

A NEW CONSTRUCTION OF SEMI-FREE ACTIONS ON MENGER MANIFOLDS

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(Communicated by Alan Dow)

ABSTRACT. A new construction of semi-free actions on Menger manifolds is presented. As an application we prove a theorem about simultaneous coexistence of countably many semi-free actions of compact metric zero-dimensional groups with the prescribed fixed-point sets: Let G be a compact metric zero-dimensional group, represented as the direct product of subgroups G_i , M a μ^n -manifold and $\nu(M)$ (resp., $\Sigma(M)$) its pseudo-interior (resp., pseudo-boundary). Then, given closed subsets $X_i, i \geq 1$, of M , there exists a G -action on M such that (1) $\nu(M)$ and $\Sigma(M)$ are invariant subsets of M ; and (2) each X_i is the fixed point set of any element $g \in G_i \setminus \{e\}$.

0. INTRODUCTION

The following remarkable theorem was proved by Iwamoto [11]:

Theorem (0.1). *Let G be a compact metric zero-dimensional group, M a μ^n -manifold and $\nu(M)$ (resp. $\Sigma(M)$) its pseudo-interior (resp., pseudo-boundary). Then for every closed subset X of M , there exists a semi-free G -action on M such that X is the fixed-point set of every element $g \in G \setminus \{e\}$, and $\nu(M)$ and $\Sigma(M)$ are invariant subsets of M .*

This theorem is a significant generalization of the Dranishnikov free action theorem [8] and Sakai's results [13], [14]. In the present paper we give a new construction of semi-free actions, and prove a simultaneous coexistence of countably many semi-free actions of compact metric zero-dimensional groups with the prescribed fixed-point sets.

Theorem (0.2). *Let G be a compact metric zero-dimensional group, represented as the direct product $\prod G_i$ of subgroups G_i , M a μ^n -manifold and $\nu(M)$ (resp., $\Sigma(M)$) its pseudo-interior (resp., pseudo-boundary). Then, given closed subsets $X_i, i \geq 1$, of M , there exists a G -action on M such that:*

- (1) $\nu(M)$ and $\Sigma(M)$ are invariant subsets of M ; and
- (2) each X_i is the fixed-point set of every element $g \in G_i \setminus \{e\} \subset \prod G_i$.

Recall that an *action* of G on a space X is a homomorphism $T: G \rightarrow \text{Aut } X$ of the group G into the group $\text{Aut } X$ of all autohomeomorphisms of X , such that the

Received by the editors May 22, 1998 and, in revised form, August 12, 1999.

1991 *Mathematics Subject Classification.* Primary 57S10, 54C55.

Key words and phrases. Semi-free action, Menger manifold, absolute extensor in finite dimension.

map $G \times X \rightarrow X$, given by $(g, x) \mapsto T(g)(x) = g \cdot x$, is continuous. A space X with a fixed action of G is called a G -space.

A subset $A \subset X$ (resp., a point $a \in X$) is said to be *invariant* (resp., *fixed*) if $G \cdot A = \{g \cdot a \mid g \in G, a \in A\} = A$ (resp., $G \cdot a = a$). A G -space X is said to be *free* (resp., *semi-free*) if $g \cdot x \neq x$, for every $x \in X$ and $g \neq e$ (resp., $g \cdot x \neq x$, for every nonfixed point x and $g \neq e$).

The proof of Theorem (0.2) is presented in detail only for the case of the Menger compactum μ^n . By Pontryagin's theorem [12], each G_i can be considered as a closed subgroup of the product $\prod_{k=1}^\infty H_{ik}$ of nontrivial finite groups with the following property: Every nontrivial element $g_i = (g_{ik}) \in G_i$ has infinitely many nontrivial coordinates.

Next, fix the Cantor compactum $\mathcal{C} = \mu^0$ with the pseudo-interior $\nu = \nu(\mu^0)$ and the pseudo-boundary $\Sigma = \Sigma(\mu^0)$. The key to the construction of the desired semi-free action lies in the canonical surjection

$$r: I^n \times H \times \mathcal{C} \rightarrow \mu^n, \text{ where } H = \prod_i H_i \text{ and } H_i = \prod_k H_{ik},$$

and a discontinuous action Ψ of H on $I^n \times H \times \mathcal{C}$ such that:

- (3) the restriction of Ψ on $G_k \subset H_k$ is a discontinuous semi-free action with respect to $r^{-1}(X_k)$;
- (4) $\Psi(h, (x, u, v)) = (x, u', v)$, where $h \in H, x \in I^n, u \in H, v \in \mathcal{C}$ (i.e. Ψ changes only the middle coordinate);
- (5) $r(I^n \times H \times \nu)$ is the pseudo-interior $\nu(\mu^n)$ and $r(I^n \times H \times \Sigma)$ is the pseudo-boundary $\Sigma(\mu^n)$;
- (6) $r^{-1}(m)$ is a rectangle subset of $I^n \times H \times \mathcal{C}$, for every $m \in \mu^n$; and
- (7) $\Phi(h, r(x, u, v)) = r(x, u', v)$, where $(x, u', v) = \Psi(h, (x, u, v))$, is a continuous action of H on μ^n , and the restriction of Φ on G_k is a continuous semi-free action with respect to X_k .

The proof of Theorem (0.2) in the general case (for an arbitrary μ^n -manifold) is analogous to the special case of μ^n and will be discussed in the last chapter.

1. PRELIMINARIES

All spaces are assumed to be separable metric and all maps to be continuous. Recall that a space X is said to be $(k - 1)$ -connected (C^{k-1}) if the homotopy group $\pi_i(X)$ is trivial, for every $i < k$. A space X is said to be *locally* $(k - 1)$ -connected (LC^{k-1}) if for every $x \in X$ and every neighborhood $U \ni x$, there exists a neighborhood $V \ni x$ with the property that every map $\alpha: S^i = \partial B^{i+1} \rightarrow V, i < k$, extends to $\tilde{\alpha}: B^{i+1} \rightarrow U$.

By the Kuratowski-Dugundji theorem [5], $X \in LC^{k-1} \cap C^{k-1}$ (resp., $X \in LC^{k-1}$) if and only if $X \in AE(k)$, i.e. X is an absolute extensor in dimension k (resp., $X \in ANE(k)$, i.e. X is an absolute neighborhood extensor in dimension k). A family $\{X_\alpha\}$ of sets $X_\alpha \subset X$ is said to be *equi*- LC^{k-1} if for every $x \in \bigcup X_\alpha$ and every neighborhood $U \ni x$, there exists a neighborhood $V \ni x$ with the property that every partial map $Z \hookrightarrow A \xrightarrow{f} V \cap X_\alpha, \dim Z \leq k$, extends to $g: Z \rightarrow U \cap X_\alpha, g|_A = f$. The following criteria are convenient for verification of connectivity properties of spaces (cf. [1] and [4], respectively):

Proposition (1.1). *Let $\{Z_\alpha\}$ be a closed cover of the compactum Z . Then the following assertions hold:*

(a) $Z \in \text{AE}(n)$ if and only if for every neighborhood $U(z)$ of $z \in Z$, there exists a neighborhood $V(z)$ (if $U(z) = Z$ then we require $V(z) = Z$), such that for every map $\varphi: S^k \rightarrow V(z), k < n$, and for every $\nu > 0$, there exists a map $\psi: B^{k+1} \rightarrow Z$, with $(\varphi, \psi|_{S^k}) < \nu$.

(b) $\{Z_\alpha\} \in \text{equi-LC}^{n-1}$, if and only if for every neighborhood $U(z)$ of $z \in Z$, there exists a neighborhood $V(z)$, such that for every map $\varphi: S^k \rightarrow V(z) \cap Z_\alpha, k < n$, and for every $\nu > 0$, there exists a map $\psi: B^{k+1} \rightarrow U(z) \cap Z_\alpha$ with $(\varphi, \psi|_{S^k}) < \nu$.

Proposition (1.2). *Let $\mathcal{P} = \{p_\alpha\}$ be a closed cover of an $\text{ANE}(r)$ -compactum Z and suppose that $p_{\alpha_1} \cap p_{\alpha_2} \cap \dots \cap p_{\alpha_t} \neq \emptyset$ implies $p_{\alpha_1} \cap p_{\alpha_2} \cap \dots \cap p_{\alpha_t} \in \text{AE}(r+1-t)$, for every $t \leq r$. Then $Z \in \text{AE}(r)$ if and only if the nerve $\mathcal{N}(\mathcal{P})$ of \mathcal{P} is $(r-1)$ -connected.*

We shall need the following three results of Bestvina [3].

Theorem (1.3). *A locally compact space X is μ^k -manifold if and only if $X \in \text{ANE}(k)$, $\dim X = k$ and X has the disjoint k -disks property, namely $X \in \text{DD}^k \text{P}$. If, in addition, X is a compact $\text{AE}(k)$, then X is homeomorphic to the Menger universal compactum μ^k .*

Theorem (1.4). *For every μ^k -manifold M^k , there exists a PL manifold R of dimension $\geq 2k + 1$ with a triangulation L and a proper map $f: R^{(k)} \rightarrow M$ which induces isomorphisms of homotopy groups of $\dim < k$ and homotopy groups of ends of $\dim < k$ (here $R^{(k)}$ is the k -skeleton with respect to L).*

Theorem (1.5). *Let $f: M_1 \rightarrow M_2$ be a proper map between μ^k -manifolds which induces isomorphisms of homotopy groups of $\dim < k$ and homotopy groups of ends of $\dim < k$. Then f is properly $(k-1)$ -homotopic to a homeomorphism.*

Let M be a μ^k -manifold. By \mathcal{Z}_M , we shall denote the collection of all Z -sets in M . A \mathcal{Z}_M -absorber A of M is called the *pseudo-boundary* $\Sigma(M)$ of M and the complement $M \setminus A$ is called the *pseudo-interior* $\nu(M)$ of M . The topological types of the pseudo-boundaries and the pseudo-interiors of M are unique [2]. The following criterion was proved in [7].

Proposition (1.6). *Let $\{A_i\}_{i=1}^\infty$ be an increasing sequence of Z -sets in a μ^k -manifold M with the following properties:*

- (1) for every $\epsilon > 0$, there exists $m > 0$ such that A_m is ϵ -dense in M ;
- (2) each A_i is a Z -set in A_{i+1} ;
- (3) $\{A_i\}_{i=1}^\infty$ is equi-LC^{k-1} and
- (4) A_i is a μ^k -manifold.

Then $\bigcup_{i=1}^\infty A_i$ is a pseudo-boundary of M .

The pseudo-boundary and the pseudo-interior of the zero-dimensional Menger manifold $\mathcal{C} = \mu^0$ can be described in the following manner:

Definition (1.7). (5) A subset $R \subset \prod_{i=1}^s T_i, 1 \leq s \leq \infty$, is called a *rectangle*, if

$$R = \prod_{i=1}^s T'_i, \text{ where } T'_i \subset T_i \text{ for every } i.$$

- (6) A rectangle subset $R_1 = \prod_{i=1}^s T'_i$ is said to have an infinite codimension in a rectangle subset $R_2 = \prod_{i=1}^s T''_i$, whenever $R_1 \subset R_2$ and the set $\{i \in \mathbb{N} \mid T''_i \setminus T'_i \neq \emptyset\}$ is infinite.

Proposition (1.8). Let $\mathcal{C} = \prod_{i=1}^{\infty} T_i$ be the Cantor compactum. Then there exists a sequence of rectangle subsets $R_1 \subset R_2 \subset \dots$ such that

- (7) $R_i \cong \mathcal{C}$ for every i ;
- (8) every R_i has infinite codimension in R_{i+1} ; and
- (9) $R = \bigcup_{i=1}^{\infty} R_i$ is a dense subset in \mathcal{C} .

According to Proposition (1.6), R is the pseudo-boundary of \mathcal{C} .

2. THE CANONICAL DECOMPOSITION OF $I^n \times \prod T_i$

Let us consider a partition $\Delta_m, m \geq 1$, of the unit segment I , determined by the subset $\delta_m = \{\frac{2i-1}{2^m} \mid 0 < i \leq 2^{m-1}\}$. The product $\underbrace{\Delta_m \times \dots \times \Delta_m}_n = (\Delta_m)^n$ of the partitions Δ_m is a partition of the cube $I^n = \underbrace{I \times \dots \times I}_n$ into $(2^{m-1} + 1)^n$ cubes of diameter $\leq \sqrt{n} \cdot 2^{1-m}$. The union F_m of all boundaries of these cubes satisfies the following properties:

- (1) $F_m = \{x = (x_1, \dots, x_n) \in I^n \mid x_i \in \delta_m, \text{ for some } i\}$; and
- (2) for every $x \in I^n$, the set $\{m \mid x \in F_m\}$ has less than $n + 1$ elements.

Fix a sequence $\{T_i, i \geq 1\}$ of finite sets T_i with $|T_i| > 1$, and construct (in a canonical way) an upper semi-continuous decomposition \mathcal{T} of $Q = I^n \times \prod_{i=1}^{\infty} T_i$. The quotient space $P = Q/\mathcal{T}$, generated by \mathcal{T} , will have properties (3)-(5) below:

- (3) P is compact and $\dim P = n$;
- (4) P satisfies the disjoint n -disks property ($DD^n P$); and
- (5) $P \in C^{n-1} \cap LC^{n-1}$.

By Theorem (1.3), P and the Menger compactum μ^n are homeomorphic. Let $x \in I^n$ and $t = (t_i)_{i=1}^m \in \prod_{i=1}^m T_i, 1 \leq m \leq \infty$. (Convention: $m + 1 = \infty$ in case $m = \infty$.) Let us denote $\mathcal{T}_i(x, t) = t_i$, if $x \notin F_i$; $\mathcal{T}_i(x, t) = T_i$, if $x \in F_i$, and $\mathcal{T}(x, t) = \prod_{i=1}^m \mathcal{T}_i(x, t)$.

Proposition (2.1). The family $\mathcal{T}_m = \{x \times \mathcal{T}(x, t) \mid x \in I^n, t \in \prod_{i=1}^m T_i\}, 1 \leq m \leq \infty$, yields an upper semi-continuous decomposition of $Q_m = I^n \times \prod_{i=1}^m T_i$:

- (6) for every (x_0, t_0) and $\epsilon > 0$, there exists $\delta > 0$ such that $\text{dist}((x, t), (x_0, t_0)) < \delta$ implies $\alpha_H(x \times \mathcal{T}(x, t), x_0 \times \mathcal{T}(x_0, t_0)) < \epsilon$ (here $\alpha_H(A, B) = \inf\{\gamma > 0 \mid A \subset N(B; \gamma)\}$ is the Hausdorff deflection; cf. [10, p. 98, (7.7.1)]).

Proof. By definition, it is evident that

- (7) $x \times \mathcal{T}(x, t) \cap x' \times \mathcal{T}(x', t') \neq \emptyset \iff x = x'$ and $t_i = t'_i$ for all $i < m + 1$ such that $x \notin F_i \iff x \times \mathcal{T}(x, t) \equiv x' \times \mathcal{T}(x', t')$ (we use a convention that $m + 1 = \infty$ in case $m = \infty$).

Fix $x_0 \in I^n, t_0 \in \prod_{i=1}^m T_i$ and $\epsilon > 0$. Let us choose $\delta > 0$ and a finite number $p \leq m$ such that

- (8) $\text{dist}(x, x') < \delta$ and the coincidence of the first p coordinates of t and t' implies $\text{dist}((x, t), (x', t')) < \epsilon/2$;
- (9) $\text{dist}((x, t), (x', t')) < \delta$ implies the coincidence of the first p coordinates of t and t' and, naturally, $\text{dist}(x, x') < \delta$; and
- (10) $\delta < \min\{\text{dist}(x_0, F_i) | x_0 \notin F_i, i \leq p\}$.

Let $\text{dist}((x, t), (x_0, t_0)) < \delta$. It easily follows from (9) and (10) that $\text{dist}(x, x_0) < \delta$, the first p coordinates of t and t_0 coincide and $\mathcal{T}_i(x, t) \subset \mathcal{T}_i(x_0, t_0)$, for every $i \leq p$. Then $\alpha_H(x \times \mathcal{T}(x, t), x_0 \times \mathcal{T}(x_0, t_0)) < \epsilon/2$ by (8). The coincidence of the first p coordinates of t and t_0 implies $\alpha_H(x_0 \times \mathcal{T}(x_0, t), x_0 \times \mathcal{T}(x_0, t_0)) < \epsilon/2$. Hence it follows that $\alpha_H(x \times \mathcal{T}(x, t), x_0 \times \mathcal{T}(x_0, t_0)) \leq \alpha_H(x \times \mathcal{T}(x, t), x_0 \times \mathcal{T}(x_0, t)) + \alpha_H(x_0 \times \mathcal{T}(x_0, t), x_0 \times \mathcal{T}(x_0, t_0)) < \epsilon$. □

Corollary (2.2). *For every $1 \leq m \leq \infty$, the following assertions hold:*

- (a) *the quotient space $P_m = Q_m/\mathcal{T}_m$ is a compactum; and*
- (b) *if $m < \infty$, then the map $p_m: P_m \rightarrow P_{m-1}$, defined by setting*

$$p_m(r_m(x, (t_1, \dots, t_m))) = r_m(x, (t_1, \dots, t_{m-1})),$$

is well-defined and continuous and the following diagram is commutative:

$$\begin{array}{ccc} Q_m & \xrightarrow{r_m} & P_m \\ q_m \downarrow & & \downarrow p_m \\ Q_{m-1} & \xrightarrow{r_{m-1}} & P_{m-1} \end{array}$$

where q_m is the projection on $Q_{m-1} = I^n \times \prod_{i=1}^{m-1} T_i$.

For $m = \infty$ we have the map $r: Q = I^n \times \prod_{i=1}^{\infty} T_i \rightarrow P = P_{\infty}$. Clearly, P is the limit of the inverse system $\{P_1 \xleftarrow{p_2} P_2 \xleftarrow{p_3} P_3 \xleftarrow{p_4} \dots\}$. Since r_m has finitely many fibers for $m < \infty$, it follows that $\dim P_m = \dim Q_m = n$. Since the dimension of the limit of the inverse system of n -dimensional spaces is less than or equal to n , we have $\dim P \leq n$.

Proposition (2.3). *$\dim P = n$ and $P \in DD^n P$.*

Proof. Every $a \in T_{m+1}$ naturally generates the embedding $s_{m,a}: P_m \rightarrow P_{m+1}$ of P_m into P_{m+1} defined by $s_{m,a}(r_m(x, t)) = r_{m+1}(x, t')$, where $t' = (t, a)$. It is evident that $s_{m,a}$ is a section of p_{m+1} . Hence P contains a copy of P_m with $\dim P_m = n$. Therefore, $\dim P = n$.

One can easily conclude from (2) that the images of the following compositions are disjoint:

$$\text{Im}(s_{m+n, a_{m+n+1}} \circ \dots \circ s_{m, a_{m+1}}) \cap \text{Im}(s_{m+n, a'_{m+n+1}} \circ \dots \circ s_{m, a'_{m+1}}) = \emptyset,$$

for every $a_{m+n+1} \neq a'_{m+n+1}, \dots, a_{m+1} \neq a'_{m+1}$. Consequently, P satisfies $DD^n P$. □

Remark (2.4). If R' is a rectangle subset of infinite codimension in a rectangle subset R'' of $\prod T_i$, then $P' = r(I^n \times R')$ is a Z -set in $P'' = r(I^n \times R'')$ and $P'' \in \text{DD}^n P$.

In the next section we shall prove that $P'' \in \text{AE}(n)$; so it will follow by the Bestvina theorem (1.3) that P'' is homeomorphic to the Menger compactum μ^n .

3. CONNECTIVITY PROPERTIES OF SUBSETS OF P

Let the factor $T_p, p < m + 1$, be represented as the disjoint union $T'_p \amalg T''_p$ (we assume $m + 1 = m$ for $m = \infty$). By replacing T_p with T'_p , we obtain the decomposition T'_m of $Q'_m = I^n \times T'_p \times \prod \{T_i | i < m + 1, i \neq p\}, 1 \leq m \leq \infty$, and the quotient map $r'_m: Q'_m \rightarrow P'_m = Q'_m/T'_m$. The decomposition T''_m of the space Q''_m and the map $r''_m: Q''_m \rightarrow P''_m$ are defined analogously. Note that $Q'_m \amalg Q''_m = Q_m$. It is clear that $r_m(Q'_m) = \hat{P}'_m \subset P_m$ and $r_m(Q''_m) = \hat{P}''_m \subset P_m$ are naturally homeomorphic to P'_m and P''_m , respectively.

Proposition (3.1). $\hat{P}'_m \cap \hat{P}''_m = r_m(F_p \times \prod_{i=1}^m T_i)$.

Definition (3.2). An *index* of $x \in I$ (briefly $\text{ind } x$) is defined to be $i \in \mathbb{N}$, if $x \in \delta_i$, and ∞ , if $x \notin \delta_j$ for every j . An index of $b = (b_1, \dots, b_n) \in I^n$ is defined to be $\text{ind}(b) = \{\text{ind}(b_i) | i \leq n\} \subset \mathbb{N} \cup \{\infty\}$.

Let J be a closed segment from the partition $\Delta_m, m < \infty$. If $J \subset \text{Int } I$, then there exists a unique point $b' \in J$ with $\text{ind}(b') < m$ (b' is the midpoint of J). If $J \cap \{0, 1\} = b'$, then J does not contain points with index less than m . We define $\text{Ind}(J) = \text{ind}(b')$ in the first case and $\text{Ind}(J) = \infty$ in the second one. In both cases point b' is said to be the *center* of J . Let $J = [\frac{2k+1}{2^m}, \frac{2k+3}{2^m}] \in \Delta_m$. Then $\text{Ind}(J) < m$ and $b' = \frac{2k+2}{2^m}$ is the center of J .

The partition $(\Delta_m)^n$ of I^n yields new partitions Q'_m of $Q_m = I^n \times \prod_{i=1}^m T_i$ and Q_m of $Q = I^n \times \prod_{i=1}^\infty T_i$ defined as follows:

$$Q'_m = \{\mathcal{E}' = \mathcal{J} \times \mathcal{T}(b, t) | \mathcal{J} = \prod_{i=1}^n J_i \in (\Delta_m)^n, b = (b_i) \in I^n, \\ t \in \prod_{i=1}^m T_i, \text{ each } b_i \text{ is the center of } J_i\}$$

and $Q_m = \{\mathcal{E}' \times \prod_{i=m+1}^\infty T_i | \mathcal{E}' \in Q'_m\}$. The partitions Q_m of Q and Q'_m of Q_m consist of regular closed subsets.

Theorem (3.3). Let $\mathcal{E}' = \mathcal{J} \times \mathcal{T}(b, t)$ be an arbitrary element of the partition $Q'_m, m < \infty$, and $\mathcal{J}_{p_1 \dots p_w} = \mathcal{J} \cap F_{p_1 \dots p_w} \neq \emptyset$, where $1 \leq p_1 < \dots < p_w \leq s, s \geq m, 0 \leq w \leq n$, and

$$F_{p_1 \dots p_w} = \begin{cases} \bigcap_{k=1}^w F_{p_k} & \text{if } w \geq 1; \\ I^n & \text{if } w = 0. \end{cases}$$

Then for every rectangle subset $R = \prod_{i=1}^s T'_i$ with the property (*):

$$“|T'_i| > 1 \text{ implies } i \geq m \text{ or } i \in \text{ind}(b)”,$$

the compactum $X = r_s(\mathcal{J}_{p_1 \dots p_w} \times R)$ is a nonempty $\text{AE}(n - w)$.

Corollary (3.4). Let $\mathcal{E}' = \mathcal{J} \times \mathcal{T}(b, t) \in \mathcal{Q}'_m, m < \infty$, and $m \leq p_1 < \dots < p_w \leq s, 0 \leq w \leq n$. Then for rectangle subsets $R_1 \subset \prod_{i=1}^m T_i$ and $R_2 \subset \prod_{i=m+1}^s T_i$, the compactum $r_s(\mathcal{J}_{p_1 \dots p_w} \times (\mathcal{T}(b, t) \cap R_1) \times R_2)$ is a nonempty $\text{AE}(n - w)$.

The following preliminary facts precede the proof of Theorem (3.3). Let us consider a finite set A represented as the disjoint union $A_1 \amalg \dots \amalg A_w$. We shall study the connectivity properties of the subpolyhedron $K = K_{A_1 \dots A_w}^v, v \leq w$, of the $(|A| - 1)$ -dimensional simplex $\Delta: K = \bigcup \{ \langle a_1 \dots a_v \rangle \mid \langle a_1 \dots a_v \rangle \text{ is a } (v - 1)\text{-dimensional simplex spanned by the vertices } a_k \in A_{i_k}, 1 \leq k \leq v, \text{ where } i_k \neq i_{k'} \text{ if } k \neq k' \}$.

Lemma (3.5). $K_{A_1 \dots A_w}^v \in C^{v-2}$ for each $v \leq w$.

Proof. We now proceed by induction on $\theta = e \cdot w + f + v$, where $e = \max |A_i|, f = |\{i : |A_i| = e\}|$. It is evident that $\theta' = e' \cdot w' + f' + v' < \theta$ if $e' < e$ and $w' \leq w, v' \leq v$ or $e' \leq e, f' < f$ and $w' \leq w, v' \leq v$.

The basis of the induction corresponds to $\theta = 1 \cdot v + v + v = 3v$. In this case K is the $(v - 1)$ -skeleton of Δ , which is $(v - 2)$ -connected. The same is valid for $e = 1$.

Assume the validity of the lemma for all $\theta' < \theta$. Suppose, without loss of generality, that $|A_1| = e > 1$ and $A_1 = A'_1 \amalg A''_1, |A'_1| \leq |A''_1| < e$. By the inductive hypothesis ($\theta' < \theta$ and $\theta'' < \theta$), $K_{A'_1 A_2 \dots A_w}^v$ and $K_{A''_1 A_2 \dots A_w}^v$ are $(v - 2)$ -connected and their intersection

$$K_{A'_1 A_2 \dots A_w}^v \cap K_{A''_1 A_2 \dots A_w}^v = K_{A_2 \dots A_w}^v \cup K_{A_2 \dots A_w}^{v-1} = \begin{cases} K_{A_2 \dots A_w}^v & \text{if } w \geq v; \\ K_{A_2 \dots A_w}^{v-1} & \text{if } w = v \geq 2. \end{cases}$$

By the Van Kampen and the Mayer-Vietoris argument [15], $K_{A_1 A_2 \dots A_w}^v \in C^{v-2}$. \square

Lemma (3.6). Let $\mathcal{J} = \prod_{i=1}^n J_i \in (\Delta_m)^n, b = (b_i) \in I^n$, each b_i is the center of J_i , and $\mathcal{J}_{p_1 \dots p_w} \neq \emptyset$ for $1 \leq p_1 < \dots < p_w, 0 \leq w \leq n$. Then $\mathcal{J}_{p_1 \dots p_w} \in \text{AE}(n - w)$.

Proof. Let $\pi_i: I^n \rightarrow I$ be the projection onto the i th factor. Then

$$F_{p_1} = \bigcup_{k=1}^n \prod_{q \in \delta_{p_1}} \Pi_{k,q}, \text{ where } \Pi_{k,q} = \pi_k^{-1}(q) = \{(x_1, \dots, x_n) \in I^n \mid x_k = q\}.$$

Hence $\mathcal{J}_{p_1 \dots p_w} = \bigcup_{k=1}^n \prod_{q \in A_k} \{\Pi'_{k,q} \mid q \in A_k\}$, where $\Pi'_{k,q} = \mathcal{J} \cap \Pi_{k,q} \cap F_{p_2 \dots p_w}, A_k = \{q \in \delta_{p_1} \mid \Pi'_{k,q} \neq \emptyset\}$. It is easy to see that $\mathcal{J}_{p_1 \dots p_w} \neq \emptyset$ implies $p_k \geq m$ or $p_k \in \text{ind}(b)$, for every $k \leq w$.

We shall omit the proof of the following fact which easily follows from the note mentioned above:

Lemma (3.7). $\bigcap_{i=1}^r \Pi'_{k_i q_i} \neq \emptyset$ for every $k_1 < k_2 < \dots < k_r, 0 \leq r \leq v = n + 1 - w, q_i \in A_{k_i}$.

We now proceed by induction on $\sigma = n + w$. The basis of induction is obvious. Assume the validity of the lemma for all $\sigma' < \sigma$. Then $\Pi'_{k,q} \in \text{AE}((n-1)-(w-1))$ and moreover, for every $k_1 < k_2 < \dots < k_r, 0 \leq r \leq v = n+1-w, q_i \in A_{k_i}$, we have $\bigcap_{i=1}^r \Pi'_{k_i,q_i} = (\mathcal{J} \cap \bigcap_{i=1}^r \Pi_{k_i,q_i}) \cap F_{p_2} \cap \dots \cap F_{p_w} \in \text{AE}((n-r)-(w-1)) = \text{AE}((n-w+1)-r)$ (since $\bigcap_{i=1}^r \Pi_{k_i,q_i}$ is an $(n-r)$ -dimensional cube, parallel the $(n-r)$ -dimensional plane $\{(x_1, \dots, x_n) | x_{k_i} = 0, i \leq r\}$ and $\mathcal{J} \cap \bigcap_{i=1}^r \Pi_{k_i,q_i}$ is a nonempty element of $(\Delta_m)^{n-r}$).

Hence the intersections of the Π_{k_i,q_i} 's are correctly connected (in the sense of Bestvina [3]) and their nerve coincides with $K_{A_1 \dots A_n}^v = K_{A_1 \dots A_n}^{n+1-w}$, which is $\text{AE}(v-1) = \text{AE}(n-w)$ by Lemma (3.5).

To complete the proof we apply Proposition (1.2) to the cover $\{\Pi'_{k,q}\}$ of $F_{p_1 \dots p_w}$. □

Proof of Theorem (3.3). We prove by induction on $\theta = s \cdot e + f - w$, where $e = \max\{|T'_i| : 1 \leq i \leq s\}, f = |\{i : |T'_i| = e, 1 \leq i \leq s, \}|$ (e, f, s and w are variables). Let us note that $\theta' = s' \cdot e' + f' - w' < \theta$ provided $e' = 1, s' = m, f' = m$ and $w' = n$ or $e' = e, s' = s, f' < f$ and $w' = w$ or $e' < e, s' = s$ and $w' = w$ or $e' = e, s' = s, f' = f$ and $w' = w$.

If $e = 1$, then each $|T'_i| = 1, i > m$, and therefore $X_{p_1 \dots p_w} \cong \mathcal{J}_{p_1 \dots p_w}$ which is $\text{AE}(n-w)$ by Lemma (3.6). Hence the basis of the induction is verified.

Assume the validity of Theorem (3.3) for all $\theta' < \theta$ and let $e > 1$. Pick $i_0 \leq s$ with $|T'_{i_0}| = e$ and represent $T'_{i_0} = \widehat{T}_{i_0} \amalg \check{T}_{i_0}, |\widehat{T}_{i_0}| \leq |\check{T}_{i_0}| < e$. Then $X_{p_1 \dots p_w}$ is the union of

$$Y' = X'_{p_1 \dots p_w} = r_s(\mathcal{J}_{p_1 \dots p_w} \times \widehat{R}) \subset \widehat{P}'_s \quad \text{and}$$

$$Y'' = X''_{p_1 \dots p_w} = r_s(\mathcal{J}_{p_1 \dots p_w} \times \check{R}) \subset \check{P}''_s,$$

where the rectangle subsets \widehat{R} and \check{R} coincide with the rectangle subsets $R \cap (\prod_{i \neq i_0} T_i \times \widehat{T}_{i_0})$ and $R \cap (\prod_{i \neq i_0} T_i \times \check{T}_{i_0})$, which satisfy the property (*) (for notation see the beginning of Section 3). By the inductive hypothesis ($\theta' \leq \theta'' < \theta$), we have $Y', Y'' \in \text{AE}(n-w)$. Let us prove that $Y' \cap Y'' \in \text{AE}(n-w-1)$. It will then follow by the Van Kampen and the Mayer-Vietoris argument [15], that $X_{p_1 \dots p_w} = Y' \cup Y'' \in \text{AE}(n-w)$.

The proof splits naturally into two parts: $i_0 \in \{p_1 \dots p_w\}$ and $i_0 \notin \{p_1 \dots p_w\}$.

Part (1). $i_0 \notin \{p_1 \dots p_w\}$. In this case, $Y' \cap Y'' \cong r_s(\mathcal{J}_{p_1 \dots p_w, i_0} \times R)$. Since $w' = w + 1$ and $e' = e, s' = s, f' = f$, the inductive variable θ' for $Y' \cap Y''$ is less than θ . By the inductive assumption $Y' \cap Y'' \in \text{AE}(n-w-1)$.

Part (2). $i_0 \in \{p_1 \dots p_w\}$. This implies that $Y' = r_s(\mathcal{J}_{p_1 \dots p_w} \times \widehat{R}) \cong r_s(\mathcal{J}_{p_1 \dots p_w} \times R) = X_{p_1 \dots p_w}$. Hence $X_{p_1 \dots p_w} \in \text{AE}(n-w)$. □

Theorem (3.8). Let \mathcal{R} be the family of all rectangle subsets $R \subset \prod_{i=1}^\infty T_i$. Then

- (a) $P_R = r(I^n \times R)$ is an $\text{AE}(n)$, for every $R \in \mathcal{R}$; and
- (b) $\{P_R = r(I^n \times R) | R \in \mathcal{R}\}$ is an equi- LC^{n-1} -family.

Proof. (a) Fix a point $(x, t) \in I^n \times R$ and $r(x, t) = z$. For every $\epsilon > 0$, it is possible to choose a number $m \geq 1$ and an element $\mathcal{E}' = \mathcal{J} \times \mathcal{T}(b, t) \in \mathcal{Q}'_m$ such that

- (1) the neighborhood $U = N(z, \epsilon) \cap P_R$ of z in P_R contains a closed neighborhood $V = r(\mathcal{J} \times (\mathcal{T}(b, t) \cap R_1) \times R_2 \times R_3)$ of z in P_R , where R is represented as the product $R_1 \times R_2 \times R_3$, $R_1 = \prod_{i=1}^m T'_i$, $R_2 = \prod_{i=m+1}^s T'_i$ and $R_3 = \prod_{i=s+1}^{\infty} T'_i$ (s is a sufficiently large number).

By Corollary (3.4), $V_s = r_s(\mathcal{J} \times (\mathcal{T}(b, t) \cap R_1) \times R_2) \subset P_s$ is $AE(n)$ for every s . Pick an arbitrary constant $\nu > 0$ and s such that Id_P and the projection p_s^∞ are ν -close. Since P_s is naturally embedded into P (cf. Proposition (2.3)), we can consider V_s as a subset of V . Now let $\varphi: S^k \rightarrow V, k < n$, be an arbitrary map. The composition $p_s^\infty \circ \varphi: S^k \rightarrow V_s$, which is ν -close to φ , is extended on $\psi: B^{k+1} \rightarrow V_s \hookrightarrow V$. By Proposition (1.1a), $P_R \in AE(n)$.

(b) The proof is analogous to (a), because the choice of the constant $\nu > 0$ does not depend on the choice of the family $\{P_R\}$. □

Corollary (3.9). $P = r(Q) \in AE(n)$ (and hence, by Section 2, is also homeomorphic to μ^n).

4. CONSTRUCTION OF THE ACTION OF THE ZERO-DIMENSIONAL COMPACT GROUP G ON P

By Pontryagin's theorem [12], every compact metric zero-dimensional group G_i can be considered as a closed subgroup of the product $\prod_{k=1}^{\infty} H_{ik}$ of nontrivial finite groups H_{ik} with the following property:

- (1) every nontrivial element $g_i = (g_{ik}) \in G_i$ has infinitely many nontrivial coordinates.

Let us identify some factors of $Q = I^n \times \prod_{i=1}^{\infty} T_i$ with finite groups: $H_{kl} \equiv T_{m_{kl}}, 1 \leq k, l < \infty$, such that \mathbb{N} can be represented as a disjoint union of $\{m_{kl} | k, l\}$ and some infinite subset. Therefore, $\prod_{i=1}^{\infty} T_i$ is represented as the product of $\mathcal{D} = \prod_{k, l \in \mathbb{N}} T_{m_{kl}}$ and the Cantor compactum \mathcal{C} . According to Proposition (1.8), there exists a sequence $R_1 \subset R_2 \subset \dots$ of rectangle subsets on \mathcal{C} , satisfying (1.8)(7)-(9). Then $\mathcal{D} \times R_1 \subset \mathcal{D} \times R_2 \subset \dots$ is the sequence of the rectangle subsets of $\mathcal{D} \times \mathcal{C}$ also satisfying (1.8)(7)-(9).

By Theorem (3.8), $\{r(I^n \times \mathcal{D} \times R_i)\}$ is an equi- LC^{n-1} -family of the Menger compactum P . By Remark (2.4), each $\{r(I^n \times \mathcal{D} \times R_i)\}$ is a Z -set in $r(I^n \times \mathcal{D} \times R_{i+1})$. Since $\bigcup_{i=1}^{\infty} r(I^n \times \mathcal{D} \times R_i)$ is a dense subset of P , it follows by Proposition (1.6) that the pseudo-boundary $\Sigma(P)$ coincides with $\bigcup_{i=1}^{\infty} r(I^n \times \mathcal{D} \times R_i)$.

To construct a semi-free action Φ of $G \subset H$ on P as in Theorem (0.2), we first define a discontinuous action Ψ of H on Q . Let $N(B, \mathcal{Q}_m)$ be the star of a subset $B \subset Q$ with respect to the partition $\mathcal{Q}_m = \{\mathcal{E}' \times \prod_{i=m+1}^{\infty} T_i | \mathcal{E}' \in \mathcal{Q}'_m\}$ of

$Q = I^n \times \prod_{i=m+1}^{\infty} T_i$. Note that $N(B, \mathcal{Q}_m)$ is a closed neighborhood of B and

- (2) $\bigcap_{m=1}^{\infty} N(B, \mathcal{Q}_m) = r^{-1}r(B)$; and

(3) if $(x, u, v) \in \text{Bd}(N(B, \mathcal{Q}_m))$, then $\text{ind}(x) = m$.

Let $B_k = r^{-1}(X_k) \subset I^n \times \mathcal{D} \times \mathcal{C}$. Now the action Ψ of H on Q is defined by setting for $h = (h_{kl}) \in H$ and $z = (x, u, v) \in I^n \times \mathcal{D} \times \mathcal{C}$,

(4) $\Psi(h, z) = (x, u', v)$, where $u' = (u'_{m_{kl}})$ and

$$u'_{m_{kl}} = \begin{cases} u_{m_{kl}} & \text{if } (x, u, v) \in \text{Int}(N(B_k, \mathcal{Q}_{m_{kl}})), \\ h_{kl} \cdot u_{m_{kl}} & \text{otherwise.} \end{cases}$$

It is obvious that

(5) $r(I^n \times \mathcal{D} \times A)$ is an invariant subset for every $A \subset \mathcal{C}$.

Let $(x, u, v) \notin B_k$ and $g = (g_{kl}) \in G_k \setminus \{e\}$. Since $\mathcal{M}_k = \{m_{kl} \mid (x, u, v) \in N(B_k, \mathcal{Q}_{m_{kl}})\}$ is finite, whereas $\mathcal{G} = \{l \mid g_{kl} \neq e\}$ is infinite, we have

(6) B_k is the fixed-point set for every $g = (g_{kl}) \in G_k \setminus \{e\}$.

Next the action Φ of H on P is defined by $\Phi(h, r(x, u, v)) = r(\Psi(h, (x, u, v)))$. As the preimage $r^{-1}(z)$ is a rectangle subset $x \times \mathcal{T}(x, t)$ of Q , the action Φ is well-defined. It follows from (5)-(6) and the formula for Φ that $\Sigma(P)$ and $\nu(P)$ are invariant subsets and X_k is the fixed-point set for every $g \in G_k \setminus \{e\}$.

To complete the proof of Theorem (0.2), we need the following fact:

Proposition (4.1). *The action $\Phi : G \times P \rightarrow P$ is continuous.*

Proof. It suffices to verify that for every $(x_0, t_0) \in Q$, $\epsilon > 0$ and $g_0 \in G$, there exist a neighborhood $O(g_0)$ of g_0 and $\delta > 0$, such that $\text{dist}((x, t), (x_0, t_0)) < \delta$ implies $\alpha_H(g \cdot (x \times \mathcal{T}(x, t)), g_0 \cdot (x_0 \times \mathcal{T}(x_0, t_0))) < \epsilon$, for every $g \in O(g_0)$.

We shall use the upper semi-continuity of the decomposition \mathcal{T} (Proposition (2.1)). Let us choose $\delta > 0$ and $p \in \mathbb{N}$, satisfying (8)-(10) in Section 2, and a neighborhood $O(g_0)$ of g_0 such that the first p coordinates of every element $g \in O(g_0)$ coincide. Then it follows that $g \cdot (x \times \mathcal{T}(x, t)) = x \times \mathcal{T}(x_0, g \cdot t)$ and $\alpha_H(x \times \mathcal{T}(x, g \cdot t), x_0 \times \mathcal{T}(x_0, g_0 \cdot t_0)) < \epsilon$, analogously to the proof of Proposition (2.1). □

5. EPILOGUE

Let M be a μ^k -manifold. By Theorem (1.4), there exist a PL manifold R of dimension $\geq 2k + 1$ with a triangulation L , and a proper map $f : R^{(k)} \rightarrow M$ which induces isomorphisms of homotopy groups of $\text{dim} < k$ and of homotopy groups of ends of $\text{dim} < k$, where $R^{(k)} = |L^{(k)}|$, $L^{(k)}$ is the k -skeleton of L . If $R^{(k)}$ would admit a cubical triangulation, then by applying Sections 2-4 word-by-word, we could construct the map $r : R^{(k)} \times \prod T_i \rightarrow (R^{(k)} \times \prod T_i)/\mathcal{T} = N^k$ into N^k , which by Theorem (1.3) is a μ^k -manifold. The natural projection $\pi' : R^{(k)} \times \prod T_i \rightarrow R^{(k)}$ generates the proper retraction $\pi : N^k \rightarrow R^{(k)}$, which would induce isomorphisms of homotopy groups of $\text{dim} < k$ and of homotopy groups of ends of $\text{dim} < k$. The composition $f \circ \pi$ would then be a proper map between μ^k -manifolds which would induce isomorphisms of homotopy groups of $\text{dim} < k$ and of homotopy groups of ends of $\text{dim} < k$. By Theorem (1.5), $f \circ \pi$ would be properly $(k - 1)$ -homotopic to a homeomorphism and therefore $(R^{(k)} \times \prod T_i)/\mathcal{T}$ would be homeomorphic to M^k . As in Section 4, we could construct the desired action of $\prod G_i$ on $N = (R^{(k)} \times \prod T_i)/\mathcal{T}$.

Problem (5.1). *Is every simplicial complex homeomorphic to a complex which admits a cubical triangulation?*

Without referring to the solvability of this problem, we give an outline of the proof of Theorem (0.2) in the case of μ^k -manifold M . To this end, we consider the handlebody decomposition $H_m = \{\text{St}(v; \beta^{m+2}L) \mid v \in (\beta^{m+1}L)^{(0)}\}$ of $R^{(k)}$ according to $\beta^m L$, where $\beta^m L$ is the m th barycentric subdivision of the triangulation L and $(\beta^{m+1}L)^{(0)}$ are the vertices of the $(m+1)$ -st barycentric subdivision.

By F_m we denote the union of all the boundaries of all elements of the partition H_m . Clearly, the analogue of (2) in Section 2 holds:

- (1) for every $x \in R^{(k)}$, the set $\{m \mid x \in F_m\}$ has less than $k+1$ elements.

All assertions from Sections 2-4 go through in our case, whereas their proofs do not differ substantially and the reader can supply the details.

Only the fact that the proper retraction $\pi: (R^{(k)} \times \prod T_i)/\mathcal{T} \rightarrow R^{(k)}$ induces isomorphisms of homotopy groups of $\dim < k$ and of homotopy groups of ends of $\dim < k$, requires a proof. Due to the limited space we must leave the details to the readers. However, we wish to point out that they do not significantly differ from the techniques used in our proofs above.

As in Section 4 we can construct the desired action of $\prod G_i$ on $(R^{(k)} \times \prod T_i)/\mathcal{T}$, where $\prod T_i$ is chosen as $\prod H_{kl} \times \mathcal{C}$.

ACKNOWLEDGEMENTS

The first author was supported in part by the INTAS grant No. 96-0712. The second author was supported in part by the Ministry of Science and Technology of the Republic of Slovenia grant No. J1-0885-0101-98. We thank the referee for a very careful reading of the manuscript and for contributing several important comments and suggestions.

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