AN ORDER CHARACTERIZATION OF COMMUTATIVITY FOR $C^*$-ALGEBRAS

WEI WU

Abstract. In this paper, we investigate the problem of when a $C^*$-algebra is commutative through operator-monotonic increasing functions. The principal result is that the function $e^{x+t}$, $t \in [0, \infty)$, is operator-monotonic increasing on a $C^*$-algebra $A$ if and only if $A$ is commutative. Therefore, $C^*$-algebra $A$ is commutative if and only if $e^{x+y} = e^x e^y$ in $A^+ + C$ for all positive elements $x, y$ in $A$.

1. Introduction

Let $A$ be a $C^*$-algebra. The sets of self-adjoint and positive elements of $A$ are denoted by $A_H$ and $A_+$, respectively. $A^+ + C$ is the $C^*$-algebra obtained from $A$ by adjoining an identity to $A$. For $x \in A$, the spectrum of $x$ in $A$ is denoted by $\sigma(x)$. If $A$ is commutative, $\Omega$ denotes the spectral space of $A$ which is locally compact in the weak * topology and $C^\infty_0(\Omega)$ denotes the set of all continuous functions on $\Omega$ vanishing at infinity. For other notations, we will follow [9].

For the commutativity of $C^*$-algebras, a lot of results have been obtained from various points of view, for example, the nilpotent and ideal characterizations given by I. Kaplansky (see [2] and [7]), the numerical characterizations given by M. J. Crabb, J. Duncan and C. M. McGregor (see [1]) and J. Duncan and P. J. Taylor (see [4]), the order characterizations given by T. Ogasawara (see [11]), S. Sherman (see [12]), M. J. Crabb, J. Duncan and C. M. McGregor (see [1]) and M. Fukamiya, M. Misonou and Z. Takeda (see [5]), *-representation characterizations given by C. F. Skau (see [13]) and S. Wright (see [14]) and spectral characterizations given by R. Nakamoto (see [14]) and Y. Kato (see [8]).

From [6], we know that some continuous real-valued functions defined on a subset $S$ of the real field $\mathbb{R}$ are operator-monotonic increasing on $S$ and others are not. Therefore, we introduce the following concept.

Definition 1. Let $A$ be a $C^*$-algebra and $f$ a continuous real-valued function defined on a subset $S$ of the real field $\mathbb{R}$. We say that the function $f$ is operator-monotonic increasing on $A$ associated with $S$ if $f(x) \leq f(y)$ in $A^+ + C$ whenever $x$ and $y$ are self-adjoint elements of $A$, $x \leq y$ and $\sigma(x) \cup \sigma(y) \subseteq S$. We denote by

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$M_A(S)$ the set of operator-monotonic increasing functions on $A$ associated with the subset $S$ of the real field $R$.

The exponential function plays an important role in the study of $C^*$-algebras (see [3], [9], etc.). In this paper, we will give another order characterization of commutativity for $C^*$-algebras through operator-monotonic increasing functions.

2. Lemmas

In this section, we derive the basic properties of operator-monotonic increasing functions on a $C^*$-algebra associated with a subset of the real field $R$.

Lemma 1. Let $A$ be a $C^*$-algebra and $S$ a subset of the real field $R$.

(i) If $f \in M_A(S)$, then $f$ is a monotonic increasing function on $S$.

(ii) $M_A(S)$ is a convex cone.

(iii) If $f$ belongs to $M_A(S)$ and takes values in some subset $T$ of the real field $R$ and $g$ belongs to $M_A(T)$, then the composition $(g \circ f)(t) = g(f(t))$ is also in $M_A(S)$.

Proof. Direct verification according to Definition 1.

Corollary 1. Let $A$ be a $C^*$-algebra. Then the function $f(t) = t^2 + at + b, a, b \in \mathbb{R}$ and $t \in [-\frac{a}{2}, \infty)$, is operator-monotonic increasing on $A$ if and only if $A$ is commutative.

Proof. Let

$$g(t) = t^2, \quad t \in [0, \infty),$$

$$h(t) = t + \frac{a}{2}, \quad t \in [-\frac{a}{2}, \infty),$$

and

$$k(t) = t + b - \frac{a^2}{4}, \quad t \in [0, \infty).$$

Then

$$f(t) = (k \circ g \circ h)(t), \quad t \in [-\frac{a}{2}, \infty),$$

and by Lemma 1(ii), the functions $h$ and $k$ are operator-monotonic increasing on $A$.

If $A$ is a commutative $C^*$-algebra, then the function $g$ is operator-monotonic increasing on $A$ by Definition 1. Therefore, the function $f$ is operator-monotonic increasing on $A$ according to Lemma 1(iii).

Conversely, if the function $f$ is operator-monotonic increasing on $A$, then, by Lemma 1(ii), the function $p(t) = (t + \frac{a}{2})^2, t \in [-\frac{a}{2}, \infty)$, is operator-monotonic increasing on $A$, and hence the function $q(t) = t^2, t \in [0, \infty)$, is operator-monotonic increasing on $A$. It follows that $A$ is a commutative $C^*$-algebra by T. Ogasawara’s order characterization theorem (see [11]).

Lemma 2. Let $A$ be a $C^*$-algebra and $f(t), f_k(t), k = 1, 2, \ldots$, continuous real-valued functions defined on a subset $S$ of the real field $R$. If $f_k \in M_A(S), k = 1, 2, \ldots$, and $\{f_k\}$ converges to $f$ uniformly on compact subsets of $S$ when $k \to \infty$, then $f$ belongs to $M_A(S)$. 

Proof. Take $x, y$ in $A_H$ such that

$$x \leq y$$

and

$$\sigma(x) \cup \sigma(y) \subseteq S.$$ 

Let $\{e_x(\lambda)\}$ and $\{e_y(\lambda)\}$ be the spectral families of $x$ and $y$, respectively. By spectral theory, we have

$$x = \int_{\sigma(x)} \lambda de_x(\lambda)$$

and

$$y = \int_{\sigma(y)} \lambda de_y(\lambda).$$

Since $\{f_k\}$ converges to $f$ uniformly on compact subsets of $S$, we get

$$f(x) = \int_{\sigma(x)} f(\lambda)de_x(\lambda)$$

$$= \lim_{k \to \infty} \int_{\sigma(x)} f_k(\lambda)de_x(\lambda)$$

$$= \lim_{k \to \infty} f_k(x)$$

and

$$f(y) = \int_{\sigma(y)} f(\lambda)de_y(\lambda)$$

$$= \lim_{k \to \infty} \int_{\sigma(y)} f_k(\lambda)de_y(\lambda)$$

$$= \lim_{k \to \infty} f_k(y)$$

(see [2] Sec. 5.2]). The premise that $\{f_k : k = 1, 2, \ldots\} \subseteq M_A(S)$ implies that

$$f_k(x) \leq f_k(y), k = 1, 2, \ldots.$$ 

Therefore, $f(x) \leq f(y)$, and hence $f \in M_A(S)$. \qed

3. Main theorem and corollary

Now we come to the main results.

**Theorem 1.** Let $\mathcal{A}$ be a C*-algebra. Then the function $f(t) = e^t, t \in [0, \infty)$, is operator-monotonic increasing on $\mathcal{A}$ if and only if $\mathcal{A}$ is a commutative C*-algebra.

**Proof.** Suppose $\mathcal{A}$ is a commutative C*-algebra. Then $\mathcal{A}$ is isometrically * isomorphic to $C_0^\infty(\Omega)$ (see [3] p. 73). Now it is clear that the function $f(t) = e^t, t \in [0, \infty)$, is operator-monotonic increasing on $\mathcal{A}$.

Conversely, assume that $x$ and $y$ are elements in $\mathcal{A}_+$ and $x \leq y$. Letting $0 < \epsilon < 1$, we have

$$x + \epsilon \leq y + \epsilon.$$
Because the function $t \mapsto -t^{-1}, t \in (0, \infty)$, is operator-monotonic increasing on $\mathcal{A}$ (see [3]), the function
\[
g_{\alpha}(t) = \frac{1}{\alpha + 1} - \frac{1}{\alpha + t}, t \in (0, \infty),
\]
where $\alpha \geq 0$, is operator-monotonic increasing on $\mathcal{A}$. For $c > 0$, let
\[
B = \max\{\frac{1}{e^c - 1} - \epsilon, 0\}.
\]
Then, when $b > B$, we have
\[
\left| \int_{b}^{\infty} g_{\alpha}(t) d\alpha \right| \leq \ln \frac{b + 1}{b + \epsilon} < c
\]
for all $t \in [\epsilon, \|y\| + \epsilon]$. So $\int_{0}^{\infty} g_{\alpha}(t) d\alpha$ converges uniformly on $[\epsilon, \|y\| + \epsilon]$, and hence the function
\[
s(t) = \ln t - \int_{0}^{\infty} g_{\alpha}(t) d\alpha
\]
is continuous on $[\epsilon, \|y\| + \epsilon]$. Let $t_0 \in [\epsilon, \|y\| + \epsilon]$ such that
\[
\left| \ln t_0 - \int_{0}^{\infty} g_{\alpha}(t_0) d\alpha \right| = \max_{t \in [\epsilon, \|y\| + \epsilon]} \left\{ \left| \ln t - \int_{0}^{\infty} g_{\alpha}(t) d\alpha \right| \right\}.
\]
Then for $\delta > 0$, there are an integer $n$ and an equidistant division
\[
0 = \alpha_0 < \alpha_1 < \ldots < \alpha_m = n
\]
of the interval $[0, n]$ such that
\[
\left| \ln t - \frac{n}{m} \sum_{k=1}^{m} \alpha_{\frac{k}{n}}(t) \right| < \delta
\]
for all $t \in [\epsilon, \|y\| + \epsilon]$. So we have
\[
\ln(x + \epsilon) \leq \frac{n}{m} \sum_{k=1}^{m} \alpha_{\frac{k}{n}}(x + \epsilon) + \delta
\]
\[
\leq \frac{n}{m} \sum_{k=1}^{m} \alpha_{\frac{k}{n}}(y + \epsilon) + \delta
\]
\[
\leq \ln(y + \epsilon) + 2\delta,
\]
which shows that $\ln(x + \epsilon) \leq \ln(y + \epsilon)$. Choose real number $K$ such that $K \geq -2 \ln(t + \epsilon), t \in [0, \infty)$. Since the function $f(t) = e^t, t \in [0, \infty)$, is operator-monotonic increasing on $\mathcal{A}$, we get
\[
e^{2\ln(x + \epsilon) + K} \leq e^{2\ln(y + \epsilon) + K},
\]
and hence
\[
(x + \epsilon)^2 \leq (y + \epsilon)^2
\]
(see [7, Sec. 3.3]). Therefore, for $\epsilon \in (0, 1)$, the functions $f_{\epsilon}(t) = t^2 + 2\epsilon t + \epsilon^2, t \in [0, \infty)$, are operator-monotonic increasing on $\mathcal{A}$. It is clear that $\{g_{\alpha}\} = \{f_{\frac{\alpha}{n}}\}$ converges to $g(t) = t^2, t \in [0, \infty)$, uniformly on compact subsets of $[0, \infty)$. By Lemma 2, the function $f(t) = t^2, t \in [0, \infty)$, is operator-monotonic increasing on
A. So \( \mathcal{A} \) is a commutative \( C^* \)-algebra by T. Ogasawara’s order characterization theorem.

**Corollary 2.** Let \( \mathcal{A} \) be a \( C^* \)-algebra. Then \( \mathcal{A} \) is commutative if and only if \( e^{x+y} = e^x e^y \) in \( \mathcal{A} + C \) for all \( x, y \in \mathcal{A}_+ \).

**Proof.** If \( \mathcal{A} \) is a commutative \( C^* \)-algebra, it is clear that \( e^{x+y} = e^x e^y \) in \( \mathcal{A} + C \) for all \( x, y \in \mathcal{A}_+ \).

Conversely, assume that \( x, y \in \mathcal{A}_+ \) and \( x \leq y \). Let \( \mathcal{B} \) be the \( C^* \)-subalgebra generated by \( (y - x) \). Then \( \mathcal{B} \) is isometrically * isomorphic to \( C^*_0(\sigma(y - x) \setminus \{0\}) \) (see [9, p. 73]). Because \( (y - x)(t) \geq 0, t \in \sigma(y - x) \setminus \{0\} \), \( e^{y-x}(t) \geq 1 \), that is, \( e^{y-x} \geq 1 \). So

\[
e^x = e^t \cdot 1 \cdot e^x \leq e^t \cdot e^{y-x} \cdot e^y = e^y.
\]

Thus the function \( f(t) = e^t, t \in [0, \infty) \), is operator-monotonic increasing on \( \mathcal{A} \). By Theorem 1, \( \mathcal{A} \) is a commutative \( C^* \)-algebra.

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**Institute of Mathematics, Academia Sinica, Beijing 100080, China**

**E-mail address:** wwu@math03.math.ac.cn

**Current address:** Department of Mathematics, East China Normal University, Shanghai 200062, China

**E-mail address:** wwu@math.ecnu.edu.cn