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GENERALIZED FRAMES AND THEIR REDUNDANCY

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Abstract. Let h be a generalized frame in a separable Hilbert space H indexed by a measure space $(M, \mathcal{S}, \mu)$, and assume its analysing operator is surjective. It is shown that h is essentially discrete; that is, the corresponding index measure space $(M, \mathcal{S}, \mu)$ can be decomposed into atoms $E_1, E_2, \cdots$ such that $L^2(\mu)$ is isometrically isomorphic to the weighted space ℓ^2_w of all sequences $\{c_i\}$ of complex numbers with $||\{c_i\}||^2 = \sum |c_i|^2 w_i < \infty$, where $w_i = \mu(E_i),\ i = 1, 2, \cdots$. This provides a new proof for the redundancy of the windowed Fourier transform as well as any wavelet family in $L^2(\mathbb{R})$.

1. Introduction

Any windowed Fourier transform $\{g_{\omega,t} : (\omega, t) \in \mathbb{R}^2\}$ defined by $g_{\omega,t}(u) = g(u - t) \exp(2\pi i \omega u)$ for a suitable g as well as the family of wavelets $\{\psi_{s,t} : (s, t) \in \mathbb{R}^2, s \neq 0\}$ defined by $\psi_{s,t}(u) = |s|^{-1/2}\psi(\frac{u-t}{s})$ for certain mother wavelets ψ are members of the class of generalized frames defined as follows. Let H be a Hilbert space and let $(M, \mathcal{S}, \mu)$ be a measure space. A generalized frame [9], [10] in H indexed by M is a family $h = \{h_m \in H : m \in M\}$ such that

(a) for any $f \in H$, the function $\tilde{f} : M \to \mathbf{C}$ defined by $\tilde{f}(m) = \langle h_m, f\rangle$ is measurable,

(b) there is a pair of constants $0 < A \leq B < \infty$ such that, for any $f \in H$,

$$A\|f\|^2_H \leq \|\tilde{f}\|^2_{L^2(\mu)} \leq B\|f\|^2_H. \tag{1}$$

A typical windowed Fourier transform is indexed by $(\mathbb{R}^2, \mathcal{B}, dxdy)$, and a typical wavelet transform is indexed by $((\mathbb{R}\backslash\{0\}) \times \mathbb{R}, \mathcal{B}, s^{-2}dsdt)$, where $\mathcal{B}$ is the class of planar Borel sets.

Each vector h_m is called a frame vector, (1) is called the frame condition, and A and B are called frame bounds. The function $\tilde{f}$ will be called the dual function of f with respect to the frame, and the map $Tf = \tilde{f}$ is called the analysing operator. If M is at most countable and μ is the counting measure, $\{h_m : m \in M\}$ is called a discrete frame [4]. (See also [2], [7], [8].) The orthogonal complement $(ImT)^{\perp}$ of the range of T can be viewed as a measure of the so-called "redundancy" of the frame. (See [11, pp. 129-138] in the discrete case.)

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For the continuous case we refer to [3], [5], [6], [11], [13], [14], [15], [16], [18] whose views are summarized in [13, p. 144] as follows: "The continuous wavelet transform is the natural extension of the *redundant* discrete wavelet transform. This transform is analogous to the Fourier transform, which is *redundant*, and results in a transform that is easier to interpret, is shift invariant, and is valuable for time-frequency/scale analysis."

Our goal is to show that continuous generalized frames, just like windowed Fourier transforms and continuous wavelet transforms, must be redundant. Aside from the intrinsic interest in the theoretical study of the frames, the study of the windowed Fourier transform as well as the wavelets in the general context of frames help us to have a new insight into these useful tools employed in diverse technological and scientific areas.

More precisely, the main result of the paper is to show that if $h \subset H$ is a generalised frame indexed by a measure space $(M, \mathcal{S}, \mu)$, and if its redundancy is zero, then $L^2(\mu)$ is isometrically isomorphic to a weighted space ℓ^2_w consisting of all sequences $\{c_i\}$ with $||\{c_i\}|| = \sum |c_i|^2 w_i < \infty$, where $w_i = \mu(E_i)$ for some μ-atoms $E_1, E_2, \cdots$. This will, in particular, provide a new proof of the fact that windowed Fourier transforms as well as the wavelet transforms are redundant.

Note that the frame condition (1) implies that ImT is closed. Also, $(ImT)^\perp \neq \{0\}$ if and only if there exists a nonzero $g \in L^2(\mu)$ such that $\displaystyle\int d\mu(m)\langle h_m, f\rangle g(m) = 0$ for all $f \in H$; this is referred to as the "linear dependence" of the generalized frame h by Kaiser [10].

Although every orthonormal basis is a discrete frame, it is not necessary for a frame to be even a set of independent vectors. An example is given in [10] in which $H = \mathbb{R}^2$ with the standard inner product, $M = \{1, 2, 3\}$, μ is the counting measure, and

$$h = \left\{ h(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, h(2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, h(3) = \begin{bmatrix} a \\ b \end{bmatrix} \right\}$$

for some $(a, b) \in \mathbb{R}^2$.

Also, note that a measurable subset E of M is called an atom if $0 < \mu(E) < \infty$ and E contains no measurable subset F such that $0 < \mu(F) < \mu(E)$ [1, p.98]. It is easy to prove the following lemma concerning atoms.

1.1. Lemma. *Let $(M, \mathcal{S}, \mu)$ be a measure space, and define $\eta = \inf\{\mu(E)) : E \in \mathcal{S}, 0 < \mu(E) < \infty\}$. The following assertions are true*:

(a) If $\eta = 0$, then there exists a sequence of disjoint measurable sets $F_1, F_2, \ldots$ such that $\mu(F_n) > 0$ and $\lim_{n\to\infty} \mu(F_n) = 0$.

(b) If $\eta > 0$, then every set of positive finite measure is a finite union of disjoint atoms.

(c) Every measurable function is μ-almost constant on an atom.

2. Main result

Our main result studies the structure of $(M, \mathcal{S}, \mu)$ when $h = \{h_m\}_{m\in M}$ has zero redundancy. In the proof, we will need the fact that if $\|h_m\|$ is a measurable function, which is the case when H is separable [12], [17], one can modify μ to a new measure $d\mu'(m) = (1 + \|h_m\|)d\mu(m)$ to replace h by a bounded frame $h'_m = h_m/(1 + \|h_m\|)^2$ in H indexed by $(M, \mathcal{S}, \mu')$ ([10, p. 82]).

To prove the main result we need the following lemma, which is obvious from (1), and is a standard fact for discrete frames.

2.1. Lemma ([10, p. 87]). *Let h be a frame in H indexed by $(M,\mathcal{S},\mu)$ and let T be the corresponding analysing operator. Then T is bijective if and only if $(ImT)^\perp = \{0\}$.*

2.2. Theorem. *Let h be a generalized frame in H indexed by $(M,\mathcal{S},\mu)$, and assume $ImT = H$, where T is the corresponding analysis operator. Then for every countable subset L of H, there exists a countable collection $\{E_i : i \in \Lambda\}$ of disjoint measurable sets such that $\tilde{f} = \sum c_{fi}\chi_i$ for all f in the closed linear span of L, where $\{c_{fi} : i \in \Lambda\}$ is a set of complex numbers depending on f, and χ_i denotes the characteristic function of E_i.*

In particular, if H is an infinite dimensional separable space, then $L^2(\mu)$ is isometrically isomorphic to the weighted space ℓ^2_w consisting of all sequences $\{c_i\}$ with $\|\{c_i\}\|^2 = \sum |c_i|^2 w_i < \infty$, where $w_i = \mu(E_i)$ for a fixed collection of disjoint μ-atoms $\{E_1, E_2, \dots\}$.

Proof. Let $\{0\} \neq L \subset H$ and assume L is countable. Let $\mathcal{S}_1$ be the σ-algebra generated by the collection of all sets of μ-measure 0 as well as the sets E_{frst} consisting of all $m \in M$ such that $r/2^t \leq |\tilde{f}(m)| < (r+1)/2^t$ and $2\pi(s-1)/2^t \leq \arg \tilde{f}(m) < 2\pi s/2^t$ for $f \in L$, $r \in \mathbf{N}$, $t \in \mathbf{N}$, and $s = 1, 2, 3, \dots, 2^t$.

Let T be the analysing operator (corresponding to h). By Lemma 2.1, T is invertible and hence the closed subspace K of H spanned by $\{T^{-1}(\chi_{frst}) : f \in L; r, t \in \mathbf{N}; s = 1, 2, \dots, 2^t\}$ is separable and contains L, where χ_{frst} denotes the characteristic function of E_{frst}. Note that $\langle h_m, f\rangle$ is $\mathcal{S}_1$-measurable for all $f \in K$.

Let $P : H \to K$ be the orthogonal projection onto K. Then, for every $f \in K$, $\langle Ph_m, f\rangle = \langle h_m, f\rangle$ for all $m \in M$, and hence Ph_m is weakly $\mathcal{S}_1$-measurable. Since K is separable, it follows from the Pettis theorem [12], [17], or from the Parseval's equality $\|Ph_m\|^2 = \sum_n |\langle h_m, e_n\rangle|^2$ for some orthonormal basis $\{e_n\}$ of K, that $\|Ph_m\|$ is $\mathcal{S}_1$-measurable. Define

$$k_m = Ph_m/(1 + \|Ph_m\|) \quad (m \in M),$$

and

$$\mu_1(E) = \int_E d\mu(m)(1 + \|Ph_m\|)^2 \quad (E \in \mathcal{S}_1).$$

Then $\langle k_m, f\rangle$ is $\mathcal{S}_1$-measurable, μ_1 is a measure on $\mathcal{S}_1$, μ_1 and $\mu|\mathcal{S}_1$ are absolutely continuous with respect to each other, and

$$A\|f\|^2 \leq \int d\mu(m)|\langle h_m, f\rangle|^2 = \int d\mu_1(m)|\langle k_m, f\rangle|^2 \leq B\|f\|^2$$

for all $f \in K$. Hence $k = \{k_m\}_{m\in M}$ is a strongly measurable frame on K indexed by $(M, \mathcal{S}_1, \mu_1)$ and $\|k_m\| \leq 1$ for all $m \in M$.

We prove k is linearly independent. Let $g \in L^2(\mu_1)$ be such that

$$\int d\mu(m)\langle k_m, f\rangle g(m) = 0$$

for all $f \in K$. Then

$$\int d\mu(m)(1 + \|Ph_m\|)\chi_{frst}(m)g(m) = 0$$

for all $f \in L$, $r \in \mathbf{N}$, and $s = 1, 2, \dots, 2^t$. Thus $g = 0$ a.e. $[\mu_1]$.

Now, due to the $\mathcal{S}_1$-measurability of $\|Ph_m\|$, we can further refine E_{frst} as the countable union of $\{E_{frstn} : n \in \mathbf{N}\}$, where

$$E_{frstn} = \{m \in E_{frst} : n - 1 \le \|Ph_m\| < n\}.$$

The set E_{frstn} has a finite μ_1-measure for all $f \in L$, $(r, t, n) \in \mathbf{N}^3$, and $1 \le s \le 2^t$. Let

$$\eta_1 = \inf\{\mu_1(E) : E \in \mathcal{S}_1, \quad 0 < \mu_1(E) < \infty\}.$$

Since $0 < \|f\| < \infty$ for some $f \in L$, at least one E_{frstn} must have a positive measure which implies that η_1 is well-defined and $0 \le \eta_1 < \infty$. We claim $\eta_1 > 0$.

If $\eta_1 = 0$, it follows from Lemma 1.1 that there exists a sequence of disjoint measurable sets $F_1, F_2, \dots$ such that $0 < \mu_1(F_n) < 5^{-n} (n = 1, 2, \dots)$. Define $g = \sum 2^n g_n$, where g_n denotes the characteristic function of F_n. Then

$$\int d\mu_1 |g(m)|^2 = \sum 4^n \mu_1(F_n) \le \sum (4/5)^n < \infty.$$

Thus $g \in L^2(\mu_1)$. Let $g = T(u)$ for some $u \in K$. Then, for μ_1-almost all $m \in F_n$,

$$2^n = g(m) = \langle k_m, u \rangle \le \|u\| \quad (n = 1, 2, \dots),$$

a contradiction. Hence $\eta_1 > 0$ and, by Lemma 1.1, every E_{frstn} is either of measure zero or a disjoint union of a finite number of μ_1-atoms. Let $\{E_i : i \in \Lambda\}$ be the collection of all such μ_1-atoms. Then Λ is countable and $\mathcal{S}_1$ is generated by $\{E_i : i \in \Lambda\}$ as well as the sets of μ-measure 0. Moreover, by ignoring subsets of measure zero, one can assume without loss of generality that $E_i \cap E_j = \phi$ for $i \ne j$. Now, for each $f \in K$, it follows from Lemma 1.1 that $\tilde{f} = \sum c_{fi} \chi_i$ for some constants $\{c_{fi} : i \in \Lambda\}$, where χ_i denotes the characteristic function of E_i.

In the special case that H is separable, it follows from the Pettis theorem, or from the Parseval's equality, that $\|h_m\|$ is measurable and hence one can directly define $\mu_1(E)$ for every $E \in \mathcal{S}_1$ as

$$\mu_1(E) = \int_E d\mu(m)(1 + \|h_m\|)^2.$$

Then $k_m = h_m/(1 + \|h_m\|)$ is a bounded frame in H indexed by $(M, \mathcal{S}, \mu_1)$ and every μ_1-atom is also a μ-atom. If $\dim H = \infty$, then Λ can be chosen to be $\mathbf{N}$. Now, the proof is concluded by observing that $g \in L^2(\mu)$ if and only if $g = \sum c_i \chi_{E_i}$ for some constants $c_1, c_2, \dots$ satisfying $\sum |c_i|^2 \mu(E_i) < \infty$. □

2.3. Corollary (Kaiser [10, pp. 79-80])**.** *Every windowed Fourier transform as well as every wavelet family is a linearly dependent generalized frame.*

The proof follows from the fact that the corresponding measures have no atoms.

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