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AN ORDER CHARACTERIZATION OF COMMUTATIVITY FOR C^*-ALGEBRAS

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Abstract. In this paper, we investigate the problem of when a C^*-algebra is commutative through operator-monotonic increasing functions. The principal result is that the function $e^t, t \in [0, \infty)$, is operator-monotonic increasing on a C^*-algebra $\mathcal{A}$ if and only if $\mathcal{A}$ is commutative. Therefore, C^*-algebra $\mathcal{A}$ is commutative if and only if $e^{x+y} = e^x e^y$ in $\mathcal{A} \dot{+} \mathbf{C}$ for all positive elements x, y in $\mathcal{A}$.

1. Introduction

Let $\mathcal{A}$ be a C^*-algebra. The sets of self-adjoint and positive elements of $\mathcal{A}$ are denoted by $\mathcal{A}_H$ and $\mathcal{A}_+$, respectively. $\mathcal{A} \dot{+} \mathbf{C}$ is the C^*-algebra obtained from $\mathcal{A}$ by adjoining an identity to $\mathcal{A}$. For $x \in \mathcal{A}$, the spectrum of x in $\mathcal{A}$ is denoted by $\sigma(x)$. If $\mathcal{A}$ is commutative, Ω denotes the spectral space of $\mathcal{A}$ which is locally compact in the weak * topology and $C_0^\infty(\Omega)$ denotes the set of all continuous functions on Ω vanishing at infinity. For other notations, we will follow [9].

For the commutativity of C^*-algebras, a lot of results have been obtained from various points of view, for example, the nilpotent and ideal characterizations given by I. Kaplansky (see [2] and [7]), the numerical characterizations given by M. J. Crabb, J. Duncan and C. M. McGregor (see [1]) and J. Duncan and P. J. Taylor (see [4]), the order characterizations given by T. Ogasawara (see [11]), S. Sherman (see [12]), M. J. Crabb, J. Duncan and C. M. McGregor (see [1]) and M. Fukamiya, M. Misonou and Z. Takeda (see [5]), *-representation characterizations given by C. F. Skau (see [13]) and S. Wright (see [14]) and spectral characterizations given by R. Nakamoto (see [10]) and Y. Kato (see [8]).

From [6], we know that some continuous real-valued functions defined on a subset S of the real field $\mathbf{R}$ are operator-monotonic increasing on S and others are not. Therefore, we introduce the following concept.

Definition 1. Let $\mathcal{A}$ be a C^*-algebra and f a continuous real-valued function defined on a subset S of the real field $\mathbf{R}$. We say that the function f is operator-monotonic increasing on $\mathcal{A}$ associated with S if $f(x) \leq f(y)$ in $\mathcal{A} \dot{+} \mathbf{C}$ whenever x and y are self-adjoint elements of $\mathcal{A}$, $x \leq y$ and $\sigma(x) \cup \sigma(y) \subseteq S$. We denote by

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$M_{\mathcal{A}}(S)$ the set of operator-monotonic increasing functions on $\mathcal{A}$ associated with the subset S of the real field $\mathbf{R}$.

The exponential function plays an important role in the study of C^*-algebras (see [3], [9], etc.). In this paper, we will give another order characterization of commutativity for C^*-algebras through operator-monotonic increasing functions.

2. Lemmas

In this section, we derive the basic properties of operator-monotonic increasing functions on a C^*-algebra associated with a subset of the real field $\mathbf{R}$.

Lemma 1. *Let $\mathcal{A}$ be a C^*-algebra and S a subset of the real field $\mathbf{R}$.*

(i) *If $f \in M_{\mathcal{A}}(S)$, then f is a monotonic increasing function on S.*

(ii) *$M_{\mathcal{A}}(S)$ is a convex cone.*

(iii) *If f belongs to $M_{\mathcal{A}}(S)$ and takes values in some subset T of the real field $\mathbf{R}$ and g belongs to $M_{\mathcal{A}}(T)$, then the composition $(g \circ f)(t) = g(f(t))$ is also in $M_{\mathcal{A}}(S)$.*

Proof. Direct verification according to Definition 1. □

Corollary 1. *Let $\mathcal{A}$ be a C^*-algebra. Then the function $f(t) = t^2 + at + b, a, b \in \mathbf{R}$ and $t \in [-\frac{a}{2}, \infty)$, is operator-monotonic increasing on $\mathcal{A}$ if and only if $\mathcal{A}$ is commutative.*

Proof. Let

$$g(t) = t^2, t \in [0, \infty),$$

$$h(t) = t + \frac{a}{2}, t \in [-\frac{a}{2}, \infty),$$

and

$$k(t) = t + b - \frac{a^2}{4}, t \in [0, \infty).$$

Then

$$f(t) = (k \circ g \circ h)(t), t \in [-\frac{a}{2}, \infty),$$

and by Lemma 1(ii), the functions h and k are operator-monotonic increasing on $\mathcal{A}$.

If $\mathcal{A}$ is a commutative C^*-algebra, then the function g is operator-monotonic increasing on $\mathcal{A}$ by Definition 1. Therefore, the function f is operator-monotonic increasing on $\mathcal{A}$ according to Lemma 1(iii).

Conversely, if the function f is operator-monotonic increasing on $\mathcal{A}$, then, by Lemma 1(ii), the function $p(t) = (t + \frac{a}{2})^2, t \in [-\frac{a}{2}, \infty)$, is operator-monotonic increasing on $\mathcal{A}$, and hence the function $q(t) = t^2, t \in [0, \infty)$, is operator-monotonic increasing on $\mathcal{A}$. It follows that $\mathcal{A}$ is a commutative C^*-algebra by T. Ogasawara's order characterization theorem (see [11]). □

Lemma 2. *Let $\mathcal{A}$ be a C^*-algebra and $f(t), f_k(t), k = 1, 2, \dots,$ continuous real-valued functions defined on a subset S of the real field $\mathbf{R}$. If $f_k \in M_{\mathcal{A}}(S), k = 1, 2, \dots,$ and $\{f_k\}$ converges to f uniformly on compact subsets of S when $k \to \infty$, then f belongs to $M_{\mathcal{A}}(S)$.*

Proof. Take x, y in $\mathcal{A}_H$ such that

$$x \leq y$$

and

$$\sigma(x) \cup \sigma(y) \subseteq S.$$

Let $\{e_x(\lambda)\}$ and $\{e_y(\lambda)\}$ be the spectral families of x and y, respectively. By spectral theory, we have

$$x = \int_{\sigma(x)} \lambda de_x(\lambda)$$

and

$$y = \int_{\sigma(y)} \lambda de_y(\lambda).$$

Since $\{f_k\}$ converges to f uniformly on compact subsets of S, we get

$$\begin{aligned} f(x) &= \int_{\sigma(x)} f(\lambda) de_x(\lambda) \\ &= \lim_{k\to\infty} \int_{\sigma(x)} f_k(\lambda) de_x(\lambda) \\ &= \lim_{k\to\infty} f_k(x) \end{aligned}$$

and

$$\begin{aligned} f(y) &= \int_{\sigma(y)} f(\lambda) de_y(\lambda) \\ &= \lim_{k\to\infty} \int_{\sigma(y)} f_k(\lambda) de_y(\lambda) \\ &= \lim_{k\to\infty} f_k(y) \end{aligned}$$

(see [7, Sec. 5.2]). The premise that $\{f_k : k = 1, 2, \dots\} \subseteq M_{\mathcal{A}}(S)$ implies that

$$f_k(x) \leq f_k(y), k = 1, 2, \dots .$$

Therefore, $f(x) \leq f(y)$, and hence $f \in M_{\mathcal{A}}(S)$. □

3. Main theorem and corollary

Now we come to the main results.

Theorem 1. *Let $\mathcal{A}$ be a C^*-algebra. Then the function $f(t) = e^t, t \in [0, \infty)$, is operator-monotonic increasing on $\mathcal{A}$ if and only if $\mathcal{A}$ is a commutative C^*-algebra.*

Proof. Suppose $\mathcal{A}$ is a commutative C^*-algebra. Then $\mathcal{A}$ is isometrically * isomorphic to $C_0^\infty(\Omega)$ (see [9, p. 73]). Now it is clear that the function $f(t) = e^t, t \in [0, \infty)$, is operator-monotonic increasing on $\mathcal{A}$.

Conversely, assume that x and y are elements in $\mathcal{A}_+$ and $x \leq y$. Letting $0 < \epsilon < 1$, we have

$$x + \epsilon \leq y + \epsilon.$$

Because the function $t \longmapsto -t^{-1}, t \in (0,\infty)$, is operator-monotonic increasing on $\mathcal{A}$ (see [6]), the function

$$g_\alpha(t) = \frac{1}{\alpha+1} - \frac{1}{\alpha+t}, t \in (0,\infty),$$

where $\alpha \geq 0$, is operator-monotonic increasing on $\mathcal{A}$. For $c > 0$, let

$$B = \max\{\frac{1-\epsilon}{e^c - 1} - \epsilon, 0\}.$$

Then, when $b > B$, we have

$$\left| \int_b^\infty g_\alpha(t) d\alpha \right| \leq \ln \frac{b+1}{b+\epsilon} < c$$

for all $t \in [\epsilon, \|y\| + \epsilon]$. So $\int_0^\infty g_\alpha(t) d\alpha$ converges uniformly on $[\epsilon, \|y\| + \epsilon]$, and hence the function

$$s(t) = \ln t - \int_0^\infty g_\alpha(t) d\alpha$$

is continuous on $[\epsilon, \|y\| + \epsilon]$. Let $t_0 \in [\epsilon, \|y\| + \epsilon]$ such that

$$\left| \ln t_0 - \int_0^\infty g_\alpha(t_0) d\alpha \right| = \max_{t \in [\epsilon, \|y\| + \epsilon]} \left\{ \left| \ln t - \int_0^\infty g_\alpha(t) d\alpha \right| \right\}.$$

Then for $\delta > 0$, there are an integer n and an equidistant division

$$0 = \alpha_0 < \alpha_1 < \ldots < \alpha_m = n$$

of the interval $[0, n]$ such that

$$\left| \ln t - \frac{n}{m} \sum_{k=1}^m g_{\alpha_k}(t) \right| < \delta$$

for all $t \in [\epsilon, \|y\| + \epsilon]$. So we have

$$\begin{aligned} \ln(x+\epsilon) &\leq \frac{n}{m} \sum_{k=1}^m g_{\alpha_k}(x+\epsilon) + \delta \\ &\leq \frac{n}{m} \sum_{k=1}^m g_{\alpha_k}(y+\epsilon) + \delta \\ &\leq \ln(y+\epsilon) + 2\delta, \end{aligned}$$

which shows that $\ln(x+\epsilon) \leq \ln(y+\epsilon)$. Choose real number K such that $K \geq -2\ln(t+\epsilon), t \in [0,\infty)$. Since the function $f(t) = e^t, t \in [0,\infty)$, is operator-monotonic increasing on $\mathcal{A}$, we get

$$e^{2\ln(x+\epsilon)+K} \leq e^{2\ln(y+\epsilon)+K},$$

and hence

$$(x+\epsilon)^2 \leq (y+\epsilon)^2$$

(see [7, Sec. 3.3]). Therefore, for $\epsilon \in (0,1)$, the functions $f_\epsilon(t) = t^2 + 2\epsilon t + \epsilon^2, t \in [0,\infty)$, are operator-monotonic increasing on $\mathcal{A}$. It is clear that $\{g_n\} = \{f_{\frac{1}{n}}\}$ converges to $g(t) = t^2, t \in [0,\infty)$, uniformly on compact subsets of $[0,\infty)$. By Lemma 2, the function $f(t) = t^2, t \in [0,\infty)$, is operator-monotonic increasing on

$\mathcal{A}$. So $\mathcal{A}$ is a commutative C^*-algebra by T. Ogasawara's order characterization theorem. $\square$

Corollary 2. *Let $\mathcal{A}$ be a C^*-algebra. Then $\mathcal{A}$ is commutative if and only if $e^{x+y} = e^x e^y$ in $\mathcal{A}\dot{+}\mathbf{C}$ for all $x, y \in \mathcal{A}_+$.*

Proof. If $\mathcal{A}$ is a commutative C^*-algebra, it is clear that $e^{x+y} = e^x e^y$ in $\mathcal{A}\dot{+}\mathbf{C}$ for all $x, y \in \mathcal{A}_+$.

Conversely, assume that $x, y \in \mathcal{A}_+$ and $x \leq y$. Let $\mathcal{B}$ be the C^*-subalgebra generated by $(y-x)$. Then $\mathcal{B}$ is isometrically * isomorphic to $C_0^\infty(\sigma(y-x) \setminus \{0\})$ (see [9, p. 73]). Because $(y-x)(t) \geq 0, t \in \sigma(y-x) \setminus \{0\}$, $e^{y-x}(t) \geq 1$, that is, $e^{y-x} \geq 1$. So

$$e^x = e^{\frac{x}{2}} \cdot 1 \cdot e^{\frac{x}{2}} \leq e^{\frac{x}{2}} \cdot e^{y-x} \cdot e^{\frac{x}{2}} = e^y.$$

Thus the function $f(t) = e^t, t \in [0, \infty)$, is operator-monotonic increasing on $\mathcal{A}$. By Theorem 1, $\mathcal{A}$ is a commutative C^*-algebra. $\square$

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