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C^*-ALGEBRAS ASSOCIATED WITH BRANCHED COVERINGS

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ABSTRACT. In this note we analyze the C^*-algebra associated with a branched covering both as a groupoid C^*-algebra and as a Cuntz-Pimsner algebra. We determine conditions when the algebra is simple and purely infinite. We indicate how to compute the K-theory of several examples, including one related to rational maps on the Riemann sphere.

1. INTRODUCTION

Given a branched covering $\sigma : X \to X$ of a locally compact space X, we define its C^*-algebra to be the C^*-algebra of the r-discrete groupoid Γ associated by Renault to the corresponding partially defined local homeomorphism T. More precisely,

$$\Gamma = \Gamma(X, \sigma) = \{(x, m-n, y) \mid m, n \in \mathbb{N}, x \in dom(T^m), y \in dom(T^n), T^m x = T^n y\},$$

where T is the restriction of σ to the nonsingular set $X \setminus S = U$. Here $dom(T^k)$ is the domain of T^k.

It turns out that this C^*-algebra is isomorphic to an augmented Cuntz-Pimsner algebra associated with a C^*-correspondence (A, E), where $A = C_0(X), E = \overline{C_c(U)}$, viewed as an A-Hilbert module via the right multiplication

$$(\xi f)(x) = \xi(x) f(\sigma(x)), \xi \in E, f \in A, x \in U,$$

and the inner product

$$\langle \xi, \eta \rangle(x) = \sum_{\sigma(y)=x} \overline{\xi(y)}\eta(y).$$

The left multiplication is given by the map

$$\varphi : A \to L(E),\ (\varphi(f)\xi)(x) = f(x)\xi(x).$$

If the singular set is empty, so that $U = X$, we recover previous results (see [7], [2]).

We consider several examples of C^*-algebras arising from branched coverings, and indicate how to compute their K-theory. These examples include some of the algebras considered by R. Exel in [9], in the case of a partial homeomorphism.

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2. Branched coverings and groupoids

We collect some facts here about branched coverings and some examples for future reference.

Definition 2.1. Let X, X' be locally compact, second countable Hausdorff spaces and let $S \subset X,\ S' \subset X'$ be closed subsets such that $U = X \setminus S$ and $V = X' \setminus S'$ are dense in X and X', respectively. A continuous surjective map $\sigma : X \to X'$ is said to be a *branched covering* with *branch sets* S (upstairs) and S' (downstairs) if:

1. the components of preimages of open sets of X' are a basis for the topology of X (in particular σ is an open map),
2. $\sigma(S) = S',\ \sigma(U) = V$, and
3. $\sigma|_U$ is a local homeomorphism.

We mention that in the original definition given by Fox (see [10]), $X \setminus S$ and $X' \setminus S'$ are supposed to be connected and S and S' are supposed to be of codimension 2 for topological reasons. We make no such restrictions here; in particular, we allow disconnected $X \setminus S$ and $X' \setminus S'$.

Examples 2.2. a) (Folding the interval) Let $X = X' = [0, 1]$, and let the map

$$\sigma(t) = \begin{cases} 2t, & 0 \le t \le 1/2, \\ 2 - 2t, & 1/2 \le t \le 1. \end{cases}$$

Then $S = \{1/2\},\ S' = \{1\}$.

b) Let $X = X' = \mathbf{D}^2$ (the closed unit disc) and $\sigma(z) = z^p$ $(p \ge 2)$. Then $S = S' = \{0\}$.

c) Let $X = \mathbf{S}^k$ be the k-sphere, $X' = \mathbf{D}^k$ be the k-disc, and let σ be the projection onto the equatorial plane. Then $S = S' = \mathbf{S}^{k-1}$ (identified with the equator).

d) Let G be a finite group acting nonfreely on a compact manifold X, let $X' = X/G$ be the corresponding orbifold, and let $\sigma : X \to X'$ be the quotient map. Then σ is a branched covering with $S =$ the set of points s with nontrivial isotropy group G_s. The previous examples are just particular cases of this, but note that not every branched covering is associated with a group action.

e) Let X be the unit circle, identified with the one-point compactification of $\mathbb{R}$. Consider $\sigma : \mathbb{R} \to X$ the usual map which wraps $\mathbb{R}$ around the circle. Here σ is not defined everywhere, but the groupoid construction will make sense.

f) Let $p : \mathbb{C} \to \mathbb{C}$ be a polynomial map of degree $k \ge 2$, and define $p(\infty) = \infty$. We get a map $p : \mathbf{S}^2 \to \mathbf{S}^2$, which is a branched covering with

$$S' = \{w \in \mathbf{S}^2 \mid \text{ the equation } p(z) = w \text{ has multiple roots}\},$$

$S = p^{-1}(S')$. In a similar way, each rational map $q : \mathbf{S}^2 \to \mathbf{S}^2$ gives a branched covering.

g) If $\sigma : U \to V$ is a homeomorphism between two open subsets of X, then we are in the situation considered by Exel in [9], by taking $A = C_0(X), I = C_0(V), J = C_0(U)$.

While one can build C^*-correspondences from any branched covering $\sigma : X \to X'$, in this note we shall restrict our attention to spaces covering themselves in a branched fashion.

For a locally compact space X and a local homeomorphism T from an open subset $dom(T)$ of X onto an open subset $ran(T)$ of X, Renault denotes by $Germ(X, T)$

the groupoid of germs of $\mathcal{G}(X,T)$, which in turn is the full pseudogroup generated by restrictions $T\mid_Y$, where Y is an open subset of X on which T is injective. He proves that the groupoid of germs coincides with the semidirect product groupoid

$$\Gamma(X,T)=\{(x,m-n,y)\ \mid\ m,n\in\mathbb{N},x\in dom(T^m),y\in dom(T^n),T^mx=T^ny\},$$

if (X,T) is essentially free in the sense that for all m,n, there is no nonempty open set on which T^m and T^n agree.

The topology on Γ is generated by the open sets

$$(Y;m,n;Z)=\{(y,m-n,z)\ \mid\ (y,z)\in Y\times Z,T^m(y)=T^n(z)\},$$

where Y and Z are open subsets on which T^m and T^n are injective, respectively. Since the range and the source maps are local homeomorphisms, Γ becomes a locally compact (Hausdorff) r-discrete groupoid, and so we may consider its C^*-algebra.

Denote by $c:\Gamma(X,T)\to\mathbb{Z}$ the cocycle defined by $c(x,k,y)=k$, i.e., c is the so-called *position cocycle*. Then $c^{-1}(0)$ is an equivalence relation $R(X,T)$, which is the increasing union of sub-relations

$$R_N=\{(x,y)\in X\times X\ \mid\ \exists\, n\le N\ \text{ with } x,y\in dom(T^n),T^nx=T^ny\}.$$

Notice that R_0 is the diagonal of $X\times X$, and each R_N is r-discrete, because the range and source maps are local homeomorphisms.

3. C^*-algebras defined by a branched covering

Definition 3.1. Given a branched covering $\sigma:X\to X$ with branch set S, consider the local homeomorphism $T:X\setminus S\to X\setminus\sigma(S)$, and assume that (X,T) is essentially free. We define $C^*(X,\sigma)$ to be $C^*(\Gamma(X,T))$.

Let $A=C_0(X)$, and let $E=\overline{C_c(U)}$, where $U=X\setminus S$, with the structure of a Hilbert A-module given by the formulae

$$(\xi f)(x)=\xi(x)f(\sigma(x)),\ \xi\in E,f\in A,x\in U,$$

$$\langle\xi,\eta\rangle(y):=\sum_{\sigma(x)=y}\overline{\xi(x)}\eta(x),y\in\sigma(U),\ \xi,\eta\in E.$$

In other words, the inner product is given by $\langle\xi,\eta\rangle=P(\bar{\xi}\eta)$, where P is the extension of $P:C_c(U)\to C_c(\sigma(U))$,

$$(P\xi)(y)=\sum_{\sigma(x)=y}\xi(x).$$

Note that the inner products generate the ideal $C_0(\sigma(U))$ in A. The left module structure on E is defined by the equation

$$\varphi:A\to L(E),\ (\varphi(f)\xi)(x)=f(x)\xi(x)\ f\in A,\xi\in E.$$

It is straightforward to verify that $\varphi(f)$ is in $L(E)$ with adjoint $\varphi(\bar{f}),f\in A$. It is also straightforward to verify that φ is injective. Thus, we may form the augmented Cuntz-Pimsner algebra $\tilde{\mathcal{O}}_E$.

Theorem 3.2. *The C^*-algebras $\tilde{\mathcal{O}}_E$ and $C^*(\Gamma(X,T))$ are isomorphic.*

The proof of the theorem will be given in several steps, using the notion of Fell bundle.

We recall briefly the Pimsner construction from [18]. A C^*-*correspondence* is a pair (E, A), where E is a (right) Hilbert module over a C^*-algebra A, and where A acts to the left on E via a $*$-homomorphism $\varphi : A \to L(E)$, from A to the bounded adjointable module maps on E. We shall always assume that our map φ is injective. The module E is not necessarily full, in the sense that the span of the inner products $\langle E, E\rangle$ may be a proper ideal of A. Given a C^*-correspondence (E, A), Pimsner constructs a C^*-algebra $\mathcal{O}_E$, which generalizes both the crossed products by $\mathbb{Z}$ and the Cuntz-Krieger algebras. The C^*-algebra generated by $\mathcal{O}_E$ and A is denoted by $\tilde{\mathcal{O}}_E$, and is called the *augmented Cuntz-Pimsner algebra* of the correspondence. The algebra $\mathcal{O}_E$ is a quotient of the generalized Toeplitz algebra $\mathcal{T}_E$ generated by the creation operators $\mathcal{T}_\xi,\ \xi \in E$ on the Fock space

$$\mathcal{E}_+ = \bigoplus_{n=0}^{\infty} E^{\otimes n}.$$

Here $E^{\otimes 0} = A$, and for $n \geq 1$, $E^{\otimes n}$ denotes the n-th tensor power of E, balanced via the map φ. The creation operators $\mathcal{T}_\xi$, $\xi \in E$, are defined by the formulae

$$\mathcal{T}_\xi a = \xi a, \quad \text{for } a \in A,$$

and

$$\mathcal{T}_\xi(\xi_1 \otimes ... \otimes \xi_n) = \xi \otimes \xi_1 \otimes ... \otimes \xi_n, \quad \text{for } \xi_1 \otimes ... \otimes \xi_n \in E^{\otimes n}.$$

A C^*-correspondence should be viewed as a generalization of an endomorphism of a C^*-algebra. Just as an endomorphism of a C^*-algebra can be "extended" to an automorphism of a larger algebra, so a general C^*-correspondence can be "extended" to an "invertible" correspondence. We will not go into much detail here except to indicate how this leads to another presentation of $\mathcal{O}_E$.

Pimsner considers a new pair $(E_\infty, \mathcal{F}_E)$, where $\mathcal{F}_E$ is the C^*-algebra generated by all the compact operators $K(E^{\otimes n})$, $n \geq 0$, in $\varinjlim L(E^{\otimes n})$, and $E_\infty = E \otimes \mathcal{F}_E$. The advantage is that E_∞ becomes an $\mathcal{F}_E$-$\mathcal{F}_E$ bimodule, such that the dual or adjoint module E^*_∞ is also an $\mathcal{F}_E$-$\mathcal{F}_E$ bimodule. More accurately, E_∞ has two $\mathcal{F}_E$-valued inner products with respect to which E_∞ satisfies all the axioms of an imprimitivity bimodule over $\mathcal{F}_E$, except possibly the one asserting that the left inner product is full. The C^*-algebra $\mathcal{O}_E$ is represented on the two-sided Fock space

$$\mathcal{E}_\infty = \bigoplus_{n\in\mathbb{Z}} E^{\otimes n}_\infty,$$

where for $n < 0$, $E^{\otimes n}_\infty$ means $(E^*_\infty)^{\otimes -n}$. It is isomorphic to the C^*-algebra generated by the multiplication operators $M_\xi \in L(\mathcal{E}_\infty)$, where for $\xi \in E_\infty$, $M_\xi\omega = \xi \otimes \omega$.

Given a branched covering $\sigma : X \to X$, we first want to identify the C^*-algebra $\mathcal{F}_E$. Recall that $T : U \to \sigma(U)$ is the local homeomorphism associated to σ.

Note that $E \otimes_\varphi E$ is a quotient of $\overline{C_c(U)} \otimes \overline{C_c(U)}$, where we identify $\xi f \otimes \eta$ with $\xi \otimes \varphi(f)\eta$ for any $\xi, \eta \in E$ and any $f \in A$. Therefore $E \otimes_\varphi E$ can be identified, as a vector space, with the completion of the compactly supported continuous functions on the set $U \cap \sigma(U)$. In a similar way, $E^{\otimes n}$ is identified (as a vector space) with $\overline{C_c(U \cap \sigma(U) \cap ... \cap \sigma^{n-1}(U))}$. The structure of a Hilbert A-module on $E^{\otimes n}$ is given by the equations

$$(\xi f)(x) = \xi(x) f(\sigma^n(x))$$

and

$$\langle \xi, \eta \rangle_n = P_n(\bar{\xi}\eta).$$

Here $P_n : C_c(U \cap \sigma(U) \cap ... \cap \sigma^{n-1}(U)) \to C_c(\sigma^n(U))$ is given by

$$P_n(\xi)(y) = \sum_{\sigma^n(x)=y} \xi(x).$$

Proposition 3.3. *The C^*-algebra $K(E)$ is isomorphic to $C^*(R(T))$, where*

$$R(T) = \{(x, y) \in U \times U \mid T(x) = T(y)\}$$

is the equivalence relation associated with $T = \sigma|_U$.

More generally, $K(E^{\otimes n}) \simeq C^*(R(T^n))$, where

$$R(T^n) = \{(x, y) \in dom(T^n) \times dom(T^n) \mid T^n(x) = T^n(y)\}$$

is the equivalence relation associated with T^n.

Proof. It is known that $K(E) = E \otimes E^*$, the tensor product balanced over A, where E^* is the adjoint of E. Since $\xi f \otimes \eta^* = \xi \otimes f\eta^*$, it follows that, as a set, $K(E) = \overline{C_c(R(T))}$. The multiplication of compact operators is exactly the convolution product on $C_c(R(T))$; therefore, as C^*-algebras, $K(E) = C^*(R(T))$.

In the same way, using the fact that $K(E^{\otimes n}) = (E^{\otimes n}) \otimes (E^{\otimes n})^*$, we get $K(E^{\otimes n}) = C^*(R(T^n))$. □

If we take

$$R_N := \bigcup_{n=0}^{N} R(T^n),$$

where R_0 is the diagonal of X, then the natural inclusion $C^*(R_N) \to C^*(R_{N+1})$ is induced by the map

$$L(E^{\otimes N}) \to L(E^{\otimes N+1}), \quad F \mapsto F \otimes I.$$

Corollary 3.4. *We have $\mathcal{F}_E = \varinjlim C^*(R_N)$. In particular, $\mathcal{F}_E$ is isomorpic to $C^*(R(X,T))$, where $R(X,T)$ is defined at the end of the previous section.*

In order to establish an isomorphism between $C^*(\Gamma)$ and $\tilde{\mathcal{O}}_E$ in our setting, we follow [8] and use the notion of a Fell bundle. We show that both $C^*(\Gamma)$ and $\tilde{\mathcal{O}}_E$ are isomorphic to the C^*-algebra associated to isomorphic Fell bundles over $\mathbb{Z}$. This point of view was suggested by Abadie, Eilers and Exel in [1]. We recall the definition of a Fell bundle and of the associated C^*-algebra (cf. [13] for a more general situation).

Definition 3.5. Consider a Banach bundle $p : \mathcal{B} \to \mathbb{Z}$. A *multiplication* on $\mathcal{B}$ is a continuous map $\mathcal{B} \times \mathcal{B} \to \mathcal{B}$ $((b_1, b_2) \to b_1 b_2)$ which satisfies the conditions:

a) $p(b_1 b_2) = p(b_1) + p(b_2), b_1, b_2 \in \mathcal{B}$,
b) $\mathcal{B}_k \times \mathcal{B}_l \to \mathcal{B}_{k+l}$ is bilinear,
c) $(b_1 b_2) b_3 = b_1 (b_2 b_3)$,
d) $\|b_1 b_2\| \le \|b_1\| \, \|b_2\|$.

An *involution* is a continuous map $\mathcal{B} \to \mathcal{B}$, $b \mapsto b^*$, which satisfies:

e) $p(b^*) = -p(b), b \in \mathcal{B}$,
f) $\mathcal{B}_k \to \mathcal{B}_{-k}$ is conjugate linear $\forall k \in \mathbb{Z}$,
g) $b^{**} = b$.

The bundle $\mathcal{B}$ together with these maps is said to be a *Fell bundle* if in addition:

h) $(b_1 b_2)^* = b_2^* b_1^*$,
i) $\|b^* b\| = \|b\|^2$, and
j) $b^* b \geq 0 \ \forall b \in \mathcal{B}$.

Note that $\mathcal{B}_0$ is a C^*-algebra. Denote by $C_c(\mathcal{B})$ the collection of compactly supported continuous sections. Of course, in our setting, because $\mathbb{Z}$ has the discrete topology, continuous, compactly supported sections are really elements of the algebraic direct sum $\sum \mathcal{B}_k$. Given $\xi, \eta \in C_c(\mathcal{B})$, define the multiplication and involution by means of the formulae

$$(\xi * \eta)(k) = \sum_l \xi(k-l)\eta(l),$$

$$\xi^*(k) = \xi(-k)^*.$$

Then $C_c(\mathcal{B})$ becomes a $*$-algebra. Let $P : C_c(\mathcal{B}) \to \mathcal{B}_0$ be the restriction map $\xi \mapsto \xi(0)$. With the inner product $\langle \xi, \eta \rangle = P(\xi^* \eta)$, $C_c(\mathcal{B})$ becomes a pre-Hilbert $\mathcal{B}_0$-module. For $\xi \in C_c(\mathcal{B})$, put $\|\xi\|_2 := \|\langle \xi, \xi \rangle\|^{1/2}$, and denote the completion of $C_c(\mathcal{B})$ with this norm by $L^2(\mathcal{B})$. Notice that

$$L^2(\mathcal{B}) = \bigoplus_{k \in \mathbb{Z}} \mathcal{B}_k$$

as Hilbert $\mathcal{B}_0$-modules. We have an embedding $C_c(\mathcal{B}) \to L(L^2(\mathcal{B}))$ given by left multiplication. Denote by $C^*(\mathcal{B})$ the completion of $C_c(\mathcal{B})$ with respect to the operator norm. The map $P : C_c(\mathcal{B}) \to \mathcal{B}_0$, $\xi \mapsto \xi(0)$ extends to a conditional expectation $P : C^*(\mathcal{B}) \to \mathcal{B}_0$.

Proof of Theorem 3.2. To the pair $(E_\infty, \mathcal{F}_E)$, we can associate the Fell bundle $\mathcal{B}$, where $\mathcal{B}_n := E_\infty^{\otimes n}, n \in \mathbb{Z}$. The multiplication is given by the tensor product, where we identify $E_\infty^* \otimes E_\infty$ with $\mathcal{F}_E$ and $E_\infty \otimes E_\infty^*$ with the ideal $\mathcal{F}_E^1$ of $\mathcal{F}_E$ generated by the (images of) $K(E^{\otimes n})$, $n \geq 1$, in $\varinjlim L(E^{\otimes n})$. (See [18].) The involution is obvious. Then

$$L^2(\mathcal{B}) = \mathcal{E}_\infty = \bigoplus_{n \in \mathbb{Z}} E_\infty^{\otimes n}.$$

Since $\mathcal{E}_\infty$ is generated by $\mathcal{F}_E$ and E_∞, it follows that the C^*-algebra generated by the operators M_ξ is isomorphic to $C^*(\mathcal{B})$. Hence, $\tilde{\mathcal{O}}_E \simeq C^*(\mathcal{B})$.

For the groupoid $\Gamma = \Gamma(X, T)$ and $l \in \mathbb{Z}$, take

$$\Gamma_l := \{(x, k, y) \in \Gamma \mid k = l\} = \{(x, y) \in X \times X \mid x_n = y_{n+l} \text{ for large } n\},$$

and $\mathcal{D}_l = \overline{C_c(\Gamma_{-l})}$ (closure in $C^*(\Gamma)$). Then it is easy to see that $C^*(\Gamma)$ is isomorphic to $C^*(\mathcal{D})$. However, $\mathcal{D}_0 = C^*(R(X, T)) \simeq \mathcal{F}_E = \mathcal{B}_0$, $\mathcal{D}_1 = \overline{C_c(\Gamma_{-1})} \simeq E_\infty = \mathcal{B}_1$, etc. Therefore the Fell bundles $\mathcal{B}$ and $\mathcal{D}$ are isomorphic.

That concludes the proof of the theorem. □

Renault [20] has nice criteria for this C^*-algebra to be simple and purely infinite. Applying them here, we obtain the following two propositions.

Proposition 3.6. *Let $\sigma : X \to X$ be an essentially free branched covering. Assume that for every nonempty open set $D \subset X$ and every $x \in X$, there exist $m, n \in \mathbb{N}$ such that $x \in dom(T^n)$ and $T^n x \in T^m D$. Then $C^*(X, \sigma)$ is simple.*

Proposition 3.7. *Assume that for every nonempty open set $D \subset X$ there exists an open set $D' \subset D$ and $m, n \in \mathbb{N}$ such that $T^n(\overline{D'})$ is strictly contained in $T^m(D')$. Then $C^*(X, \sigma)$ is purely infinite.*

The C^*-algebra of the equivalence relation $R(X, T)$ plays the role of the AF-algebra. It is an inductive limit of unital noncontinuous trace algebras, similar to the dimension drop algebras considered by Elliott et al.

4. K-theory computations and examples

We use the six-term exact sequence obtained by Pimsner to compute the K-theory of the C^*-algebras associated to a branched covering.

Theorem 4.1. *Let $\sigma : X \to X$ be a branched covering of a locally compact space X with singular sets S and S'. Let $U = X \setminus S$. We have an exact sequence*:

$$\begin{array}{ccccc} K_0(C_0(U \cap \sigma(U))) & \longrightarrow & K_0(C_0(X)) & \longrightarrow & K_0(C^*(X, \sigma)) \\ \uparrow & & & & \downarrow \\ K_1(C^*(X, \sigma)) & \longleftarrow & K_1(C_0(X)) & \longleftarrow & K_1(C_0(U \cap \sigma(U))) \end{array}$$

Proof. Theorem 4.9 of [18] for the augmented C^*-algebra $\tilde{\mathcal{O}}_E$ yields the following six-term exact sequence:

$$\begin{array}{ccccc} K_0(I) & \overset{\otimes(\iota_I - [E])}{\longrightarrow} & K_0(C_0(X)) & \overset{i_*}{\longrightarrow} & K_0(C^*(X, \sigma)) \\ \delta \uparrow & & & & \downarrow \delta \\ K_1(C^*(X, \sigma)) & \overset{i_*}{\longleftarrow} & K_1(C_0(X)) & \overset{\otimes(\iota_I - [E])}{\longleftarrow} & K_1(I) \end{array}$$

where I is the ideal $I = \varphi^{-1}(K(E)) \cap C_0(\sigma(U))$ in $A = C_0(X)$, and where, recall, $C^*(X, \sigma) \simeq \tilde{\mathcal{O}}_E$. The maps δ are the boundary maps corresponding to the Toeplitz extension for E described in [18]. The map i_* is the map on K-theory induced by the inclusion of $C_0(X)$ in $C^*(X, \sigma)$, and the two instances of $\otimes(\iota_I - [E])$ are the maps induced by Kasparov multiplication by $\iota_I - [E] \in KK(I, C_0(X))$. (Here, $K_*(C_0(X))$ is identified with $KK_*(\mathbb{C}, C_0(X))$, $K_*(\mathcal{O}_E)$ is identified with $KK_*(\mathbb{C}, \mathcal{O}_E)$, and $K_*(I)$ is identified with $KK_*(\mathbb{C}, I)$.) However, $\varphi^{-1}(K(E)) = C_0(U)$; therefore $I = C_0(U \cap \sigma(U))$. ☐

Example 4.2. Consider the folding of the interval $\sigma : [0, 1] \to [0, 1]$. Then

$$C^*(R_0) = C([0, 1]), C^*(R_1) = \{f : [0, 1] \to \mathbf{M}_2 \ \mid \ f(\tfrac{1}{2}) \in \mathbb{C} \otimes I_2\},$$

$$C^*(R_2) = \{f : [0, 1] \to \mathbf{M}_4 \ \mid \ f(0), f(\tfrac{1}{4}), f(\tfrac{3}{4}), f(1) \in \mathbf{M}_2 \otimes I_2, \ f(\tfrac{1}{2}) \in \mathbb{C} \otimes I_4\},$$

etc.

Since $([0, 1], \sigma)$ is essentially free, we may consider $C^*([0, 1], \sigma)$ defined above. Note that this C^*-algebra is not simple, since the orbit of 0 is $\{0, 1\}$ which is not dense. To compute its K-theory, note that $U = [0, 1/2) \cup (1/2, 1], \sigma(U) = [0, 1)$; therefore $I = C_0([0, 1/2) \cup (1/2, 1))$. The exact sequence becomes

$$\begin{array}{ccccc} 0 & \to & \mathbb{Z} & \to & K_0 \\ \uparrow & & & & \downarrow \\ K_1 & \leftarrow & 0 & \leftarrow & \mathbb{Z} \end{array}$$

hence $C^*([0, 1], \sigma)$ has $K_0 = \mathbb{Z} \oplus \mathbb{Z}$ and $K_1 = 0$.

Example 4.3. Let

$$q(z) = \frac{(z^2+1)^2}{4z(z^2-1)}, \quad z \in \mathbf{S}^2 = \mathbb{C} \cup \{\infty\}.$$

Then q is a rational map of degree 4, which is a local homeomorphism, except at the points z such that the equation $q(z) = w$ has double roots. We find

$$U = \mathbf{S}^2 \setminus \{\pm i, \pm(\sqrt{2} \pm 1)\}, \quad q(U) = \mathbf{S}^2 \setminus \{-1, 0, 1\}.$$

It is known (Lattès 1918) that the Julia set of q is the whole Riemann sphere, and that every backward orbit $\bigcup_{n \geq 0} q^{-n}(z)$ is dense. From Theorem 4.2.5 in [3], it follows that for any nonempty open set W of $\mathbf{S}^2$, there is $N \geq 0$ such that $q^N(W) = \mathbf{S}^2$. Notice that the forward orbits of all the points in the singular set $\{\pm i, \pm(\sqrt{2} \pm 1)\}$ are finite. Denote by F the union of these orbits. Given any open set D, we can find an open subset $D' \subset D$ such that the closure $\overline{D'}$ does not intersect the finite set F. We can find a positive integer M such that $q^M \mid_{D'}$ is a local homeomorphism and $q^M(D') = \mathbf{S}^2$; hence $\overline{D'}$ is strictly contained in $q^M(D')$. In particular, $(\mathbf{S}^2, q)$ satisfies the hypotheses of Propositions 3.6 and 3.7. Hence $C^*(\mathbf{S}^2, q)$ is simple and purely infinite. We have

$$I \simeq C_0(\mathbf{S}^2 \setminus \{\pm i, \pm(\sqrt{2} \pm 1), 0, \pm 1\}).$$

To compute the K-theory of I, we use the more general short exact sequence

$$0 \to C_0(\mathbf{S}^2 \setminus \{p_1, p_2, ..., p_n\}) \to C(\mathbf{S}^2) \xrightarrow{j} \mathbb{C}^n \to 0.$$

That gives

$$\begin{array}{ccccc} K_0 & \to & \mathbb{Z}^2 & \xrightarrow{j_*} & \mathbb{Z}^n \\ \uparrow & & & & \downarrow \\ 0 & \leftarrow & 0 & \leftarrow & K_1 \end{array}$$

Using the fact that j_* takes the Bott element into 0, it follows that

$$ker j_* \simeq \mathbb{Z}, \ coker j_* \simeq \mathbb{Z}^{n-1}.$$

We obtain

$$K_0(C_0(\mathbf{S}^2 \setminus \{p_1, p_2, ..., p_n\})) = \mathbb{Z}, \ K_1(C_0(\mathbf{S}^2 \setminus \{p_1, p_2, ..., p_n\})) = \mathbb{Z}^{n-1}.$$

The exact sequence for our C*-algebra becomes

$$\begin{array}{ccccc} \mathbb{Z} & \to & \mathbb{Z}^2 & \to & K_0(C^*(\mathbf{S}^2, q)) \\ \uparrow & & & & \downarrow \\ K_1(C^*(\mathbf{S}^2, q)) & \leftarrow & 0 & \leftarrow & \mathbb{Z}^8 \end{array}$$

Note that for a general rational map q of degree ≥ 2, the Julia set may not be the entire sphere. Its complement, the Fatou set, will provide an ideal in $C^*(\mathbf{S}^2, q)$ such that the quotient will be a simple C^*-algebra. For more details about the dynamics of a rational map, we refer to [3].

Example 4.4. Consider $\mathbf{S}^1 = \mathbb{R} \cup \{\infty\}$, and let $\sigma : \mathbb{R} \to \mathbf{S}^1, \sigma(t) = exp(2\pi i t)$. Then $I = C_0(\mathbb{R})$ with $K_0 = 0$ and $K_1 = \mathbb{Z}$; therefore we have the exact sequence

$$\begin{array}{ccccc} 0 & \to & \mathbb{Z} & \to & K_0(C^*(\mathbf{S}^1, \sigma)) \\ \uparrow & & & & \downarrow \\ K_1(C^*(\mathbf{S}^1, \sigma)) & \leftarrow & \mathbb{Z} & \leftarrow & \mathbb{Z} \end{array}$$

Since the map $\mathbb{Z} \to \mathbb{Z}$ in the exact sequence is $id - id$, hence the zero map, it follows that

$$K_0(C^*(\mathbf{S}^1, \sigma)) \simeq \mathbb{Z}^2, \;\; K_1(C^*(\mathbf{S}^1, \sigma)) \simeq \mathbb{Z}.$$

Note that $C^*(\mathbf{S}^1, \sigma)$ is simple, since every orbit is dense.

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