PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 4, Pages 1113–1121
S 0002-9939(00)05660-4
Article electronically published on December 7, 2000

# STABILITY OF WAVELET FRAMES AND RIESZ BASES, WITH RESPECT TO DILATIONS AND TRANSLATIONS

JING ZHANG

(Communicated by Christopher D. Sogge)

*Dedicated to the memory of Professor Long Ruilin*

Abstract. We consider the perturbation problem of wavelet frame (Riesz basis) $\{\psi_{j,k,a_0,b_0}\} = \{a_0^{nj/2}\psi(a_0^j x - kb_0)\}$ about dilation and translation parameters $a_0$ and $b_0$. For wavelet functions whose Fourier transforms have small supports, we give a method to determine whether the perturbation system $\{\psi_{j,k,a,b_0}\}$ is a frame (Riesz basis) and prove the stability about dilation parameter $a_0$ on Paley-Wiener space. For a great deal of wavelet functions, we give a definite answer to the stability about translation $b_0$. Moreover, if the Fourier transform $\hat{\psi}$ has small support, we can estimate the frame bounds about the perturbation of translation parameter $b_0$. Our methods can be used to handle nonhomogeneous frames (Riesz basis).

## 1. Introduction

Frame is an important topic in both harmonic analysis and wavelet theory. For a general introduction, see [B], [BHW], [BW], [D1], [D2], [HW] and [Y]. A set of vectors $\{f_j\}_{j\in N}$ in a separable Hilbert space $\mathcal{H}$ is called a frame if there are constants $A$ and $B$ such that for every $f \in \mathcal{H}$

$$A\|f\|^2 \le \sum_{j\in N} |\langle f, f_j\rangle|^2 \le B\|f\|^2.$$

The constants $A$ and $B$ are called frame bounds. If only the right-hand inequality is satisfied for all $f \in \mathcal{H}$, then $\{f_j\}$ is called a Bessel sequence with bound $B$. Many people considered the stability of frames. Duffin and Schaeffer introduced frames and the frame algorithm which makes it possible to reconstruct any element $f \in \mathcal{H}$ from the sequence of coefficients $\langle f, f_j\rangle$ uniquely and stably. Their algorithm is stable if the elements $f_j$ are exactly known. Favier and Zalik [FZ] and Zhang [Z1] considered the wavelet frames $\{a^{nj/2}\psi(a^j x - kb)\}_{j\in Z,k\in Z^n}$ and Gabor frames $\{e^{i\langle jb,x\rangle}\psi(x - ka)\}_{j,k\in Z^n}$, which are useful in many areas of mathematics, physics and engineering, e.g., harmonic analysis, quantum mechanics, scattering theory, signal and image processing. They studied under what conditions the frames are stable if $\psi$ is replaced by another function $\phi$ or the $k$ are replaced by approximations $\{\lambda_{j,k}\}$. Balan studied the perturbation of translation parameter $b$. This problem

Received by the editors July 13, 1998 and, in revised form, July 1, 1999.
2000 *Mathematics Subject Classification.* Primary 42C15; Secondary 41A30.
*Key words and phrases.* Frames, stability, wavelets.

was first considered by Daubechies and Tchamichian for Meyer orthogonal wavelet basis in 1990 [D2]. We quote Balan's result as follows.

**Theorem** ([Ba])**.** *Let $\{\psi_{j,k,a_0,b_0}\}$ be a wavelet Riesz basis on $L^2(R)$ with frame bounds $A$ and $B$. Furthermore, suppose that $\hat{\psi}$, the Fourier transform of $\psi$, satisfies the following requirements: $\hat{\psi}$ is in the class $C^1$ on $R$ and both $\hat{\psi}$ and $\hat{\psi}'$ are bounded by*

$$|\hat{\psi}(\xi)|, |\hat{\psi}'(\xi)| \leq C\frac{|\xi|^\alpha}{(1+|\xi|)^\gamma}$$

*for some $C > 0$, $\gamma > 1+\alpha > 1$. Then there exists an $\varepsilon > 0$ such that for any $b$ with $|b-b_0| < \varepsilon$, the set $\{\psi_{j,k,a_0,b}\}$ is a Riesz basis, where $\psi_{j,k,a,b}(x) = a^{j/2}\psi(a^j x - kb)$.*

Christensen, Heil and Walnut studied the stability of frames in Hilbert space from the point of view of a more general stability theory [C], [CH], [HW]. Feichtinger and Gröchenig studied the stability theory for atoms in Banach spaces [FG].

In this paper, we consider the stability problems, that is, the perturbation of dilation $a$ and translation $b$.

We first explain our notations in this paper. We use $C$ to denote constant and do not distinguish different constants. $N$ is the set of all positive integers, and $Z$ is the set of all integers. We define the Fourier transform, inner product and norm by

$$\hat{f}(\xi) = \int f(x)e^{-ix\cdot\xi}dx, \quad \langle f, g\rangle = \int f(x)\overline{g(x)}dx, \quad \|f\| = \langle f, f\rangle^{\frac{1}{2}},$$

so that

$$\int f(x)\overline{g(x)}dx = (2\pi)^{-n}\int \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi, \quad \|f\|^2 = (2\pi)^{-n}\int |\hat{f}(\xi)|^2 d\xi,$$

where we omit the integral area $R^n$.

We need four lemmas; for the details, see [CH], [FZ], [Z1] and [Z2].

**Lemma 1** ([CH], [FZ])**.** *Let $\{f_j\}$ be a frame (Riesz basis) in Hilbert space $\mathcal{H}$ with frame bounds $A$ and $B$. Assume $\{g_j\} \subset \mathcal{H}$ and $\{f_j - g_j\}$ is a Bessel sequence with bound $M < A$. Then $\{g_j\}$ is a frame (Riesz basis) with frame bounds $A[1-(M/A)^{\frac{1}{2}}]^2$ and $B[1+(M/B)^{\frac{1}{2}}]^2$.*

Note that in Lemma 2, the coefficients are different from Theorem 2 in [Z1] because we give a different inner product.

**Lemma 2** ([CS], [D1], [L], [Z1])**.** *If $\{\psi_{j,k,a,b}\}$ is a wavelet frame with frame bounds $A$ and $B$, then*

$$b^n A \leq \sum_{j\in Z} |\hat{\psi}(a^j\xi)|^2 \leq b^n B \qquad a.e. \quad \xi \in R^n,$$

*where $\psi_{j,k,a,b}(x) = a^{nj/2}\psi(a^j x - kb)$.*

**Definition.** If $\phi,\ \psi \in L^2(R^n)$, $\int \psi(x)dx = 0$ but $\int \phi(x)dx \neq 0$, $\{\phi_{k,b}(x), \psi_{j,k,a,b}(x)\}_{j\in N, k\in Z^n}$ is a frame (Riesz basis) on $L^2(R^n)$, then we call it a non-homogeneous frame (Riesz basis), where $\phi_{k,b}(x) = \phi(x-kb)$.

**Lemma 3** ([Z2]). *If $\{\phi_{k,b}(x), \psi_{j,k,a,b}(x)\}_{j\in N, k\in Z^n}$ is a nonhomogeneous frame on $L^2(R^n)$ with frame bounds $A$ and $B$, we have a necessary condition*

$$Ab^n \le |\hat{\phi}(\xi)|^2 + \sum_{j\in N} |\hat{\psi}(a^{-j}\xi)|^2 \le Bb^n \qquad a.e. \quad \xi \in R^n.$$

Using the triangle inequality, the next lemma obviously holds.

**Lemma 4.** *Assume that $\{f_j\}_{j\in N}$ is a frame on $L^2(R^n)$ with frame bounds $A$ and $B$. If*

$$|\sum_j |\langle f, f_j\rangle|^2 - \sum_j |\langle f, g_j\rangle|^2| \le M\|f\|^2 < A\|f\|^2,$$

*then $\{g_j\}$ is a frame with frame bounds $A-M$ and $B+M$.*

We first consider the perturbation of translation parameter $b$ in Theorem 1 and Theorem 2, and then dilation $a$ in Theorem 3. We prove similar results for nonhomogeneous frames in Theorem 4 and Theorem 5.

**Theorem 1.** *If $\{\psi_{j,k,a_0,b_0}\}$ is a frame (Riesz basis) on $L^2(R^n)$ with frame bounds $A$ and $B$, $\hat{\psi}$ is continuous and bounded by*

$$|\hat{\psi}(\xi)| \le C\frac{|\xi|^\alpha}{(1+|\xi|)^{n+\gamma}},$$

*where $\gamma > \alpha > 0$. Then there exists a $\delta > 0$ such that for any $b$ with $|b-b_0| < \delta, \{\psi_{j,k,a_0,b}\}$ is a frame (Riesz Basis) on $L^2(R^n)$.*

If $\hat{\psi}$ has a small support, we can compute the frame bounds as follows.

**Theorem 2.** *Suppose that $\{\psi_{j,k,a_0,b_0}\}$ is a frame on $L^2(R^n)$ with frame bounds $A$ and $B$. Then there exists a $\delta > 0$ such that for any $b$ with $|b-b_0| < \delta$ and supp $\hat{\psi} \subset [-\pi/(b_0 \vee b), \pi/(b_0\vee b)]^n$, $\{\psi_{j,k,a_0,b}\}$ is a frame with frame bounds $A-M$ and $B+M$, where $M = |1-(\frac{b_0}{b})^n|B < A$, $b_0 \vee b = \max(b_0, b)$.*

**Theorem 3.** *Suppose that $\{\psi_{j,k,a_0,b_0}\}$ is a frame on $L^2(R^n)$, supp $\hat{\psi} \subset [-\pi/b_0, \pi/b_0]^n$, and*

$$m = \text{esssup}|\sum_j (|\hat{\psi}(a_0^{-j}\xi)|^2 - \hat{\psi}(a^{-j}\xi)|^2)| < b_0^n A.$$

*Then $\{\psi_{j,k,a,b_0}\}$ is a frame on $L^2(R^n)$ with frame bounds $A - b_0^{-n}m$ and $B + b_0^{-n}m$. In particular, if $\hat{\psi}$ is continuous and*

$$|\hat{\psi}(\xi)| \le C\frac{|\xi|^\alpha}{(1+|\xi|)^{\gamma}}, \qquad \gamma > \alpha > 0.$$

*Then $\{\psi_{j,k,a,b_0}\}$ is a frame on Paley-Wiener space for all $a$ in some neighborhood of $a_0$.*

**Theorem 4.** *Let $\{\phi_{k,b_0}, \psi_{j,k,a_0,b_0}\}$ be a nonhomogeneous frame (Riesz basis) on $L^2(R^n)$ with frame bounds $A$ and $B$. Suppose $\hat{\phi}$ and $\hat{\psi}$ are continuous and*

$$|\hat{\phi}(\xi)|, |\hat{\psi}(\xi)| \le C\frac{1}{(1+|\xi|)^{n+\gamma}}, \qquad \gamma > 0.$$

*Then there exists a $\delta > 0$ such that for any $b$ with $|b-b_0| < \delta, \{\phi_{k,b}, \psi_{j,k,a_0,b}\}$ is a frame (Riesz basis). Moreover, if supp $\hat{\phi} \subset [-\pi/(b_0\vee b), \pi/(b_0 \vee b)]^n$, supp $\hat{\psi} \subset$*

*$[-\pi/(b_0\vee b),\pi/(b_0\vee b)]^n$, and for $b$ with $M=|1-(\frac{b_0}{b})^n|B<A$, then $\{\phi_{k,b},\psi_{j,k,a_0,b}\}$ is a frame on $L^2(R^n)$ with frame bounds $A-M$ and $B+M$.*

**Theorem 5.** *Let $\{\phi_{k,b_0},\psi_{j,k,a_0,b_0}\}$ be a nonhomogeneous frame on $L^2(R^n)$ with frame bounds $A$ and $B$, supp $\hat\phi\subset[-\pi/b_0,\pi/b_0]^n$, supp $\hat\psi\subset[-\pi/b_0,\pi/b_0]^n$, if*

$$m=\operatorname{esssup}|\sum_{j\in N}|\hat\psi(a_0^{-j}\xi)|^2-\sum_{j\in N}|\hat\psi(a^{-j}\xi)|^2|<Ab_0^n.$$

*Then $\{\phi_{k,b_0},\psi_{j,k,a,b_0}\}$ is a frame on $L^2(R^n)$ with frame bounds $A-m/b_0^n$ and $B+m/b_0^n$. Especially, if $\hat\psi$ is continuous and*

$$|\hat\psi(\xi)|\le C|\xi|^\alpha,\qquad \alpha>0,$$

*then there exists a $\delta>0$ such that for any $a$ with $|a-a_0|<\delta$, $\{\phi_{k,b_0},\psi_{j,k,a,b_0}\}$ is a frame on Paley-Wiener space.*

## 2. Proof of main results

*Proof of Theorem* 1. We use Balan's idea to define a unitary operator

$$U_b:\quad L^2(R^n)\to L^2(R^n),\quad (U_b\psi)(x)=(\frac{b}{b_0})^{n/2}\psi(\frac{b}{b_0}x)=\phi(x).$$

Then it is easy to see that

$$\hat\phi(\xi)=(\frac{b}{b_0})^{-n/2}\hat\psi(\frac{b_0}{b}\xi),\quad U_b\psi_{j,k,a_0,b}=\phi_{j,k,a_0,b_0}.$$

Therefore, $\{\psi_{j,k,a_0,b}\}$ is a frame if and only if $\{\phi_{j,k,a_0,b_0}\}$ is a frame. For all $f,g\in L^2(R^n)$, by Plancherel Theorem and Parseval identity, we have

$$\begin{aligned}
&\sum_{j\in Z}\sum_{k\in Z^n}|\langle f,g_{j,k,a,b}\rangle|^2\\
&=(2\pi)^{-2n}\sum_{j\in Z}\sum_{k\in Z^n}|\int\hat f(\xi)a^{-nj/2}\overline{\hat g(a^{-j}\xi)}e^{ia^{-j}bk\cdot\xi}d\xi|^2\\
&=(2\pi b)^{-2n}\sum_{j\in Z}\sum_{k\in Z^n}a^{nj}|\int\hat f(a^j\xi/b)\overline{\hat g(\xi/b)}e^{ik\cdot\xi}d\xi|^2\\
&=(2\pi b)^{-2n}\sum_{j\in Z}\sum_{k\in Z^n}a^{nj}|\int_{[-\pi,\pi]^n}\sum_{l\in Z^n}\hat f(a^j(\xi+2l\pi)/b)\overline{\hat g((\xi+2l\pi)/b)}e^{ik\cdot\xi}d\xi|^2\\
&=(2\pi)^{-n}b^{-2n}\sum_{j\in Z}a^{nj}\int_{[-\pi,\pi]^n}|\sum_{l\in Z^n}\hat f(a^j(\xi+2l\pi)/b)\overline{\hat g((\xi+2l\pi)/b)}|^2d\xi\\
&=(2\pi)^{-n}b^{-2n}\sum_{j\in Z}a^{nj}\int\hat f(a^j\xi/b)\overline{\hat g(\xi/b)}\sum_{l\in Z^n}\overline{\hat f(a^j(\xi+2l\pi)/b)}\hat g((\xi+2l\pi)/b)d\xi\\
&=(2\pi b)^{-n}\sum_{j\in Z}\sum_{l\in Z^n}\int\hat f(\xi)\overline{\hat g(a^{-j}\xi)}\overline{\hat f(\xi+a^j2l\pi/b)}\hat g(a^{-j}\xi+2l\pi/b)d\xi.
\end{aligned}$$

Thus,

$$\begin{aligned}
(1)\quad &\sum_{j\in Z}\sum_{k\in Z^n}|\langle f,g_{j,k,a,b}\rangle|^2\\
&=(2\pi b)^{-n}\sum_{j\in Z}\sum_{l\in Z^n}\int\hat f(\xi)\overline{\hat g(a^{-j}\xi)}\overline{\hat f(\xi+a^j2l\pi/b)}\hat g(a^{-j}\xi+2l\pi/b)d\xi.
\end{aligned}$$

By the Cauchy-Schwarz inequality we have

$$
\begin{aligned}
&\sum_{j\in Z}\sum_{k\in Z^n}|\langle f, g_{j,k,a,b}\rangle|^2\\
&\quad\le (2\pi b)^{-n}\sum_{j\in Z}\sum_{l\in Z^n}\Big(\int|\hat f(\xi)|^2|\hat g(a^{-j}\xi)\hat g(a^{-j}\xi+2l\pi/b)|d\xi\Big)^{1/2}\\
&\qquad\Big(\int|\hat f(\xi+a^j 2l\pi/b)|^2|\hat g(a^{-j}\xi)\hat g(a^{-j}\xi+2l\pi/b)|d\xi\Big)^{1/2}\\
&\quad\le (2\pi b)^{-n}\Big(\sum_{j\in Z}\sum_{l\in Z^n}\int|\hat f(\xi)|^2|\hat g(a^{-j}\xi)\hat g(a^{-j}\xi+2l\pi/b)|d\xi\Big)^{1/2}\\
&\qquad\Big(\sum_{j\in Z}\sum_{l\in Z^n}\int|\hat f(\xi)|^2|\hat g(a^{-j}\xi-2l\pi/b)\hat g(a^{-j}\xi)|d\xi\Big)^{1/2}\\
&\quad= (2\pi b)^{-n}\sum_{j\in Z}\sum_{l\in Z^n}\int|\hat f(\xi)|^2|\hat g(a^{-j}\xi)\hat g(a^{-j}\xi+2l\pi/b)|d\xi\\
&\quad\le b^{-n}\sup_{\xi\in R^n}\sum_{j\in Z}\sum_{l\in Z^n}|\hat g(a^{-j}\xi)\hat g(a^{-j}\xi+2l\pi/b)|\|f\|^2\\
&\quad= b^{-n}\sup_{1\le|\xi|\le a}\sum_{j\in Z}\sum_{l\in Z^n}|\hat g(a^{-j}\xi)\hat g(a^{-j}\xi+2l\pi/b)|\|f\|^2.
\end{aligned}
$$

Let $g=\psi-\phi$. We have

$$
\begin{aligned}
&\sum_{j\in Z}\sum_{k\in Z^n}|\langle f,(\psi-\phi)_{j,k,a_0,b_0}\rangle|^2\\
&\quad\le b_0^{-n}\sup_{1\le|\xi|\le a_0}\sum_{j\in Z}\sum_{l\in Z^n}|\hat\psi(a_0^{-j}\xi)-\hat\phi(a_0^{-j}\xi)|\\
&\qquad\qquad\cdot|\hat\psi(a_0^{-j}\xi+2l\pi/b_0)-\hat\phi(a_0^{-j}\xi+2l\pi/b_0)|\|f\|^2\\
&\quad= b_0^{-n}\sup_{1\le|\xi|\le a_0}\sum_{j\in Z}|\hat\psi(a_0^{-j}\xi)-\hat\phi(a_0^{-j}\xi)|\\
&\qquad\qquad\cdot\sum_{l\in Z^n}|\hat\psi(a_0^{-j}\xi+2l\pi/b_0)-\hat\phi(a_0^{-j}\xi+2l\pi/b_0)|\|f\|^2.
\end{aligned}
$$

For all $j$ and $\xi$,

$$
\begin{aligned}
&\sup_{1\le|\xi|\le a_0}\sum_{l\in Z^n}|\hat\psi(a_0^{-j}\xi+2l\pi/b_0)|\\
&\quad\le C\sup_{1\le|\xi|\le a_0}\sum_{l\in Z^n}\frac{1}{(1+|a_0^{-j}\xi+2l\pi/b_0|)^{n+\gamma-\alpha}}\\
&\quad\le C\sum_{l\in Z^n}\frac{1}{(1+|l|)^{n+\gamma-\alpha}}\le C<\infty.
\end{aligned}
$$

For the same reason, the second term is bounded uniformly, that is,

$$
\sup_{1\le|\xi|\le a_0}\sum_{l\in Z^n}|\hat\phi(a_0^{-j}\xi+2l\pi/b_0)|\le C<\infty.
$$

For all $J \in N$,

$$
\begin{aligned}
&\sup_{1\le|\xi|\le a_0}\sum_{j\in Z}|\hat{\psi}(a_0^{-j}\xi)-\hat{\phi}(a_0^{-j}\xi)|\\
&\quad\le \sup_{1\le|\xi|\le a_0}\sum_{|j|\le J}|\hat{\psi}(a_0^{-j}\xi)-(\frac{b}{b_0})^{-n/2}\hat{\psi}(a_0^{-j}\frac{b_0}{b}\xi)|\\
&\qquad+\sup_{1\le|\xi|\le a_0}\sum_{j<-J}(|\hat{\psi}(a_0^{-j}\xi)|+|\hat{\phi}(a_0^{-j}\xi)|)\\
&\qquad+\sup_{1\le|\xi|\le a_0}\sum_{j>J}(|\hat{\psi}(a_0^{-j}\xi)|+|\hat{\phi}(a_0^{-j}\xi)|)\\
&\quad= I+II+III.
\end{aligned}
$$

For every $\varepsilon>0$, choose $J$ such that $a_0^{-J}<\varepsilon$. Since $1\le|\xi|\le a_0$, $|j|\le J$, and $\hat{\psi}(a_0^{-j}\xi)$ is continuous uniformly on $\xi$, we can choose $\delta$ small enough so that if $|b-b_0|<\delta$, then

$$
|\hat{\psi}(a_0^{-j}\xi)-\hat{\psi}(a_0^{-j}\frac{b_0}{b}\xi)|<\varepsilon,\quad \forall|j|\le J.
$$

Thus,

$$
\begin{aligned}
I&\le\sum_{|j|\le J}\sup_{1\le|\xi|\le a_0}(|1-(\frac{b}{b_0})^{-n/2}||\hat{\psi}(a_0^{-j}\xi)|+(\frac{b}{b_0})^{-n/2}||\hat{\psi}(a_0^{-j}\xi)-\hat{\psi}(a_0^{-j}\frac{b_0}{b}\xi)|)\\
&\le C(2J+1)(|1-(\frac{b}{b_0})^{-n/2}|+(\frac{b}{b_0})^{-n/2}\varepsilon)=o(1),\qquad b\to b_0.
\end{aligned}
$$

We now turn to II. We will just estimate the first term in the series, since the other term can be handled similarly.

$$
\begin{aligned}
\sup_{1\le|\xi|\le a_0}\sum_{j<-J}|\hat{\psi}(a_0^{-j}\xi)|&\le\sup_{1\le|\xi|\le a_0}C\sum_{j<-J}\frac{|a_0^{-j}\xi|^{\alpha}}{(1+|a_0^{-j}\xi|)^{n+\gamma}}\\
&\le\sup_{1\le|\xi|\le a_0}C\sum_{j<-J}\frac{1}{(1+|a_0^{-j}\xi|)^{n+\gamma-\alpha}}\le C\sum_{j<-J}a_0^{j(n+\gamma-\alpha)}\\
&\le Ca_0^{-J(n+\gamma-\alpha)}=o(1),\qquad J\to+\infty.
\end{aligned}
$$

Finally, the last part is also small, since if $J$ is large,

$$
\sup_{1\le|\xi|\le a_0}\sum_{j>J}|\hat{\psi}(a_0^{-j}\xi)|\le C\sum_{j>J}|a_0^{-j}a_0|^{\alpha}\le Ca_0^{-J\alpha}=o(1),\qquad J\to+\infty.
$$

From the estimates above, we conclude that for every $\varepsilon>0$, there exists $\delta>0$ such that for all $|b-b_0|<\delta$,

$$
\sum_{j\in Z}\sum_{k\in Z^n}|\langle f,(\psi-\phi)_{j,k,a_0,b_0}\rangle|^2\le\varepsilon\|f\|^2,
$$

which shows $\{\phi_{j,k,a_0,b_0}\}$ is a frame (Riesz basis) on $L^2(R^n)$ for $b$ sufficiently close to $b_0$ by Lemma 1. The proof is finished.

*Proof of Theorem* 2. Since $supp\ \hat{\psi} \subset [-\pi/(b_0 \vee b), \pi/(b_0 \vee b)]^n$,

$$\begin{aligned}|\sum_{j\in Z}\sum_{k\in Z^n} &|\langle f, \psi_{j,k,a_0,b_0}\rangle|^2 - \sum_{j\in Z}\sum_{k\in Z^n} |\langle f, \psi_{j,k,a_0,b}\rangle|^2| \\ &= (2\pi)^{-n}|b_0^{-n} - b^{-n}| \int |\hat{f}(\xi)|^2 \sum_j |\hat{\psi}(a_0^{-j}\xi)|^2 d\xi.\end{aligned}$$

By Lemma 2 the last term above is dominated by

$$|b_0^{-n} - b^{-n}|b_0^n B\|f\|^2 = |1 - (\frac{b_0}{b})^n|B\|f\|^2.$$

Choosing those $b$ such that $|1-(\frac{b_0}{b})^n|B < A$ and applying Lemma 4, the conclusion follows. □

*Proof of Theorem* 3. By the same computation as in the proof of Theorem 2, we have

$$\begin{aligned}|\sum_{j\in Z}\sum_{k\in Z^n} &|\langle f, \psi_{j,k,a_0,b_0}\rangle|^2 - \sum_{j\in Z}\sum_{k\in Z^n} |\langle f, \psi_{j,k,a,b_0}\rangle|^2| \\ &= (2\pi b_0)^{-n}|\int |\hat{f}(\xi)|^2 (\sum_j |\hat{\psi}(a_0^{-j}\xi)|^2 - \sum_j |\hat{\psi}(a^{-j}\xi)|^2) d\xi|,\end{aligned}$$

and the first assertion in the theorem follows. If $f$ is a function in Paley-Wiener space, more precisely, there exist two constants $0 < m < M < \infty$, such that $supp\hat{f} \subset \{\xi : m \le |\xi| \le M\}$, then for big enough $J$, if $|j| > J$, we have

$$\begin{aligned}\sum_{j<-J} |\hat{\psi}(a^{-j}\xi)|^2 &\le C \sum_{j<-J} \frac{|a^{-j}\xi|^{2\alpha}}{(1+|a^{-j}\xi|)^{2\gamma}} \\ &\le C \sum_{j<-J} (a^{-j}m)^{2(\alpha-\gamma)} \le Ca^{-2J(\gamma-\alpha)} = o(1), \quad J \to -\infty, \\ \sum_{j>J} |\hat{\psi}(a^{-j}\xi)|^2 &\le C \sum_{j>J} |a^{-j}M|^{2\alpha} \le Ca^{-2J\alpha} = o(1), \quad J \to \infty.\end{aligned}$$

For $|j| \le J$, we have

$$\begin{aligned}|\sum_{|j|\le J} &(|\hat{\psi}(a_0^{-j}\xi)|^2 - |\hat{\psi}(a^{-j}\xi)|^2| \\ &\le \sum_{|j|\le J} (|\hat{\psi}(a_0^{-j}\xi)| + |\hat{\psi}(a^{-j}\xi)|)|\hat{\psi}(a_0^{-j}\xi) - \hat{\psi}(a^{-j}\xi)| \\ &\le C(2J+1)\varepsilon = o(1), \qquad |a_0 - a| < \delta,\end{aligned}$$

where we used the fact that if $|j| \le J$, $m < |\xi| < M$, then $\hat{\psi}(a^{-j}\xi)$ is uniformly continuous, so for every $\varepsilon > 0$, there exists $\delta$, such that for all $|a_0 - a| < \delta$, $|\hat{\psi}(a_0^{-j}\xi) - \hat{\psi}(a^{-j}\xi)| < \varepsilon$. Theorem 3 is proved.

*Proof of Theorem* 4. The idea of the proof of Theorem 4 is the same as in the proof of Theorem 1. We just describe the main steps. We again use the unitary operator $U_b$ defined by

$$U_b : \quad L^2(R^n) \to L^2(R^n), \quad (U_b\psi)(x) = (\frac{b}{b_0})^{n/2}\psi(\frac{b}{b_0}x) = \Psi(x), \quad (U_b\phi)(x) = \varphi(x).$$

By a similar computation as in Theorem 1, we have

$$\begin{aligned}
&\sum_{k\in Z^n} |\langle f, g_{k,b_0}\rangle|^2 + \sum_{j\in N}\sum_{k\in Z^n} |\langle f, h_{j,k,a_0,b_0}\rangle|^2 \\
&\quad = (2\pi b_0)^{-n}\Big(\sum_{l\in Z^n} \int \hat{f}(\xi)\overline{\hat{f}(\xi+2l\pi/b_0)}\overline{\hat{g}(\xi)}\hat{g}(\xi+2l\pi/b_0)d\xi \\
&\qquad + \sum_{j\in N}\sum_{l\in Z^n} \int \hat{f}(\xi)\overline{\hat{f}(\xi+a_0^j 2l\pi/b_0)\hat{h}(a_0^{-j}\xi)}\hat{h}(a_0^{-j}\xi+2l\pi/b_0)d\xi\Big) \\
&\quad \le (2\pi b_0)^{-n}\Big(\sum_{l\in Z^n} \int |\hat{f}(\xi)|^2|\hat{g}(\xi)\hat{g}(\xi+2l\pi/b_0)|d\xi \\
&\qquad + \sum_{j\in N}\sum_{l\in Z^n} \int |\hat{f}(\xi)|^2|\hat{h}(a_0^{-j}\xi)\hat{h}(a_0^{-j}\xi+2l\pi/b_0)|d\xi\Big).
\end{aligned}$$

Let $g=\phi-\varphi$, $h=\psi-\Psi$. Then

$$\begin{aligned}
&\sum_{k\in Z^n} |\langle f, (\phi-\varphi)_{k,b_0}\rangle|^2 + \sum_{j\in N}\sum_{k\in Z^n} |\langle f, (\psi-\Psi)_{j,k,a_0,b_0}\rangle|^2 \\
&\quad \le b_0^{-n}\|f\|^2\Big(\sup|\hat{\phi}(\xi)-\hat{\varphi}(\xi)|\sum_{l\in Z^n}|\hat{\phi}(\xi+2l\pi/b_0)-\hat{\varphi}(\xi+2l\pi/b_0)| \\
&\qquad\qquad + \sup_{1\le|\xi|\le a_0}\sum_{j\in N}|\hat{\psi}(a_0^{-j}\xi)-\hat{\Psi}(a_0^{-j}\xi)| \\
&\qquad\qquad \cdot \sum_{l\in Z^n}|\hat{\psi}(a_0^{-j}\xi+2l\pi/b_0)-\hat{\Psi}(a_0^{-j}\xi+2l\pi/b_0)|\Big).
\end{aligned}$$

By the continuity and decay conditions, we obtain our first claim as in Theorem 1. We obtain our second claim by Lemma 3 since

$$\begin{aligned}
&\Big|\Big(\sum_{k\in Z^n} |\langle f, \phi_{k,b_0}\rangle|^2 + \sum_{j\in N}\sum_{k\in Z^n} |\langle f, \psi_{j,k,a_0,b_0}\rangle|^2\Big) \\
&\quad - \Big(\sum_{k\in Z^n} |\langle f, \phi_{k,b}\rangle|^2 + \sum_{j\in N}\sum_{k\in Z^n} |\langle f, \psi_{j,k,a_0,b}\rangle|^2\Big)\Big| \\
&\quad = (2\pi)^{-n}|b_0^{-n}-b^{-n}|\int |\hat{f}(\xi)|^2\Big(|\hat{\phi}(\xi)|^2+\sum_{j\in N}|\hat{\psi}(a^{-j}\xi)|^2\Big)d\xi.
\end{aligned}$$

*Proof of Theorem* 5. The first result is easily seen by

$$\begin{aligned}
&\Big|\sum_{j\in N}\sum_{k\in Z^n} |\langle f, \psi_{j,k,a_0,b_0}\rangle|^2 - \sum_{j\in N}\sum_{k\in Z^n} |\langle f, \psi_{j,k,a,b_0}\rangle|^2\Big| \\
&\qquad = (2\pi b_0)^{-n}\Big|\int |\hat{f}(\xi)|^2\sum_{j\in N}\big(|\hat{\psi}(a_0^{-j}\xi)|^2-|\hat{\psi}(a^{-j}\xi)|^2\big)d\xi\Big|.
\end{aligned}$$

The second part is similar to the proof of Theorem 3. We leave these details to the reader.

We complete our proof of Theorem 1–Theorem 5.

## Acknowledgement

I would like to thank the referee for his comments.

## References

[Ba] Balan, R., Stability Theorems for Fourier Frames and Wavelet Riesz Basis, The Journal of Fourier Analysis and Applications, 3(1997),499-504. MR **99f:**42060

[B] Benedetto, J. J, Irregular sampling and frames, in "Wavelets, a Tutorial in Theory and Applications" (Chui, C.K.Ed), 445-507, Academic Press, San Diego, 1992. MR **93c:**42030

[BHW] Benedetto, J. J, Heil, C. and Walnut, D. F, Differentiation and the Balian-Low Theorem. J. Fourier Anal 1(1995),355-402. MR **96f:**42002

[BW] Benedetto, J. J and Walnut, D. F, Gabor frames for $L^2$ and related spaces. in "Wavelets:Mathematics and Applications" (J. J. Benedetto and M. Frazier, Eds), 97-162. CRC Press, Boca Raton, FL, 1994. MR **94i:**42040

[C] Christensen, O., Frame perturbation, Proc. of Amer. Math. Soc. 123(1995), 1217-1220.

[CH] Christensen, O. and Heil, C. E., Perturbations of Banach Frames and atomic decompositions, Math. Nachr. 185(1997), 33–47. MR **98m:**42061

[CS] Chui, C. K and Shi, X., Inequalities of Littlewood-Paley type for frames and wavelets, SIAM J. Math. Anal. 24(1993), 263-277. MR **94d:**42039

[D1] Daubechies, I., "Ten Lectures on Wavelets" SIAM, Philadelphia, 1992. MR **93e:**42045

[D2] Daubechies, I., The wavelet transform, time-frequency localization and sigual analysis, IEEE Trans. Inform. Theory 36(1990), 961-1005. MR **91e:**42038

[FG] Feichtinger, H. G. and Grächenig, Banach spaces related to integrable group representations and their atomic decompositions,1, J. Funct. Anal. 86(1989), 307-340. MR **91g:**43011

[FZ] Favier, S. J. and Zalik, R. A, On the stability of frames and Riesz basis, Appl. Comput. Harm. Anal. 2(1995), 160-173. MR **96e:**42030

[HW] Heil, C. E. and Walnut, D. F., Continuous and discrete wavelet transforms, SIAM Rev. 31(1989), 628-666. MR **91c:**42032

[L] Long Ruilin, "Hidimensional Wavelet Analysis", High Educational Press, Beijing (in Chinese)1995.

[Y] Young, R. M., "An Introduction to Nonharmonic Fourier Series", Academic Press, New York, 1980. MR **81m:**42027

[Z1] Zhang Jing, On the stability of wavelet and Gabor frames (Riesz bases), J of Fourier Anal. 1(1999), 105-125. MR **2000a:**42055

[Z2] Zhang Jing, The existence of frames due to nonhomogeneous Calderón reproducing formula, preprint.

Institute of Mathematics, Academia Sinica, Beijing, People's Republic of China 100080

*Current address*: Department of Mathematics, Washington University, Campus Box 1146, One Brookings Drive, St. Louis, Missouri 63130-4899

*E-mail address*: `zhj@math.wustl.edu`