \leq C_M (k)^{-M} \|\varphi\|_{L^\infty}.
\end{equation}
Hence, we have measures $\mu_k$ on $(-\pi, \pi)$, supported on $[-a, a]$ for some $a < \pi$, such that
\begin{equation}
\|\mu_k\|_{\mathcal{M}(I)} \leq C_M (k)^{-2M}, \quad \omega = \sum_k e^{ik \cdot x'} \mu_k,
\end{equation}
where the norm denotes the total variation of $\mu_k$. Hence
\begin{equation}
g(x', y) = \int_{-\pi}^{\pi} \sum_k e^{ik \cdot x'} \, d\mu_k(t),
\end{equation}
so
\begin{align}
f(x) &= \sum_k e^{ik \cdot u(x)} \int_{-\pi}^{\pi} \, d\mu_k(t) \\
&= \varphi(x) \sum_k e^{ik \cdot u(x)} \int_{-\pi}^{\pi} \, d\mu_k(t) \\
&= \sum_k \varphi(x) g_k(x) \int_{-\pi}^{\pi} \, d\mu_k(t),
\end{align}
where we choose $\varphi \in C_0^\infty(\Omega)$ equal to 1 on the support of $f$, and set $g_k(x) = e^{ik \cdot u(x)}$, with $u(x) = (u_1(x), \ldots, u_{n-1}(x))$. The estimates done above imply convergence in sup-norm of the infinite series, to a function $\tilde{f}(x)$ equal a.e. to $f(x)$. The analysis done above also shows that, for
\begin{equation}
f_k(x) = \varphi(x) g_k(x) \int_{-\pi}^{\pi} \, d\mu_k(t),
\end{equation}
we have
\begin{equation}
S_R f_k(x) \to f_k(x), \quad x \in \mathbb{R}^n \setminus \mathcal{O},
\end{equation}
and, for each compact $K \subset \mathbb{R}^n \setminus \mathcal{O}$,
\begin{equation}
\sup_{R \in (0, \infty), x \in K} |S_R f_k(x)| \leq C_K \|g_k\|_{H^1(\Omega)} \|\mu_k\|_{\mathcal{M}(I)}.
\end{equation}
Now
\[ \|g_k\|_{H^s(\Omega)} \leq C(k)^s, \]
so, given \( N \), we can produce \( M = M(\ell, N) \) and apply (26) to obtain
\[ \sup_{R \in (0, \infty), x \in K} |S_R f_k(x)| \leq C_{kN}(\ell)^{-N}. \]
Thus, from (27), we have, for \( x \in K \),
\[ \lim_{R \to \infty} S_R f(x) = \sum_k f_k(x) = \tilde{f}(x), \]
and the theorem is proven.

It is clear what sort of representative of the class of \( f \in \mathcal{B}(\Omega, F) \) the function \( \tilde{f}(x) \) is. If \( x_0 \in \Sigma_c \subset \Omega \), then \( \tilde{f}(x_0) \) is the mean of the limit of \( \tilde{f}(x) \) as \( x \to x_0 \) from within \( \{ v(x) > c \} \) and as \( x \to x_0 \) from within \( \{ v(x) < c \} \). In particular, for each \( x_0 \in \Omega \),
\[ \tilde{f}(x_0) = \lim_{r \to 0} \frac{1}{V_{n_0}} \int_{|y| < r} f(x_0 + y) \, dy, \]
where \( V_{n_0} \) is the volume of the unit ball in \( \mathbb{R}^n \).

There are other Riemannian manifolds \( M \) besides \( \mathbb{R}^n \) for which there are analogues of Theorem 1 with
\[ S_R f(x) = \chi_R(\sqrt{-\Delta}) f(x), \]
where \( \Delta \) is the Laplace-Beltrami operator on \( M \) and \( \chi_R(\lambda) \) is 1 for \( |\lambda| < R \), 0 for \( |\lambda| > R \), and 1/2 for \( |\lambda| = R \). One class of examples is the class of "strongly scattering manifolds," in the terminology of [PT], §10. Using the "compactification" trick from §6 of [PT], we can extend Theorem 1 to the case where \( M \) is a sphere \( S^n \), or other compact rank-one symmetric space. Using results of [BC], we can extend Theorem 1 to compact 2-dimensional manifolds (and then \( \mathcal{O} \) is empty). Using Theorem 5.4 of [PT], we can extend Theorem 1 to the case \( M = \mathbb{T}^3 \), as long as all the leaves \( \Sigma_c \) of \( \mathcal{F} \) have nonzero Gauss curvature in \( \Omega \).

References