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COMPLETE ORTHOGONAL DECOMPOSITION HOMOMORPHISMS BETWEEN MATRIX ORDERED HILBERT SPACES

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ABSTRACT. The purpose of this paper is to show that a complete order homomorphism and a complete orthogonal decomposition homomorphism between the non-commutative L^2-spaces induce respectively an isomorphism and a $*$-isomorphism between the associated reduced von Neumann algebras.

1. INTRODUCTION

In [C] Connes studied an order isomorphism on a Hilbert space and introduced an orientable homogeneous selfdual cone to construct a von Neumann algebra. On the other hand, Schmitt and Wittstock [SW] introduced a matrix ordered Hilbert space to handle a non-commutative order and characterized it using the face property of the family of selfdual cones. From the point of view of the complete positivity of the maps on a matrix ordered Hilbert space, we showed in [M2] the relationship between an order isomorphism or an orthogonal decomposition isomorphism defined by Yamamuro [Y] and an isomorphism of a von Neumann algebra. In the present article we shall generalize their results to the case where a complete order homomorphism is not necessarily bijective.

We shall use the notation as introduced in [SW] with respect to the matrix ordered standard forms.

Let M_n and $M_{n,m}$ be respectively a set of all $n \times n$ and $n \times m$ matrices over $\mathbb{C}$. For a Hilbert space H and $n \in \mathbb{N}$, put $H_n = H \otimes M_n$. Let $(H, H_n^+, n \in \mathbb{N})$, where H_n^+ denotes a selfdual cone in H_n, be a matrix ordered Hilbert space, and let $(\hat{H}_n, \hat{H}_n^+, n \in \mathbb{N})$ be another one. Let h be a bounded linear map of H into $\hat{H}$. A bijective linear map h is called an order isomorphism if $hH^+ = \hat{H}^+$. We call h a complete order isomorphism if $h_n H_n^+ = \hat{H}_n^+$ for every $n \in \mathbb{N}$. We call h an o.d. (orthogonal decomposition) homomorphism if h is 1-positive and $(h\xi, h\eta) = 0$ whenever $\xi, \eta \in H^+$ and $(\xi, \eta) = 0$. If h_n is an o.d. homomorphism for every $n \in \mathbb{N}$, we call h a complete o.d. homomorphism. A bijective map h is called a complete o.d. isomorphism if both h and h^{-1} are complete o.d. homomorphisms.

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From now on, let $(M, H, H_n^+, n \in \mathbb{N})$ and $(\hat{M}, \hat{H}, \hat{H}_n^+, n \in \mathbb{N})$ be matrix ordered standard forms of von Neumann algebras. Here we use the notation

$$\mathrm{Ad}(h) : x \in M \mapsto hxh^{-1} \in B(\hat{H})$$

for the invertible map $h : H \to \hat{H}$.

Throughout this paper, we assume a Hilbert space to be separable.

2. Results

The main results are as follows:

Theorem A. *Let $(M, H, H_n^+, n \in \mathbb{N})$ and $(\hat{M}, \hat{H}, \hat{H}_n^+, n \in \mathbb{N})$ be matrix ordered standard forms. Suppose that h is a complete order homomorphism of H into $\hat{H}$ with support projection e and range projection f. Put $N = M \cap \{e\}'$ and $\hat{N} = \hat{M} \cap \{f\}'$. If e is completely positive and hH^+ is a selfdual cone in the closed range space of h, then we obtain the following properties:*

1. *f is completely positive.*
2. *$(eM|_{eH}, eH, e_nH_n^+, n \in \mathbb{N})$ and $(f\hat{M}|_{f\hat{H}}, f\hat{H}, f_n\hat{H}_n^+, n \in \mathbb{N})$ are matrix ordered standard forms.*
3. *$h|_{eH}$ is a complete order isomorphism of eH onto $f\hat{H}$, and $\mathrm{Ad}(h|_{eH})$ is an isomorphism of eMe onto $f\hat{M}f$.*

Theorem B. *With $(M, H, H_n^+, n \in \mathbb{N})$ and $(\hat{M}, \hat{H}, \hat{H}_n^+, n \in \mathbb{N})$ as before, let h be a completely positive o.d. homomorphism of H into $\hat{H}$ with support projection e and range projection f. If h has the closed range, then we obtain the following properties:*

1. *e belongs to $M \cap M'$.*
2. *f is completely positive.*
3. *$(f\hat{M}|_{f\hat{H}}, f\hat{H}, f_n\hat{H}_n^+, n \in \mathbb{N})$ is a matrix ordered standard form.*
4. *$h|_{eH}$ is a complete o.d. isomorphism of eH onto $f\hat{H}$, and $\mathrm{Ad}(h|_{eH})|_{M_e}$ is a $*$-isomorphism of M_e onto $f\hat{M}f$.*

We need some lemmata to prove Theorem A.

Lemma 1. *Let (M, H, J, H^+) be a standard form, and let $\hat{H}$ be a Hilbert space with a selfdual cone $\hat{H}^+$. Suppose that h is a linear bijection of H onto $\hat{H}$ such that $hH^+ = \hat{H}^+$. Then, for the polar decomposition $h = u|h|$ of h, u is a 1-positive isometry of H onto $\hat{H}$, and there exists a positive invertible operator k in M such that $|h| = kJkJ$.*

Proof. Since for every $\xi \in \hat{H}^+$, $(h^*\xi, \eta) = (\xi, h\eta) \geq 0$ holds for all $\eta \in H^+$, it follows from the selfduality of H^+ that $h^*\hat{H}^+ \subset H^+$. Hence $h^*hH^+ \subset H^+$. Since $(h^{-1})^* = (h^*)^{-1}$, $h^*hH^+ = H^+$. By [C, Theorem 3.3] there exists a positive invertible operator k in M such that $h^*h = k^2Jk^2J$. Note that we may assume $H^+ = \mathcal{P}_{\xi_0}^{\natural}$ with a cyclic and separating vector $\xi_0 \in H^+$ by the unicity of the standard form. Then $|h| = kJ_{H^+}kJ_{H^+}$, and

$$uH^+ = hk^{-1}Jk^{-1}JH^+ = hH^+ = \hat{H}^+.$$

This completes the proof. □

Lemma 2. *With $(M, H, H_n^+, n \in \mathbb{N})$ a matrix ordered standard form and $(\hat{H}, \hat{H}_n^+, n \in \mathbb{N})$ a matrix ordered Hilbert space, let h be an order isomorphism of H onto $\hat{H}$. If h is completely positive, then h is a complete order isomorphism. In addition, there exists a von Neumann algebra $\hat{M}$ such that $(\hat{M}, \hat{H}, \hat{H}_n^+, n \in \mathbb{N})$ is a matrix ordered standard form, and $\mathrm{Ad}(h)$ is an isomorphism of M onto $\hat{M}$.*

Proof. Let $h = u|h|$ be the polar decomposition of h. By Lemma 1, $|h|$ can be written as $|h| = kJ_{H^+}kJ_{H^+}$ for some positive invertible operator $k \in M$. Hence

$$|h_n|H_n^+ = (k \otimes 1_n)J_{H_n^+}(k \otimes 1_n)J_{H_n^+}H_n^+ = H_n^+.$$

Then $u_nH_n^+ = h_nH_n^+ \subset \hat{H}_n^+$. Since u is unitary, u is a complete order isomorphism of H onto $\hat{H}$. Thus we see that h is a complete order isomorphism and u is a complete o.d. isomorphism. By [M2], Proposition 2.6, Theorem 2.7] we obtain the desired result. □

Lemma 3. *With $(M, H, H_n^+, n \in \mathbb{N})$ a matrix ordered standard form, let e be a completely positive projection on H. Then there exists a von Neumann algebra A such that $(A, eH, e_nH_n^+, n \in \mathbb{N})$ is a matrix ordered standard form. In addition, if $N = M \cap \{e\}'$, then*

$$A = eM|_{eH} = N|_{eH}.$$

Proof. Put $J = J_{H^+}, K = eH, K_n = e_nH_n, K^+ = eH^+, K_n^+ = e_nH_n^+$ for every $n \in \mathbb{N}$. There exists by [M1, Lemma 1] a von Neumann algebra A such that(A, K, K_n^+) is a matrix ordered standard form. The inclusion $eM|_K \subset A$ follows from the first part of the proof of [M1, Lemma 2]. We prove that $A \subset N|_K$. Note that in a standard form (M, H, J, H^+) the map $q \mapsto qJqJH^+$ is an order isomorphism of the set of all projections in M onto the set of all closed faces in H^+ (see [C, Theorem 4.2 c)]). If p is a projection in A, then $[\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}]J_{K_2^+}[\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}]J_{K_2^+}K_2^+$, which shall be denoted by F, is a closed face in K_2^+ and

$$P_F = \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} J_{K_2^+} \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} J_{K_2^+}.$$

Here P_F denotes the projection of K_2 onto the (closed) linear span of F in K_2. There then exists a projection $P = [\begin{smallmatrix} a & b \\ b^* & c \end{smallmatrix}]$ in $M_2(M)$ such that $P_{\langle F \rangle} = PJ_2PJ_2$, where $P_{\langle F \rangle}$ denotes the projection of H_2 onto the closed linear span of the face $\langle F \rangle$ generated by F in H_2^+. It follows from [I, Lemma II.1.7] that $P_F\Xi = e_2P_{\langle F \rangle}\Xi = P_{\langle F \rangle}\Xi$ for all $\Xi \in K_2$. By setting $\Xi = [\begin{smallmatrix} 0 & 0 \\ 0 & \xi \end{smallmatrix}]$ we have

$$\begin{bmatrix} 0 & 0 \\ 0 & \xi \end{bmatrix} = \begin{bmatrix} bJbJ\xi & bJcJ\xi \\ cJbJ\xi & cJcJ\xi \end{bmatrix}$$

for all $\xi \in K$. It follows from [SW, Corollary 3.3] that $b\xi = 0$ for all $\xi \in K$. Using both equalities $\xi = cJcJ\xi$ and $b^*b + c^2 = c$ since P is a projection, we have

$$c\xi = c^2JcJ\xi = (c - b^*b)JcJ\xi = cJcJ\xi = \xi.$$

Moreover, by setting $\Xi = [\begin{smallmatrix} 0 & \xi \\ 0 & 0 \end{smallmatrix}]$ we have

$$p\xi = eaJcJ\xi$$

for all $\xi \in K$. Therefore, when ξ is an element of K^+, $p\xi = eaJc\xi = ea\xi$. Hence, $p\xi = ea\xi$ for all $\xi \in K$ because K is spanned by K^+. Since $e_2P_{\langle F \rangle} = P_{\langle F \rangle}e_2$,

$ea = ae$. Consequently, we obtain

$$N|_K \subset eM|_K \subset A \subset N|_K.$$

Therefore, we obtain the desired equalities. □

Remark. In the above lemma, if we assume that $(eH, e_nH_n^+, n \in \mathbb{N})$ is a matrix ordered Hilbert space having the conditions for a matrix ordered standard form and e is 2-positive instead of the complete positivity, then Lemma 3 holds. Namely, if there exists a matrix ordered standard form $(A, K, K_n^+, n \in \mathbb{N})$ and $eH^+ = K^+, e_2H_2^+ = K_2^+$, then $A = eM|_K = N|_K$.

Proof of Theorem A. Put $K = eH, K^+ = eH^+, K_n^+ = e_nH_n^+, \hat{K} = hH, \hat{K}^+ = hH^+$ and $\hat{K}_n^+ = h_nH_n^+$. Let $h = u|h|$ and $h_0 = u_0|h_0|$ be the polar decompositions of h and $h_0 = h|_K$, respectively. Since e is a completely positive projection, it follows from Lemma 3 that $(eM|_K, K, K_n^+)$ is a matrix ordered standard form. By assumption h_0 is a order isomorphism of K onto $\hat{K}$. Hence by Lemma 2 $|h_0|$ is a complete order isomorphism on K. Therefore, $|h_0|_nK_n^+$ is a selfdual cone in K_n, and so is $\hat{K}_n^+$ in $\hat{K}_n$ because of the complete positivity of h. Since f_n is the support projection of h_n, it follows that $\hat{K}_n^+ \subset f_n\hat{H}_n^+$. If $\xi \in \hat{K}_n^+, \eta \in \hat{H}_n^+$, then $(\xi, f_n\eta) = (\xi, \eta) \geq 0$. Hence $\hat{K}_n^+ \subset (f_n\hat{H}_n^+)'$ (in $\hat{K}_n$). Therefore, $\hat{K}_n^+ = \hat{K}_n^{+\prime} \supset (f_n\hat{H}_n^+)'' \supset f_n\hat{H}_n^+$ (in $\hat{K}_n$). Hence $\hat{K}_n^+ = f_n\hat{H}_n^+$, which means that f is completely positive. Therefore, by Lemma 3 $(f\hat{M}|_{K_n}, \hat{K}, \hat{K}_n^+)$ is a matrix ordered standard form and $\mathrm{Ad}(h_0)$ is an isomorphism of $eM|_K$ onto $f\hat{M}|_{\hat{K}}$. □

Now, we examine the properties of o.d. homomorphisms between two ordered Hilbert spaces.

Proposition 4 (cf. [DY, (2.1)]). *Let (M, H, J, H^+) be a standard form, and let $\hat{H}$ be a Hilbert space with a selfdual cone $\hat{H}^+$. Then h is an o.d. homomorphism of H into $\hat{H}$ if and only if $hH^+ \subset \hat{H}^+$ and $|h| \in M \cap M'$.*

One can give the similar proof to that of Dang-Yamamuro.

Proposition 5 (cf. [DY, (3.1)]). *With (M, H, J, H^+) and $\hat{H}, \hat{H}^+$ as before, if h is a bijective o.d. homomorphism of H to $\hat{H}$, then h is an o.d. isomorphism.*

Proof. Let $h = u|h|$ be the polar decomposition of h. Using the argument in the proof of Proposition 4, we see that h is an order isomorphism. By Lemma 1, $|h|$ can be written as $|h| = kJkJ$ for some positive invertible operator $k \in M$. Since $u = hk^{-1}Jk^{-1}J$, it follows that $uH^+ \subset \hat{H}^+$. Hence u is an o.d. homomorphism, and so u is an o.d. isomorphism. Hence $|h|$ is an o.d. homomorphism. By [Y, (3.4)], k belongs to $M \cap M'$. This means that $|h|^{-1}$ is an o.d. homomorphism. Therefore, h is an o.d. isomorphism. This completes the proof. □

Lemma 6. *With $(M, H, H_n^+, n \in \mathbb{N})$ a matrix ordered standard form and $(\hat{H}, \hat{H}_n^+, n \in \mathbb{N})$ a matrix ordered Hilbert space, let h be a completely positive o.d. homomorphism of H into $\hat{H}$. Then $\overline{h_nH_n^+}$ is a selfdual subcone of $\hat{H}_n^+$ and $(\overline{hH}, \overline{h_nH_n^+}, n \in \mathbb{N})$ is a matrix ordered Hilbert space, and h is a complete o.d. homomorphism.*

Proof. Let $h = u|h|$ be the polar decomposition of h. Using Proposition 4, we see that $|h|$ belongs to $M \cap M'$. Hence $|h_n|$ belongs to $M_n \cap M_n'$. This implies that h_n is an o.d. homomorphism, i.e., h is a complete o.d. homomorphism. To complete the proof, it suffices to show that $\overline{h_n H_n^+}$ is selfdual; hence $\overline{|h_n| H_n^+}$ is selfdual. Recall that for every $n \in \mathbb{N}$ the selfdual cone H_n^+ is generated by the elements $[x_i J_{H^+} x_j J_{H^+} \xi]_{i,j=1}^n, x_1, \cdots, x_n \in M, \xi \in H^+$. If we set $h_\varepsilon = |h| + \varepsilon 1, \varepsilon > 0$, then for such elements x_i, ξ, we get

$$\lim_{\varepsilon \to 0} |h_n| [h_\varepsilon^{-\frac{1}{2}} x_i J_{H^+} h_\varepsilon^{-\frac{1}{2}} x_j J_{H^+} \xi]_{i,j=1}^n = \lim_{\varepsilon \to 0} [|h| h_\varepsilon^{-1} x_i J_{H^+} x_j J_{H^+} \xi]_{i,j=1}^n$$

$$= [e x_i J_{H^+} x_j J_{H^+} \xi]_{i,j=1}^n = e_n [x_i J_{H^+} x_j J_{H^+} \xi]_{i,j=1}^n,$$

where e denotes the support projection of $|h|$ and it belongs to the center of M. This implies that $|h_n| H_n^+$ is dense in the selfdual cone $e_n H_n^+$. □

Proof of Theorem B. We apply Proposition 4, Proposition 5, Lemma 6 and [M2, Theorem 2.7]. □

Applying Lemma 3 and Theorem B, we obtain the following corollary:

Corollary 7. *For matrix ordered standard forms* $(M, H, H_n^+, n \in \mathbb{N})$ *and* $(\hat{M}, \hat{H}, \hat{H}_n^+, n \in \mathbb{N})$, *suppose that* u *is a completely positive part ial isometry with initial projection* e *and final projection* f. *Put* $\rho(x) = uxu^*$ *for all* $x \in eMe$. *Then* $(eM|_{eH}, eH, e_n H_n^+, n \in \mathbb{N})$ *and* $(f\hat{M}|_{f\hat{H}}, f\hat{H}, f_n \hat{H}_n^+, n \in \mathbb{N})$ *are matrix ordered standard forms, and* ρ *is a* $*$*-isomorphism of* eMe *onto* $f\hat{M}f$.

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References

[C] A. Connes, *Caractérisation des espaces vectoriels ordonnées sous-jacents aux algèbres de von Neumann*, Ann. Inst. Fourier **24** (1974), 121–155. MR **51:**13705

[DY] T. B. Dang and S. Yamamuro, *On homomorphisms of an orthogonally decomposable Hilbert space*, J. Funct. Anal. **68** (1986), 366–373. MR **87m:**46112

[I] B. Iochum, *Cônes Autopolaires et Algèbres de Jordan*, Lecture Notes in Mathematics, 1049, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1984. MR **86m:**46067

[M1] Y. Miura, *Completely positive projections on a Hilbert space*, Proc. Amer. Math. Soc. **124** (1996), 2475–2478. MR **96j:**46060

[M2] ———, *Complete order isomorphisms between non-commutative L^2-spaces in Math. Scand.* (to appear).

[SW] L. M. Schmitt and G. Wittstock, *Characterization of matrix-ordered standard forms of W^*-algebras*, Math. Scand. **51** (1982), 241-260. MR **84i:**46062

[Y] S. Yamamuro, *Absolute values in orthogonally decomposable spaces*, Bull. Austral. Math. Soc. **31** (1985), 215–233. MR **86i:**46010

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