CARDINAL ELEMENTARY EXTENSIONS

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Abstract. We determine the consistency strength of some model theoretic extension properties for cardinals.

Let $M, N$ be two nonempty sets. We say that $N$ extends $M$ iff for every structure $\mathfrak{M} = (M, \varepsilon, A_1, \ldots, A_n)$ there are $B_1, \ldots, B_n$ such that $\mathfrak{M} = (N, \varepsilon, B_1, \ldots, B_n)$ is an elementary extension of $\mathfrak{M}$. In this note we make some observations on this property for cardinals. More precisely we shall determine the consistency strength of the two properties “$\kappa$ extends $\kappa$” and “$\kappa^+$ extends $\kappa$”. First note that if some $\tau > \kappa$ extends $\kappa$, then $\kappa$ is weakly $\Pi^1_1$-indescribable.

Now let us look at the property “$\kappa^+ \kappa$ extends $\kappa$” which is easy to analyze. Its consistency strength over ZFC is simply given by “for some regular $\tau > \kappa$ $V_\tau$ extends $V_\kappa$”. Let us call a $\kappa$ with the latter property sublime (with witness $\tau$).

Clearly, a witness $\tau$ for the sublimity of $\kappa$ is inaccessible. So if we force with $\text{Col}(\kappa^+, < \tau)$ we get a model where $\kappa^+ \kappa$ extends $\kappa$. This gives one direction of our previous claim. For the other one just assume that some regular $\tau > \kappa$ extends $\kappa$. We shall show that $\kappa$ is sublume in $L \kappa$ with witness $\tau$. To get this we only need to know that $\tau$ extends $\kappa$ in $L$. For then $L_\tau$ extends $L_\kappa$ which yields that $\kappa$ is sublime in $L$. To see that $\tau$ extends $\kappa$ in $L$ we need a small but familiar argument. Since we shall need it once more we make it a lemma.

Lemma 1. Assume that $\tau > \kappa$ extends $\kappa$ and $\text{cf}(\tau) > \omega$. Let $M$ be a transitive set with $\kappa \in M$ and $|M| = \kappa$. Then there is an elementary embedding $j : M \rightarrow N$ with $\kappa$ transitive, $j \mathrel{|} \kappa = \text{id} \mathrel{|} \kappa$ and $j(\kappa) = \tau$.

Proof. Let $\pi$ be an isomorphism of $(M, \varepsilon)$ onto $(\kappa, E)$. Build a structure $\mathfrak{A} = (\kappa, \varepsilon, E, A)$ such that $A$ witnesses that $E|\alpha$ is well founded for every $\alpha < \kappa$. Now choose some elementary extension $\mathfrak{B} = (\tau, \varepsilon, E, B)$ of $\mathfrak{A}$. Then $E|\alpha$ is well founded for every $\alpha < \tau$, so $E$ is well founded since $\text{cf}(\tau) > \omega$. Also $E$ is extensional. So let $\sigma : (\tau, E) \rightarrow (N, \varepsilon)$ where $N$ is transitive. Finally set $j = \sigma \circ \pi$.

Now let us return to the situation before the lemma. We have some regular $\tau > \kappa$ which extends $\kappa$ and want to show that $\tau$ extends $\kappa$ in $L$. Let $\mathfrak{A} = (\kappa, \varepsilon, A_1, \ldots, A_n)$ be a structure in $L$. Choose some $\alpha < \kappa^+$ such that $\mathfrak{A} \in L_\alpha$. By the lemma let $j : L_\alpha \rightarrow N$ be an elementary embedding with $N$ transitive, $j \mathrel{|} \kappa = \text{id} \mathrel{|} \kappa$, $j(\kappa) = \tau$. Let $\mathfrak{B} = j(\mathfrak{A})$. Because $N = L_\gamma$ for some $\gamma$ we have that $\mathfrak{B} \in L$. However, $\mathfrak{B}$ is an elementary extension of $\mathfrak{A}$ with universe $\tau$.

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1. $ZFC + \exists \kappa \kappa^{++} \text{ extends } \kappa$
2. $ZFC + \exists \kappa \kappa \text{ is sublume}$

We are left with the vague task of defending our implicit claim that sublimity is a natural large cardinal property. We can only offer the following result which at least locates the strength of sublimity.

**Proposition 1.** (a) Assume $V = L$. Then every sublume cardinal is sublume. Hence there are many sublume cardinals below any sublume one.

(b) Let $\kappa$ be weakly ineffable. Then some $\alpha > \kappa$ is sublume.

**Proof.** (a) Let $\kappa$ be sublume with witness $\tau$. Assume that $\tau$ is not sublume, so there is a sequence $\bar{S} = \langle S_\alpha | \alpha \in C \rangle$, a club in $\kappa$, $S_\alpha \subseteq \alpha$, such that $S_\alpha \neq S_\beta \cap \alpha$ for all $\alpha, \beta \in C$, $\alpha < \beta$. Let $\bar{S}$ be the $<_{L^\gamma}$-least such that $\bar{S} \in L_\gamma$ where $\gamma < \kappa^+$. By the lemma let $j: L_\gamma \rightarrow L_\delta$ be an elementary embedding with $j | \kappa = \text{id} | \kappa$ and $j(\kappa) = \tau$. Let $j(\bar{S}) = \langle \bar{S}_\alpha | \alpha \in D \rangle$. Clearly, $j(\bar{S}) \upharpoonright \kappa = \bar{S}$ and $D$ is closed in $\tau$, so $\bar{S} \in D$ and $\bar{S}_\alpha \subseteq \kappa$. So let $\bar{S}_\kappa \in L_\delta$ and $\bar{S} \in L_\delta$ where $\delta < \kappa^+$ and choose an elementary embedding $k: L_\delta \rightarrow L_\delta$ with $k | \kappa = \text{id} | \kappa$ and $k(\kappa) = \tau$. Now we see that $j(\bar{S}) = k(\bar{S})$ since they are both equal to the $<_{L^\gamma}$-least sequence which shows that $\tau$ is not sublume. But then $k(\bar{S}_\kappa) \cap \kappa = \bar{S}_\kappa$ yields that $\bar{S}_\kappa \cap \alpha = S_\alpha$ for many $\alpha \in C$ which is a contradiction. So $\kappa$ must be sublume. By reflection many $\alpha < \kappa$ are sublume.

(b) We shall show that some $\alpha < \kappa$ is sublume with witness $\kappa$. Assume not. Let $I = \{ \alpha < \kappa | \alpha \text{ is inaccessible} \}$. For $\alpha \in I$ choose $S_\alpha \subseteq V_\alpha$ such that $(V_\alpha, \in, S_\alpha)$ has no elementary extension of the form $(V_\kappa, \in, S)$. Let $C_\alpha = \{ \beta < \alpha | (V_\beta, \in, S_\alpha \cap V_\beta) \prec (V_\alpha, \in, S_\alpha) \}$. So $C_\alpha$ is club in $\alpha$. Since $\kappa$ is weakly ineffable there are $S \subseteq V_\kappa$ and $C \subseteq \kappa$ such that $X = \{ \alpha \in I | S_\alpha = S \cap V_\alpha \text{ and } C_\alpha = C \cap \alpha \}$ is unbounded in $\kappa$. However, then it is easy to see that $(V_\alpha, \in, S \cap V_\alpha) \prec (V_\kappa, \in, S)$ for any $\alpha \in C$ and $X \subseteq C$. So we have $(V_\alpha, \in, S_\alpha) \prec (V_\kappa, \in, S)$ for any $\alpha \in X$ which is a contradiction.\[\square\]

We now turn to our analysis of the property "$\kappa^+ \text{ extends } \kappa$". This is much stronger, for it implies that $\theta^+ \text{ exists}. To see this observe that if $\kappa^+ \text{ extends } \kappa$, then $\kappa$ is weakly $\Pi^1_1$-indescribable and $(\kappa^+)^L < \kappa^+$ because $L_\kappa < L_{\kappa^+}$. This is known to imply the existence of $O^\#$\!. We shall later give a proof for this. On the other hand it is well known that any cardinal $\lambda > \kappa$ extends $\kappa$ if $\kappa$ is Ramsey. A special proof of this will be useful for us. For this we need the following concept.

**Definition.** Let $\mathfrak{A} = (X, \varepsilon, A_1, \ldots, A_n)$ be a structure where $X$ is a transitive set and $\kappa = \text{On} \cap X$. An $\mathfrak{A}$-mouse is a structure $M = (J^{\mathfrak{A}, \varepsilon, B, U})$ such that $\beta > \kappa$, $\mathfrak{A} \in M$, $U$ is a normal measure on $\kappa$ in $M$ and $M$ is iterable with respect to $U$.

Now let $\kappa$ be Ramsey and let $\mathfrak{A} = (\kappa, \varepsilon, A_1, \ldots, A_n)$ be a structure. Then an $\mathfrak{A}$-mouse $M$ exists. This is part of the folklore. We may assume that $|M| = \kappa$, so given any cardinal $\lambda > \kappa$ we can iterate $M \lambda$ times to get an embedding $\pi: M \rightarrow N$ which is $\Sigma_1$-preserving. However, then $\pi(\mathfrak{A})$ is an elementary extension of $\mathfrak{A}$ with universe $(\lambda, \varepsilon)$. This shows that $\lambda$ extends $\kappa$. Moreover, this argument motivates the following concept.
Definition. \( \kappa \) is a Keisler cardinal iff every structure \( \mathfrak{A} = (\kappa, \varepsilon, A_1, \ldots, A_n) \) has an elementary extension \( \mathfrak{B} = (\tau, \varepsilon, B_1, \ldots, B_n) \) for which a \( \mathfrak{B} \)-mouse exists.

An easy condensation argument shows that we can always assume that \( \tau \) has cardinality \( \kappa \) and a \( \mathfrak{B} \)-mouse of cardinality \( \kappa \) exists.

**Theorem 2.** The following theories are equiconsistent:

1. ZFC + \( \exists \kappa \kappa^+ \) extends \( \kappa \).
2. ZFC + \( \exists \kappa \kappa^+ \) is Keisler.

**Proof.** (2)\( \rightarrow \) (1) Here we actually have that if \( \kappa \) is Keisler, then any cardinal \( \lambda > \kappa \) extends \( \kappa \). Given \( \mathfrak{A} \) choose a \( \mathfrak{B} \)-mouse \( M \) of cardinality \( \kappa \) with \( \mathfrak{A} \prec \mathfrak{B} \) and iterate \( M \) \( \lambda \)-times to find an elementary extension of \( \mathfrak{B} \) (hence \( \mathfrak{A} \)) with universe \( (\lambda, \varepsilon) \).

(1)\( \rightarrow \) (2) Let \( \kappa^+ \) extend \( \kappa \). Since a measurable cardinal is Keisler we may assume that there is no inner model with a measurable cardinal. We shall show that \( \kappa \) is Keisler in the Dodd-Jensen core model \( K \) (see [1]). Using our assumption \( -L^\mu \) we get that \( (\kappa^+)^K = \kappa^+ \). To see this set \( \tau = (\kappa^+)^K \). If we had that \( \tau \prec \kappa^+ \), then the lemma would give us an elementary embedding of \( K \) into a transitive set \( N \). Since \( \kappa \geq \omega_2 \) results in [1] would give us \( L^\mu \). So we have \( \tau = \kappa^+ \). Now let \( \mathfrak{A} = (\kappa, \varepsilon, A_1, \ldots, A_n) \) be a structure in \( K \). Since \( \tau \) extends \( \kappa \) there are a transitive set in \( K \) and \( B_1, \ldots, B_n \) such that \( On \cap K = \tau \) and \( (K, \varepsilon, A_1, \ldots, A_n) \prec (\kappa, \varepsilon, B_1, \ldots, B_n) \). Set \( \mathfrak{B} = (\tau, \varepsilon, B_1, \ldots, B_n) \). Hence \( \mathfrak{A} \prec \mathfrak{B} \). Since \( \kappa \) cannot be the largest cardinal in \( K \) we must have that \( K \neq K_\tau \). In fact there is a mouse \( M \in K_\tau \) such that \( M \not\in K \). Let \( N \) be the \( \tau \)-th iterate of \( M \). Standard arguments give that \( N \) is a \( \mathfrak{B} \)-mouse.

It can be shown that every Keisler cardinal is inaccessible. We now turn to the question of how large a Keisler cardinal is. We shall show that there is one below any \( \omega_1 \)-Erdős cardinal. However, since the concept of a Keisler cardinal looks rather exotic we want to locate its position in the hierarchy of large cardinals quite precisely. For this we have to recall some definitions from [2].

**Definition.**

(a) Let \( f : [A]<\omega \rightarrow On, A \subseteq On \). Assume that \( X \) is an infinite homogeneous set for \( f \). Then set \( \text{typ}_f(X) = \{ \delta_n | n < \omega \} \), where \( f^n[X]^n = \{ \delta_n \} \).

(b) Let \( f : [A]<\omega \rightarrow On, A \subseteq On \). A sequence \( (X_\alpha | \alpha < \tau) \) is called homogeneous for \( f \) iff every \( X_\alpha \) is homogeneous for \( f \) and \( \text{typ}_f(X_\alpha) = \text{typ}_f(X_\beta) \) for all \( \alpha, \beta < \tau \).

(c) \( \kappa \) is \( \tau \)-Erdős iff for every regressive \( f : [C]<\omega \rightarrow \kappa, C \) club in \( \kappa \), there is a homogeneous sequence \( (X_\alpha | \alpha < \tau) \) for \( f \) such that \( \text{otp}(X_\alpha) \geq \max\{\omega, \alpha\} \).

(d) \( \kappa \) is nearly \( \tau \)-Erdős iff for every pair of functions \( f, g \) with \( f : [C]<\omega \rightarrow \lambda \) and \( g : [C]<\omega \rightarrow \kappa \) regressive, \( C \subseteq \kappa \) club, \( \lambda < \kappa \), there is a sequence \( (X_\alpha | \alpha < \tau) \), \( \text{otp}(X_\alpha) \geq \max\{\omega, \alpha\} \), such that \( (X_\alpha | \alpha < \tau) \) is homogeneous for \( f \) and every set \( X_\alpha \) for \( \alpha < \tau \) is homogeneous for \( g \).

Clearly, \( \omega_1 \)-Erdős cardinals and nearly \( \omega_1 \)-Erdős cardinals are very close to each other. So the following result is quite precise. We assume that the reader knows the standard techniques of translating the combinatorial definitions above into their model theoretic counterparts.

**Proposition 2.**

(a) Every Keisler cardinal is nearly \( \omega_1 \)-Erdős. Hence there are nearly \( \omega_1 \)-Erdős cardinals below any Keisler cardinal.

(b) Let \( \kappa \) be \( \omega_1 \)-Erdős. Then some \( \tau < \kappa \) is a Keisler cardinal.
Proof. (a) Let $\kappa$ be a Keisler cardinal. Let $f, g$ be a pair of functions with $f : [C]^{<\omega} \to \lambda$ and $g : [C]^{<\omega} \to \kappa$ regressive where $C \subseteq \kappa$ is club and $\lambda < \kappa$. Consider the structure $\mathfrak{A} = (V_\kappa, \varepsilon, f, g)$. Since $\kappa$ is Keisler there is some elementary extension $\mathfrak{B}$ of $\mathfrak{A}$ for which a $\mathfrak{B}$-mouse $M$ exists. Now let $\pi : M \to N$ be the $\omega_1$-th iteration map and let $\pi(\mathfrak{B}) = (Z, \varepsilon, f, g)$. Let $I$ consist of the first $\omega_1$ iteration points of $M$. Then $I$ is homogeneous for $\pi$ and $\bar{g}$. Now set $a = \text{typ}_f(I)$. Then $a \in V_\kappa$ since $\text{rng}(\bar{f}) \subseteq \kappa$. Now let $\omega \leq \xi < \omega_1$ and fix a bijection $h_\xi : \omega \to \xi$. Now there is a canonical tree $T = T(h_\xi, a, f, g)$ of height $\omega$ such that any infinite branch of $T$ gives a set $X$ such that $\text{otp}(X) = \xi$, $X$ is homogeneous for $\pi$ and $\bar{g}$ and $\text{typ}_f(X) = a$.

Since $\pi(\mathfrak{B})$ is a model of $\text{ZFC}$ (and $\text{cf}(\text{On} \cap Z) > \omega$) there is some $X \in Z$ such that $\text{otp}(X) = \xi$, $X$ is homogeneous for $\pi$ and $\bar{g}$ and $\text{typ}_f(X) = a$. Now use that $\mathfrak{A}$ is an elementary submodel of $\pi(\mathfrak{B})$. This shows that $\kappa$ is nearly $< \omega_1$-Erdős. Since $\kappa$ is also weakly compact this property reflects.

(b) Let $<^*$ be a well ordering of $V_\kappa$. Assume that no $\tau < \kappa$ is a Keisler cardinal. For notational reasons let us assume that for every $\tau < \kappa$ there is an $A_\tau \subseteq \tau$ such that $(\tau, \varepsilon, A_\tau)$ shows that $\tau$ is not Keisler. Let $A_\kappa$ be the $<^*$-least such. We shall derive a contradiction. Consider the structure $\mathfrak{M} = (V_\kappa, \varepsilon, <^*)$. Since $\kappa$ is $< \omega_1$-Erdős there is a sequence $\{I_\alpha | \alpha < \omega_1\}$ such that $\text{otp}(I_\alpha) = \omega(1 + \alpha)$, $I_\alpha$ is a remarkable set of indiscernibles for $\mathfrak{M}$, and setting $\delta_\alpha = \sup(\text{Hull}_{\mathfrak{M}}(I_\alpha) \cap \text{min} I_\alpha)$ we have that $\delta_\alpha = \delta_\beta$ for all $\alpha, \beta < \omega_1$ and $I_\alpha, I_\beta$ have the same indiscernibility type in $\mathfrak{M}$ with parameters less than $\delta$, the common value of all the $\delta_\alpha$. For $\alpha < \omega_1$ let $\pi_\alpha : \text{Hull}_{\mathfrak{M}}(I_\alpha) \to \mathfrak{M}_\alpha$ be the transitive collapse and let $\bar{I}_\alpha = \pi^\alpha_\alpha I_\alpha$. Then we have $\mathfrak{M}_\alpha \prec \mathfrak{M}_\beta$ for $\alpha < \beta < \omega_1$. So let $\mathfrak{B}$ be the union of this chain and set $\bar{I} = \bigcup_{\alpha < \omega_1} \bar{I}_\alpha$. Then $\bar{I}$ is a club set of indiscernibles for $\mathfrak{M}_\delta = \text{min} \bar{I}$ and $\bar{I}$ generates $\mathfrak{M}$ with parameters less than $\delta$. Let $\eta = (\delta^+)^\mathfrak{M}$ and set $H = H^\eta_\mathfrak{M}$. By well-known results the uncountable club generating set of indiscernibles $I$ gives us some $U$ such that $M = (H, \varepsilon, U)$ thinks that $U$ is a normal measure on $\delta$ and such that $M$ is amenable and iterable with respect to $U$. Now let $\gamma$ be the first element of $I_0$ and set $B = \pi_0(A_\gamma) \subseteq \delta$. Set $\mathfrak{B} = (\delta, \varepsilon, B)$. Clearly, we have $(\gamma, \varepsilon, A_\gamma) \prec (\rho, \varepsilon, A_\rho)$ for any $\rho \in I_0$. Hence there is some $\tau < \gamma$ such that $(\tau, \varepsilon, A_\tau) \prec (\gamma, \varepsilon, A_\gamma)$. So using $\pi_0$ we find some $\tau < \delta$ such that $(\tau, \varepsilon, A_\tau) \prec \mathfrak{B}$. However, this contradicts the definition of $A_\tau$ since $M$ is essentially a $\mathfrak{B}$-mouse. Formally, we can replace $M$ by $(J_0^{B, \mathfrak{B}}, \varepsilon, B, U \cap J_0^{B, \mathfrak{B}})$. \qed

Let us conclude with some remarks on other extension properties. Obviously, the proof of Theorem 1 essentially treats all the cases "$\tau$ extends $\kappa$" where $\tau > \kappa$ is regular, $\tau \neq \kappa^+$ and $\tau$ is not the successor of a singular cardinal. On the other hand successors of singular cardinals behave like $\kappa^+$. Just use the covering lemma. For extension properties of the form "some or some specific singular cardinal extends $\kappa$" one has to introduce versions of sublimity with singular witnesses. These are actually slightly weaker.

References


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