

COEFFICIENT IDEALS AND THE COHEN-MACAULAY PROPERTY OF REES ALGEBRAS

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ABSTRACT. Let A be a local ring and let $I \subset A$ be an ideal of positive height. If $J \subset I$ is a reduction of I , then the coefficient ideal $\mathfrak{a}(I, J)$ is by definition the largest ideal \mathfrak{a} such that $I\mathfrak{a} = J\mathfrak{a}$. In this article we study the ideal $\mathfrak{a}(I, J)$ when the Rees algebra $R_A(I)$ is Cohen-Macaulay.

1. INTRODUCTION

Let (A, \mathfrak{m}) be a local ring of dimension d having an infinite residue field, and let $I \subset A$ be an ideal of height h . An ideal $J \subset I$ is called a reduction of I if there exists an integer $r \geq 0$ such that $I^{r+1} = JI^r$. The least such integer r is called the reduction number of I with respect to J and is denoted by $r_J(I)$. Every minimal reduction is generated by the same number of elements. This number $\ell = \ell(I)$ is called the analytic spread of I . If $J \subset I$ is a reduction of I , Aberbach and Huneke defined in [1] the coefficient ideal $\mathfrak{a}(I, J)$ as the largest ideal \mathfrak{a} such that $I\mathfrak{a} = J\mathfrak{a}$. The importance of this notion stems from the role it plays in Briançon-Skoda type theorems. The Briançon-Skoda theorem states that if the ring A is regular, then $\overline{I^{n+\ell-1}} \subset J^n$ for all minimal reductions $J \subset I$ and all integers $n \geq 0$ (see [2], [17], [18] and [8]). Recall here that given an ideal $I \subset A$, the integral closure of I , \overline{I} , is the ideal of A consisting of all elements $x \in A$ satisfying an equation of type $x^k + a_1x^{k-1} + \cdots + a_k = 0$ where $a_i \in I^i$ ($i = 1, \dots, k$). Aberbach and Huneke could now prove a theorem that they called a Briançon-Skoda theorem with coefficients saying that if the ring A contains a field and the ideal I is \mathfrak{m} -primary, then $\overline{I^{n+d-1}} \subset \mathfrak{a}(I, J)J^n$ ([1, Theorem 2.7]).

On the other hand, assuming that A is a regular local ring essentially of finite type over a field of characteristic zero, Lipman showed in his article [15] that $\overline{I^{n+\ell-1}} \subset \text{adj}(I^{\ell-1})J^n$ where $\text{adj}(I^{\ell-1})$ denotes the adjoint ideal of $I^{\ell-1}$ (see [15, p. 745 and p. 747]). The adjoint $\text{adj}(I^{\ell-1})$ can be defined as $\Gamma(Y, I^{\ell-1}\omega_Y)$ where ω_Y denotes the canonical sheaf of Y and $Y \rightarrow \text{Spec } A$ is any desingularization such that $I\mathcal{O}_Y$ is invertible. The A -module $\Gamma(Y, I^{\ell-1}\omega_Y)$ is considered here as an ideal of A by means of the trace homomorphism $\Gamma(Y, \omega_Y) \rightarrow \omega_A = A$. It turns out that $\text{adj}(I^{\ell-1}) \subset \mathfrak{a}(I, J)$. Moreover, in the case $d = 2$ and I is an integrally closed \mathfrak{m} -primary ideal, Lipman proved in [15, Proposition 3.3] that $\mathfrak{a}(I, J) = \text{adj}(I) = J : I$. Observe that in this case the Rees algebra $R_A(I) = \bigoplus_{n \geq 0} I^n$ is always known to

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have rational singularities (see [13, Proposition 1.2] and [9, Proposition 2.1]). In particular, it is Cohen-Macaulay.

The purpose of this article is to investigate the coefficient ideal $\mathfrak{a}(I, J)$ under the assumption that $R_A(I)$ is Cohen-Macaulay. We suppose, moreover, that the ring A is Gorenstein, the ideal I satisfies the condition G_ℓ (i.e. $\mu(I_{\mathfrak{p}}) \leq \text{ht } \mathfrak{p}$ for every $\mathfrak{p} \in V(I)$ with $\text{ht } \mathfrak{p} \leq \ell - 1$), and that $\text{depth } A/I^n \geq d - h - n + 1$ for $n = 1, \dots, \ell - h$.

Our main result, Theorem 3.4, now says that for any minimal reduction $J \subset I$ the coefficient ideal $\mathfrak{a}(I, J)$ coincides with the ideal $\Gamma(X, I^{h-1}\omega_X) \subset A$ where $X = \text{Proj } R_A(I)$. In fact, when $h > 1$, this means that $\mathfrak{a}(I, J)$ can be identified with the homogeneous component $[\omega_{R_A(I)}]_{h-1}$ of the canonical module $\omega_{R_A(I)}$. It follows in particular that $\mathfrak{a}(I, J)$ is independent of the minimal reduction J . Moreover, if $r = r_J(I)$, then $\mathfrak{a}(I, J) = J^r : I^r$, and in the case $r \leq \ell - h + 1$ we even get $\mathfrak{a}(I, J) = J : I = \dots = JI^{r-1} : I^r$. As a corollary of Theorem 3.4 we can show that if A is a regular local ring essentially of finite type over a field of characteristic zero and I is an equimultiple ideal such that $R_A(I)$ is normal and Cohen-Macaulay, then $\mathfrak{a}(I, J) = \text{adj}(I^{h-1})$ if and only if $R_A(I)$ has rational singularities (see Corollary 3.5).

Finally, we observe in Proposition 3.7 that when A has pseudorational singularities, the ideals considered above in fact satisfy a sharpened version of the Briançon-Skoda theorem with coefficients saying that $\overline{I^{n+h-1}} \subset \mathfrak{a}(I, J)J^n$ for all reductions $J \subset I$ and all integers $n \geq 0$.

2. PRELIMINARIES

In the following all rings and schemes are assumed to be Noetherian. Moreover, all local rings are assumed to have an infinite residue field.

Let (A, \mathfrak{m}) be a local ring of dimension d which is a homomorphic image of a Gorenstein local ring, and let $I \subset A$ be an ideal of positive height. We first want to recall some results about the structure of the canonical module of a Rees algebra needed in the sequel. The canonical module of the Rees algebra $R_A(I)$ and especially its relationship with that of the form ring $G_A(I)$ have been investigated by several people (see e.g. [7], [3], [11] and [20]). For the convenience of the reader we have collected the necessary results into Theorem 2.2 below. At the same time we want to point out that it is very natural to consider in this context also the canonical sheaf ω_X of the scheme $X = \text{Proj } R_A(I)$.

The sheaf ω_X is then equal to the associated sheaf of the canonical module $\omega_{R_A(I)}$. In general it is defined as $H^{-\dim X}(\mathcal{R}_X^\bullet)$ where \mathcal{R}_X^\bullet is the dualizing complex of X . By definition $\mathcal{R}_X^\bullet = f^!(\widetilde{R}_A^\bullet)$ where $f: X \rightarrow \text{Spec } A$ is the canonical projection and R_A^\bullet is a normalized dualizing complex of A . The canonical sheaf ω_X is defined up to an isomorphism. The general theory of duality provides a trace morphism $\mathbf{R}\Gamma(X, \mathcal{R}_X^\bullet) \rightarrow R_A^\bullet$ (see [4, Chapter VII, Corollary 3.4]). As $\dim X = d$, taking cohomology gives a trace homomorphism $\Gamma(X, \omega_X) \rightarrow \omega_A$. Recall that this corresponds to the canonical homomorphism $H_{\mathfrak{m}}^d(A) \rightarrow H_E^d(X, \mathcal{O}_X)$ where $E = X \times_A A/\mathfrak{m}$ via the isomorphisms

$$\text{Hom}_A(\Gamma(X, \omega_X), E_A(k)) = H_E^d(X, \mathcal{O}_X) \quad \text{and} \quad \text{Hom}_A(\omega_A, E_A(k)) = H_{\mathfrak{m}}^d(A)$$

(see [16, the proof of Lemma 4.2] or [18, p. 110]).

Note the following elementary fact:

Lemma 2.1. *Let A be a ring and let $I \subset A$ be an ideal of positive height. Set $X = \text{Proj } R_A(I)$. Let \mathcal{F} be an \mathcal{O}_X -module such that $\text{grade}(I\mathcal{O}_{X,x}, \mathcal{F}_x) > 0$ for all $x \in X$.*

Then $\mathcal{F}(n) = I^n \mathcal{F}$ for all $n \geq 0$. Moreover, $\text{Hom}_X(J\mathcal{O}_X, \mathcal{F}) = \text{Hom}_A(J, \Gamma(X, \mathcal{F}))$ for any ideal $J \subset A$.

Proof. Since the sheaf $I^n \mathcal{O}_X$ is locally principal, it is easily checked that the natural epimorphism $\mathcal{F} \otimes \mathcal{O}_X(n) = \mathcal{F} \otimes I^n \mathcal{O}_X \rightarrow I^n \mathcal{F}$ is an isomorphism. In order to prove the second claim, note first that a morphism $J\mathcal{O}_X \rightarrow \mathcal{F}$ determines a homomorphism $J \rightarrow \Gamma(X, J\mathcal{O}_X) \rightarrow \Gamma(X, \mathcal{F})$. Conversely, any homomorphism $J \rightarrow \Gamma(X, \mathcal{F})$ induces homomorphisms $J I^n \rightarrow I^n \Gamma(X, \mathcal{F}) \rightarrow \Gamma(X, \mathcal{F}(n))$ ($n \in \mathbb{N}$), and therefore a morphism $J\mathcal{O}_X \rightarrow \mathcal{F}$. \square

Recall that if S is a graded ring defined over a local ring, then the a -invariant

$$a(S) = \sup\{n \in \mathbb{Z} \mid [H_{\mathfrak{M}}^{\dim S}(S)]_n \neq 0\}$$

where \mathfrak{M} denotes the homogeneous maximal ideal of S .

Theorem 2.2. *Let (A, \mathfrak{m}) be a local ring which is a homomorphic image of a Gorenstein local ring, and let $I \subset A$ be an ideal of positive height. Set $R = R_A(I)$, $G = G_A(I)$ and $X = \text{Proj } R_A(I)$. Then the following hold:*

- (a) $\omega_R = \bigoplus_{n \geq 1} \Gamma(X, I^n \omega_X)$;
- (b) $\Gamma(X, \omega_X) \supset \Gamma(X, I\omega_X) \supset \Gamma(X, I^2\omega_X) \supset \dots$;
- (c) $\Gamma(X, I^n \omega_X) = \text{Hom}_A(I, \Gamma(X, I^{n+1}\omega_X))$ for all $n \geq 0$;
- (d) The trace homomorphism $\Gamma(X, \omega_X) \rightarrow \omega_A$ is always injective. It is bijective, if $a(G) < 0$;
- (e) When A is Cohen-Macaulay, there exists a monomorphism

$$0 \rightarrow \bigoplus_{n \geq 1} \Gamma(X, I^{n-1}\omega_X) / \Gamma(X, I^n \omega_X) \rightarrow \omega_G$$

which is an isomorphism if R is Cohen-Macaulay. In the case $a(G) < 0$, this gives

$$\omega_A = \Gamma(X, \omega_X) = \Gamma(X, I\omega_X) = \dots = \Gamma(X, I^{-a(G)-1}\omega_X)$$

so that $I^{n+a(G)+1}\omega_A \subset \Gamma(X, I^n \omega_X)$ for $n \geq 0$.

Proof. (a) Observe first that if S is any homogeneous graded ring defined over a local ring, then there are isomorphisms $[\omega_S]_n = \Gamma(\text{Proj } S, \omega_{\text{Proj } S}(n))$ for all $n \geq 1$ (see [5, Corollary 2.9 and its proof]). Also note that $[\omega_R]_n = 0$ for $n < 1$, because $a(R) = -1$ (see [3, Part I, 6.3]). Because $\omega_{X,x}$ is the canonical module of $\mathcal{O}_{X,x}$ for all $x \in X$, we know that $\text{Ass } \omega_{X,x} \subset \text{Min } \mathcal{O}_{X,x}$. Hence $\text{grade}(I\mathcal{O}_{X,x}, \omega_{X,x}) > 0$ for all $x \in X$ so that Lemma 2.1 is applicable. Hence

$$\omega_R = \bigoplus_{n \geq 1} [\omega_R]_n = \bigoplus_{n \geq 1} \Gamma(X, \omega_X(n)) = \bigoplus_{n \geq 1} \Gamma(X, I^n \omega_X).$$

(b) This is clear, because $\omega_X \supset I\omega_X \supset I^2\omega_X \supset \dots$.

(c) As $\Gamma(X, \omega_X(n)) = \text{Hom}_X(\mathcal{O}_X(1), \omega_X(n+1)) = \text{Hom}_X(I\mathcal{O}_X, \omega_X(n+1))$, the claim is a consequence of Lemma 2.1.

(d) Set $d = \dim A$. Consider the Sancho de Salas sequence

$$(1) \quad \dots \rightarrow [H_{\mathfrak{M}}^d(R)]_n \rightarrow H_{\mathfrak{m}}^d(R_n) \rightarrow H_E^d(X, \mathcal{O}_X(n)) \rightarrow [H_{\mathfrak{M}}^{d+1}(R)]_n \rightarrow 0$$

where \mathfrak{M} denotes the homogeneous maximal ideal of R and $E = X \times_A A/\mathfrak{m}$ (see [16, p. 150]). Because $[H_{\mathfrak{M}}^{d+1}(R)]_0 = 0$, we observe that the canonical homomorphism $H_{\mathfrak{m}}^d(A) \rightarrow H_E^d(X, \mathcal{O}_X)$ is surjective. But this means that the trace homomorphism $\Gamma(X, \omega_X) \rightarrow \omega_A$ is injective. We also observe that to prove the surjectivity, it is

enough to show that $[H_{\mathfrak{m}}^d(R)]_0 = 0$. By looking at the long exact sequences of cohomology corresponding to the exact sequences

$$0 \longrightarrow R^+ \longrightarrow R \longrightarrow A \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow R^+(1) \longrightarrow R \longrightarrow G \longrightarrow 0,$$

it is now easy to check that this is indeed the case when $a(G) < 0$.

(e) Consider the exact sequence

$$(2) \quad 0 \longrightarrow I\mathcal{O}_X \longrightarrow \mathcal{O}_X \longrightarrow j_*\mathcal{O}_Y \longrightarrow 0$$

where $Y = \text{Proj } G$ and $j: Y \longrightarrow X$ is the inclusion. It is possible to express R as a quotient of a graded Cohen-Macaulay ring R' such that $\dim R' = \dim R$. Let $i: X \longrightarrow X'$ be the corresponding closed imbedding of schemes. Then $i_*\omega_X = \mathcal{H}om_{X'}(i_*\mathcal{O}_X, \omega_{X'})$ and $(ij)_*\omega_Y = \mathcal{E}xt_{X'}^1((ij)_*\mathcal{O}_Y, \omega_{X'})$. An application of the functor $\mathcal{H}om_{X'}(i_*(\cdot), \omega_{X'}(n))$ to (2) now gives us for all $n \in \mathbb{Z}$ an exact sequence

$$(3) \quad 0 \longrightarrow i_*(\omega_X(n)) \longrightarrow i_*(\omega_X(n-1)) \longrightarrow (ij)_*(\omega_Y(n)).$$

By taking global sections we get the exact sequence

$$(4) \quad 0 \longrightarrow \Gamma(X, \omega_X(n)) \longrightarrow \Gamma(X, \omega_X(n-1)) \longrightarrow \Gamma(Y, \omega_Y(n)).$$

These sequences give the desired monomorphism, because $\Gamma(Y, \omega_Y(n)) = [\omega_G]_n$ for all $n \geq 1$. If R is Cohen-Macaulay, then we can take $R' = R$ so that (3) becomes short exact. When $n \geq 1$, the same also holds for (4) since we now have $H^1(X, \omega_X(n)) = 0$ for all $n \geq 1$. To see this, note first that by using the sequence (1) we obtain $H_E^{d-1}(X, \mathcal{O}_X(n)) = [H_{\mathfrak{m}}^d(R)]_n = 0$ for $n \leq -1$. Since

$$\text{Hom}_A(H^1(X, \omega_X(n)), E_A(k)) = H_E^{d-1}(X, \mathcal{O}_X(-n))$$

by the local-global duality of Lipman ([14, p. 188]), we get $H^1(X, \omega_X(n)) = 0$ for all $n \geq 1$. As now $[\omega_G]_n = 0$ for $n \leq 0$, the claim follows. The remaining two statements are immediate. □

Remark 2.1. The module $\Gamma(X, \omega_X)$ coincides with the degree one component \mathcal{D}_1 of the fundamental divisor $\mathcal{D} = \omega_{IR}$ introduced in [22]. To see this, note first that $\mathcal{O}_X = I\mathcal{O}_X(-1)$. The Sancho de Salas sequence for the module IR then gives us an isomorphism $H_E^d(X, \mathcal{O}_X) \longrightarrow [H_{\mathfrak{m}}^{d+1}(IR)]_{-1}$. By taking the Matlis-duals we get an isomorphism $[\omega_{IR}]_1 \times_A \hat{A} \longrightarrow \Gamma(X, \omega_X) \times_A \hat{A}$, and thus also an isomorphism $[\omega_{IR}]_1 \longrightarrow \Gamma(X, \omega_X)$.

Given an ideal $I \subset A$, a minimal reduction $J \subset I$ can be considered as a “simplification” of I . Nevertheless, it is not always easy to say how the algebraic properties of the corresponding blowups $\text{Proj } R_A(I)$ and $\text{Proj } R_A(J)$ are related to each another. However, in the following Proposition 2.3 we can utilize the existence of a finite morphism $\text{Proj } R_A(I) \longrightarrow \text{Proj } R_A(J)$ to compare their canonical sheaves.

Proposition 2.3. *Let A be a local ring which is a homomorphic image of a Gorenstein local ring, and let $I \subset A$ be an ideal of positive height. Let ω_X be a fixed canonical sheaf of $X = \text{Proj } R_A(I)$. Let $J \subset A$ be any reduction of I , and let $r \geq 0$ be any integer such that $I^{r+1} = JI^r$. Set $Y = \text{Proj } R_A(J)$. Then*

$$\Gamma(X, I^n \omega_X) = \Gamma(Y, J^{n+r} \omega_Y) :_{\omega_A} I^r$$

for all $n \geq 0$ as submodules of ω_A .

Proof. Let $\pi: X \rightarrow Y$ be the canonical morphism. Since π is finite, we know that there exists an isomorphism $\pi_*\omega_X \cong \mathcal{H}om_Y(\pi_*\mathcal{O}_X, \omega_Y)$. As $I^n = I^r J^{n-r}$ for $n \geq r$, we have $\pi_*\mathcal{O}_X = I^r \mathcal{O}_Y(-r)$. Then

$$\pi_*(\omega_X(n)) \cong \mathcal{H}om_Y(\pi_*\mathcal{O}_X, \omega_Y)(n) = \mathcal{H}om_Y(I^r \mathcal{O}_Y, J^{n+r} \omega_Y).$$

An application of Lemma 2.1 thus gives

$$\Gamma(X, I^n \omega_X) \cong \text{Hom}_Y(I^r \mathcal{O}_Y, J^{n+r} \omega_Y) = \text{Hom}_A(I^r, \Gamma(Y, J^{n+r} \omega_Y)).$$

The general theory of trace (see [4, Chapter III, Theorem 10.5]) implies that there exists a commutative diagram

$$\begin{array}{ccc} \Gamma(X, \omega_X) & \xrightarrow{\text{trace}} & \omega_A \\ \wr \downarrow & & \uparrow \text{trace} \\ \text{Hom}_Y(\pi_*\mathcal{O}_X, \omega_Y) & \xrightarrow{e} & \Gamma(Y, \omega_Y) \end{array}$$

where e equals to the homomorphism

$$\text{Hom}_Y(\pi_*\mathcal{O}_X, \omega_Y) \rightarrow \text{Hom}_Y(\mathcal{O}_Y, \omega_Y) = \Gamma(Y, \omega_Y)$$

induced by the morphism $\mathcal{O}_Y \rightarrow \pi_*\mathcal{O}_X$. By the above this can be identified with

$$\text{Hom}_A(I^r, \Gamma(Y, J^r \omega_Y)) \rightarrow \text{Hom}_A(J^r, \Gamma(Y, J^r \omega_Y)) = \Gamma(Y, \omega_Y).$$

Hence every element of $\text{Hom}_A(I^r, \Gamma(Y, J^{n+r} \omega_Y)) \subset \text{Hom}_A(I^r, \Gamma(Y, J^r \omega_Y))$ comes from multiplication by some element of ω_A , which proves the claim. \square

3. THE MAIN RESULTS

We begin with the following technical lemma.

Lemma 3.1. *Let (A, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension d , and let $I \subset A$ be an ideal of positive height such that $R_A(I)$ is Cohen-Macaulay. Set $X = \text{Proj } R_A(I)$. Then there is for all $0 < i \leq d - 1$ and $n \in \mathbb{Z}$ an isomorphism*

$$H_{\mathfrak{m}}^{d-1-i}(A/I^{i-n}) = \text{Hom}_A(H^i(X, \omega_X(n-i)), E_A(k)).$$

Proof. Indeed, by the local-global duality of Lipman ([14, p. 188])

$$\text{Hom}_A(H^i(X, \omega_X(n-i)), E_A(k)) = H_E^{d-i}(X, \mathcal{O}_X(i-n))$$

where $E = X \times_A A/\mathfrak{m}$. Let \mathfrak{M} denote the homogeneous maximal ideal of $R_A(I)$. Because $R_A(I)$ is Cohen-Macaulay, the Sancho de Salas sequence

$$\dots \rightarrow [H_{\mathfrak{M}}^{d-i}(R_A(I))]_{i-n} \rightarrow H_{\mathfrak{m}}^{d-i}(I^{i-n}) \rightarrow H_E^{d-i}(X, \mathcal{O}_X(i-n)) \rightarrow \dots$$

(see [16, p. 150]) gives $H_E^{d-i}(X, \mathcal{O}_X(i-n)) = H_{\mathfrak{m}}^{d-i}(I^{i-n})$. Finally, by looking at the cohomology sequence of the exact sequence $0 \rightarrow I^{i-n} \rightarrow A \rightarrow A/I^{i-n} \rightarrow 0$ we get $H_{\mathfrak{m}}^{d-i}(I^{i-n}) = H_{\mathfrak{m}}^{d-i-1}(A/I^{i-n})$ as needed. \square

Now recall from [1, Definition 1.2] that given two ideals $J \subset I$ in a ring A , the *coefficient ideal* of I relative to J , $\mathfrak{a}(I, J)$, is the largest ideal \mathfrak{a} such that $\mathfrak{a}I = \mathfrak{a}J$. It is easy to see that the coefficient ideal always exists. Moreover, if $\text{grade } \mathfrak{a}(I, J) > 0$, then J is necessarily a reduction of I .

Let A be a Gorenstein local ring, and let $I \subset A$ be an ideal of positive height. Set $X = \text{Proj } R_A(I)$. In the following we shall always consider the A -module $\Gamma(X, \omega_X)$ as an ideal of A by means of the trace homomorphism $\Gamma(X, \omega_X) \rightarrow \omega_A = A$.

The next observation is now crucial for the proof of our main Theorem 3.4.

Proposition 3.2. *Let (A, \mathfrak{m}) be a Gorenstein local ring of dimension d . Let $I \subset A$ be an ideal with positive height h and analytic spread ℓ . Suppose that $\text{depth } A/I^n \geq d - h - n + 1$ for $n = 1, \dots, \ell - h$. Set $X = \text{Proj } R_A(I)$. If $R_A(I)$ is Cohen-Macaulay, then $\Gamma(X, I^{m+1}\omega_X) = J\Gamma(X, I^m\omega_X)$ as ideals of A for all reductions $J \subset I$ and all integers $m \geq h - 1$. This means in particular that $\Gamma(X, I^{h-1}\omega_X) \subset \mathfrak{a}(I, J)$.*

Proof. The assumption $\text{depth } A/I^n \geq d - h - n + 1$ for $n = 1, \dots, \ell - h$ implies by Lemma 3.1 that $H^i(X, \omega_X(m-i)) = 0$ for $0 < i < \ell$. We can now proceed as in [15, p. 747]. For the convenience of the reader, we sketch here the argument. We may assume that J is a minimal reduction of I . Consider the sheaves $I\mathcal{O}_X$ and $J\mathcal{O}_X$. Because J is a reduction of I , we clearly have $I\mathcal{O}_X = J\mathcal{O}_X$. Let $\{a_1, \dots, a_\ell\}$ be a generating set of J . The corresponding global sections of $I\mathcal{O}_X$ then determine an epimorphism $\mathcal{O}_X^{\oplus \ell} \rightarrow I\mathcal{O}_X$ and so also an epimorphism $\sigma: \mathcal{N} \rightarrow \mathcal{O}_X$ where $\mathcal{N} = ((I\mathcal{O}_X)^{-1})^{\oplus \ell}$. This means that the Koszul complex $K(\sigma, I^{m+1}\omega_X)$:

$$0 \rightarrow \bigwedge^\ell \mathcal{N} \otimes I^{m+1}\omega_X \rightarrow \dots \rightarrow \bigwedge^1 \mathcal{N} \otimes I^{m+1}\omega_X \rightarrow I^{m+1}\omega_X \rightarrow 0$$

is exact. Note that

$$\bigwedge^j \mathcal{N} \otimes I^{m+1}\omega_X = (\omega_X(m+1-j))^{\oplus \binom{\ell}{j}} \quad (j = 0, \dots, \ell).$$

So $H^{j-1}(X, \bigwedge^j \mathcal{N} \otimes I^{m+1}\omega_X) = 0$ for $1 < j \leq \ell$. Arguing as in [18, Lemma 5.1], we then see that the homomorphism $\Gamma(X, \bigwedge^j \mathcal{N} \otimes I^{m+1}\omega_X) \rightarrow \Gamma(X, I^{m+1}\omega_X)$ is surjective, which implies the first claim. The last claim is now immediate, because $\Gamma(X, I^h\omega_X) = J\Gamma(X, I^{h-1}\omega_X) = I\Gamma(X, I^{h-1}\omega_X)$. \square

For the proof of Theorem 3.4 we have to identify the associated primes of the ideal $\Gamma(X, I^{h-1}\omega_X) \subset A$. Therefore we still need

Lemma 3.3. *Let (A, \mathfrak{m}) be a Gorenstein local ring and let $I \subset A$ be an ideal of positive height. Set $X = \text{Proj } R_A(I)$. If $\Gamma(X, \omega_X) = A$, then $\text{Ass } A/\Gamma(X, I^n\omega_X) \subset \{\mathfrak{p} \in \text{Spec } A \mid \text{ht } \mathfrak{p} = \ell(I_{\mathfrak{p}})\}$ for all $n \geq 0$.*

Proof. Set $\Omega_n = \Gamma(X, I^n\omega_X)$ for all $n \geq 0$. Let $\mathfrak{p} \in \text{Ass } A/\Omega_n$. Observe that $(\Omega_n)_{\mathfrak{p}} = \Gamma(X_{\mathfrak{p}}, I_{\mathfrak{p}}^n\omega_{X_{\mathfrak{p}}})$ where $X_{\mathfrak{p}} = \text{Proj } R_{A_{\mathfrak{p}}}(I_{\mathfrak{p}})$. By change of notation we can therefore assume that $\mathfrak{p} = \mathfrak{m}$. We now have $\text{depth } A/\Omega_n = 0$. Let us show that $\text{depth } \Omega_{k-1}/\Omega_k = 0$ for some $k \geq 1$. Suppose the contrary. The exact sequences

$$0 \rightarrow \Omega_{k-1}/\Omega_k \rightarrow A/\Omega_k \rightarrow A/\Omega_{k-1} \rightarrow 0$$

would then imply that $\text{depth } A/\Omega_k > 0$ for all $k \geq 0$ which is a contradiction. But from this we obtain that $\text{grade}(\mathfrak{m}, M) = 0$ where $M = \bigoplus_{k \geq 1} \Omega_{k-1}/\Omega_k$. According to Theorem 2.2 (e) there is now a monomorphism $0 \rightarrow M \rightarrow \omega_G$ where $G = G_A(I)$. From this we see that $\text{grade}(\mathfrak{m}, \omega_G) = 0$. Then $\mathfrak{m}G \subset Q$ for some $Q \in \text{Ass } \omega_G \subset \text{Min } G$. But G being formally equidimensional (use [6, Corollary (18.24)]), we have $\text{ht } \mathfrak{m}G = \dim G - \ell(I)$. Therefore $\ell(I) = \text{ht } \mathfrak{m}$ as wanted. \square

Theorem 3.4. *Let (A, \mathfrak{m}) be a Gorenstein local ring of dimension d . Let $I \subset A$ be an ideal with positive height h and analytic spread ℓ . Suppose that I satisfies G_ℓ and that $\text{depth } A/I^n \geq d - h - n + 1$ for $n = 1, \dots, \ell - h$. Set $X = \text{Proj } R_A(I)$. If $R_A(I)$ is Cohen-Macaulay, then $\mathfrak{a}(I, J) = \Gamma(X, I^{h-1}\omega_X)$ for all minimal reductions $J \subset I$. In particular, $\mathfrak{a}(I, J)$ is independent of the minimal reduction J . Moreover, $\mathfrak{a}(I, J) = J^r : I^r$ where $r = r_J(I)$. In the case $r \leq \ell - h + 1$, we even have $\mathfrak{a}(I, J) = J : I = \dots = JI^{r-1} : I^r$.*

Proof. Set $\Omega = \Gamma(X, I^{h-1}\omega_X)$. By virtue of Proposition 3.2 we need to prove that $\mathfrak{a}(I, J) \subset \Omega$. As $I\mathfrak{a}(I, J) = J\mathfrak{a}(I, J)$, we clearly have $I^r\mathfrak{a}(I, J) = J^r\mathfrak{a}(I, J)$ so that $\mathfrak{a}(I, J) \subset J^r : I^r$. Let us verify that $a(G_A(J)) \leq -h$. Indeed, because of [12, Theorem 2.2 and Remark 2.7] we may apply [12, Lemma 2.3 b)] to find a generating sequence a_1, \dots, a_ℓ of J which is a d -sequence. But as explained in [22, p. 33] this implies that $a(G_A(J)) \leq -h$ as wanted. Set $Y = \text{Proj } R_A(J)$. We now obtain from Theorem 2.2 (e) that $J^r \subset \Gamma(Y, J^{h-1+r}\omega_Y)$. But then Proposition 2.3 implies that $\mathfrak{a}(I, J) \subset J^r : I^r \subset \Gamma(Y, J^{h-1+r}\omega_Y) : I^r = \Omega$.

Finally suppose that $r \leq \ell - h + 1$. Since $\mathfrak{a}(I, J) \subset J : I \subset \dots \subset JI^{r-1} : I^r$, it is enough to show that $JI^{r-1} : I^r \subset \Omega$. By Lemma 3.3 we can check this locally at prime ideals $\mathfrak{p} \in \text{Spec } A$ satisfying $\text{ht } \mathfrak{p} = \ell(I_{\mathfrak{p}})$. We first observe that by [12, Remark 2.7] $I_{\mathfrak{p}} = J_{\mathfrak{p}}$ for all $\mathfrak{p} \in V(I)$ with $\text{ht } \mathfrak{p} < \ell$. We then have $a(G_{A_{\mathfrak{p}}}(I_{\mathfrak{p}})) \leq -h$. By Theorem 2.2 (e) this gives $\Omega_{\mathfrak{p}} = A_{\mathfrak{p}}$. Now take $\mathfrak{p} \in \text{Spec } A$ with $\text{ht } \mathfrak{p} = \ell(I_{\mathfrak{p}})$. If $\text{ht } \mathfrak{p} < \ell$, then by the above there is nothing to prove. Thus, suppose that $\text{ht } \mathfrak{p} = \ell$. By change of notation we can assume that $\mathfrak{p} = \mathfrak{m}$. Set $R = R_A(I)$ and $G = G_A(I)$. It is well-known that the Cohen-Macaulayness of R implies that of G (see e.g. [21, Theorem 5.1.23]). Since $\Omega_{\mathfrak{q}} = A_{\mathfrak{q}}$ for $\mathfrak{q} \neq \mathfrak{m}$, we observe that either $\Omega = A$ or Ω is \mathfrak{m} -primary. In the first case there is nothing to prove. In the second case we necessarily have $a(G) = -h + 1$. Indeed, by [19, Theorem 3.5] $a(G) = -h$ or $a(G) = -h + 1$, and by Theorem 2.2 (e) only the latter case is possible. Let \mathfrak{M} denote the homogeneous maximal ideal of G . Because $[H_{\mathfrak{M}}^i(G)]_n = 0$ for all $n > -h + 1$ and $i \geq 0$, we have $[H_{G^+}^i(G)]_n = 0$ for $n > -h + 1$ and $i \geq 0$ (see e.g. [10, Lemma 1.1]). The exact sequences

$$0 \longrightarrow R^+ \longrightarrow R \longrightarrow A \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow R^+(1) \longrightarrow R \longrightarrow G \longrightarrow 0$$

easily imply that $[H_{R^+}^i(R)]_n = 0$ for $n > -h + 1$ and $i > 1$. Then $H^i(X, \mathcal{O}_X(n)) = 0$ for $n > -h + 1$ and $i > 0$. We also obtain $[H_{R^+}^0(R)]_n = 0$ and $[H_{R^+}^1(R)]_n = 0$ for $n \geq 0$ so that $\Gamma(X, \mathcal{O}_X(n)) = I^n$ for $n \geq 0$. In particular, we have

$$H^1(X, \mathcal{O}_X(\ell - h - 1)) = \dots = H^{\ell-2}(X, \mathcal{O}_X(-h + 2)) = 0$$

and

$$H^1(X, \mathcal{O}_X(\ell - h)) = \dots = H^{\ell-1}(X, \mathcal{O}_X(-h + 2)) = 0$$

It then follows from [18, Remark 5.2], that

$$\begin{aligned} H^{\ell-1}(X, \mathcal{O}_X(-h + 1)) &= \Gamma(X, \mathcal{O}_X(\ell - h + 1)) / J\Gamma(X, \mathcal{O}_X(\ell - h)) \\ &= I^{\ell-h+1} / JI^{\ell-h}. \end{aligned}$$

By a version of the local-global duality of Lipman ([14, p. 188], see also [5, Proposition 2.3])

$$H_E^1(X, \omega_X(h - 1)) = \text{Hom}_A(H^{d-1}(X, \mathcal{O}_X(-h + 1)), E_A(k)).$$

Recall from Theorem 2.2 (a) that $\omega_R = \bigoplus_{n \geq 1} \Gamma(X, I^n \omega_X)$. Because ω_R is Cohen-Macaulay, the Sancho de Salas sequence for ω_R (see [16, p. 150]) gives us an isomorphism $H_{\mathfrak{m}}^1(\Omega) = H_E^1(X, \omega_X(h - 1))$. By looking at the cohomology sequence of the exact sequence $0 \longrightarrow \Omega \longrightarrow A \longrightarrow A/\Omega \longrightarrow 0$, we obtain that $H_{\mathfrak{m}}^1(\Omega) = H_{\mathfrak{m}}^0(A/\Omega)$. Since Ω is \mathfrak{m} -primary, it finally follows that

$$\Omega = \text{Ann } H_{\mathfrak{m}}^0(A/\Omega) = \text{Ann } I^{\ell-h+1} / JI^{\ell-h} = JI^{\ell-h} : I^{\ell-h+1}.$$

So $JI^{r-1} : I^r \subset \Omega$ as wanted. □

Remark 3.1. Suppose $h > 1$. Combined with Theorem 2.2 and Proposition 3.2 Theorem 3.4 then implies that

$$\omega_{R_A(I)} = \bigoplus_{n=1}^{h-2} (\mathfrak{a}(I, J) : I^{h-1-n}) \oplus \bigoplus_{n \geq h-1} I^{n-h+1} \mathfrak{a}(I, J).$$

Note that we have $I^{h+a} \subset \mathfrak{a}(I, J)$ where $a = a(G_A(I))$.

Let A be a regular local ring essentially of finite type over a field of characteristic zero. We now recall the definition of the adjoint of an ideal from [15, p. 740]. Let K denote the fraction field of A . Let v be a valuation of K whose valuation ring A_v contains A . If \mathfrak{m}_v is the maximal ideal of A_v , set $\mathfrak{p} = \mathfrak{m}_v \cap A$. The valuation v is called a prime divisor of A if A_v/\mathfrak{m}_v has transcendence degree $\text{ht } \mathfrak{p} - 1$ over the subfield $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. It is equivalent to say that A_v is essentially of finite type over A . Let $I \subset A$ be an ideal. The *adjoint* of I is

$$\text{adj}(I) = \bigcap_v \{r \in K \mid v(r) \geq v(I) - v(J_{A_v/A})\}$$

where the intersection is taken over all prime divisors of A , and the Jacobian ideal $J_{A_v/A}$ is the 0th Fitting ideal of the A_v -module $\Omega_{A_v/A}^1$ of Kähler differentials. Alternatively, $\text{adj}(I) = \Gamma(X, I\omega_X)$ where $X \rightarrow \text{Spec } A$ is any proper birational morphism such that X has rational singularities and that $I\mathcal{O}_X$ is invertible (see [15, Proposition 1.3.1]).

Corollary 3.5. *Let (A, \mathfrak{m}) be a regular local of dimension d essentially of finite type over a field of characteristic zero. Let $I \subset A$ be an ideal with positive height h and analytic spread ℓ such that I satisfies G_ℓ and that $\text{depth } A/I^n \geq d - h - n + 1$ for $n = 1, \dots, \ell - h$. If $R_A(I)$ has rational singularities, then $\mathfrak{a}(I, J) = \text{adj}(I^{h-1})$ for all minimal reductions $J \subset I$. Moreover, when I is an equimultiple ideal such that $R_A(I)$ is normal and Cohen-Macaulay, also the converse holds.*

Proof. Set $X = \text{Proj } R_A(I)$. Observe first that when $R_A(I)$ is normal and Cohen-Macaulay, it has rational singularities if and only if X has rational singularities (see e.g. [9, Proposition 2.1]). Let $f: Y \rightarrow X$ be a desingularization. By definition X now has rational singularities if and only if $f_*\omega_Y = \omega_X$. Noting that $\Gamma(X, f_*\omega_Y(n)) = \Gamma(Y, I^n\omega_Y)$ for all $n \geq 0$, we obtain that $f_*\omega_Y = \omega_X$ if and only if $\text{adj}(I^n) = \Gamma(Y, I^n\omega_Y) = \Gamma(X, I^n\omega_X)$ for $n \gg 0$. Observe here that the trace homomorphism $f_*\omega_Y \rightarrow \omega_X$ induces an inclusion $\Gamma(Y, I^n\omega_Y) \subset \Gamma(X, I^n\omega_X)$. By Theorem 2.2 (c) we have $\text{Hom}_A(I, \Gamma(X, I^n\omega_X)) = \Gamma(X, I^{n-1}\omega_X)$. In fact, by using the same argument we also obtain that $\text{Hom}_A(I, \Gamma(Y, I^n\omega_Y)) = \Gamma(Y, I^{n-1}\omega_Y)$. This implies that if $R_A(I)$ has rational singularities, then $\text{adj}(I^n) = \Gamma(X, I^n\omega_X)$ for all $n \geq 0$. An application of Theorem 3.4 then implies that $\mathfrak{a}(I, J) = \text{adj}(I^{h-1})$ for any minimal reduction $J \subset I$. Conversely, suppose that I is equimultiple and that $\mathfrak{a}(I, J) = \text{adj}(I^{h-1})$ for some minimal reduction $J \subset I$. By [15, p. 745 and p. 747] we know that $\text{adj}(I^n) = I^{n-\ell+1} \text{adj}(I^{\ell-1})$ for $n \geq \ell - 1$. On the other hand, using Proposition 3.2 we have $\Gamma(X, I^n\omega_X) = I^{n-h+1} \mathfrak{a}(I, J)$ for $n \geq h - 1$. Hence $\Gamma(X, I^n\omega_X) = \text{adj}(I^n)$ for $n \geq \ell - 1$ which means that X and so also $R_A(I)$ have rational singularities. \square

In the situation of Corollary 3.5 the coefficient ideal is always integrally closed, because that is known to be true for the adjoint ideal (see [15, p. 741 (b)]). In fact, this holds even more generally:

Proposition 3.6. *Let A be a Gorenstein local domain of dimension d . Let $I \subset A$ be an ideal with positive height h and analytic spread ℓ such that I satisfies G_ℓ and that $\text{depth } A/I^n \geq d - h - n + 1$ for $n = 1, \dots, \ell - h$. Suppose that $R_A(I)$ is Cohen-Macaulay. If $\text{Proj } R_A(I)$ is normal, then $\mathfrak{a}(I, J)$ is integrally closed for any minimal reduction $J \subset I$.*

Proof. Set $X = \text{Proj } R_A(I)$. Consider ω_X as a subsheaf of the constant sheaf K where K is the quotient field of A . Since ω_X is reflexive, we know that

$$\Gamma(X, I^{h-1}\omega_X) = \bigcap_{\text{codim } \overline{\{x\}}=1} I^{h-1}\omega_{X,x}.$$

But then $\Gamma(X, I^{h-1}\omega_X)$ is integrally closed as an intersection of integrally closed A -submodules of K (see [23, p. 349]). \square

Finally, we want to point out that the ideals we have considered in this article satisfy the following sharpened version of the Briançon-Skoda theorem with coefficients.

Proposition 3.7. *Let (A, \mathfrak{m}) be a Gorenstein local ring of dimension d having pseudorational singularities. Let $I \subset A$ be an ideal of positive height h and analytic spread ℓ . Suppose that $\text{depth } A/I^n \geq d - h - n + 1$ for $n = 1, \dots, \ell - h$. If $R_A(I)$ is Cohen-Macaulay, then*

$$\overline{I^{n+h-1}} \subset \mathfrak{a}(I, J)J^n$$

for all minimal reductions $J \subset I$ and all integers $n \geq 0$.

Proof. We may clearly assume that A is a domain. Let $f: Y \rightarrow X$ be the normalization of $X = \text{Proj } R_A(I)$. Because A has pseudorational singularities, we know that $\Gamma(Y, \omega_Y) = A$. This implies that $\mathcal{O}_Y \subset \omega_Y$ as subsheaves of the constant sheaf K where K is the quotient field of A . Then $\Gamma(Y, I^n \mathcal{O}_Y) \subset \Gamma(Y, I^n \omega_Y)$ for all $n \geq 0$. Moreover, the trace homomorphism $f_* \omega_Y \rightarrow \omega_X$ gives an inclusion $\Gamma(Y, I^n \omega_Y) \subset \Gamma(X, I^n \omega_X)$. Now recall that $\Gamma(Y, I^n \mathcal{O}_Y) = \overline{I^n}$ (see e.g. [18, p. 100]). The claim follows, because by Proposition 3.2 we have $\Gamma(X, I^n \omega_X) = J^{n-h+1} \Gamma(X, I^{h-1} \omega_X) \subset J^{n-h+1} \mathfrak{a}(I, J)$ for $n \geq h - 1$. \square

REFERENCES

- [1] I. M. Aberbach and C. Huneke, *A theorem of Briançon-Skoda type for regular local rings containing a field*, Proc. Amer. Math. Soc. **124** (1996), 707–713. MR **96f**:13039
- [2] J. Briançon and H. Skoda, *Sur la clôture intégrale d'un idéal de germes de fonctions holomorphes en un point de C^n* , C.R. Acad. Sci Paris (A) (1974), 949–951. MR **49**:5394
- [3] S. Goto and K. Nishida, *The Cohen-Macaulay and Gorenstein Rees algebras associated to filtrations*, Mem. Amer. Math. Soc. **526** (1994). MR **95b**:13001
- [4] R. Hartshorne, *Residues and duality*, Springer Lecture Notes, vol. 20, Springer-Verlag, Berlin-Heidelberg-New York, 1966. MR **36**:5145
- [5] M. Herrmann, E. Hyry, and T. Korb, *On Rees algebras with a Gorenstein Veronese subring*, J. Algebra **200** (1998), 279–311. MR **98m**:13006
- [6] M. Herrmann, S. Ikeda, and U. Orbanz, *Equimultiplicity and blowing up*, Springer-Verlag, Berlin-Heidelberg-New York, 1988. MR **89g**:13012
- [7] J. Herzog, A. Simis, and W.V. Vasconcelos, *On the canonical module of the Rees algebra and the associated graded ring of an ideal*, J. Algebra **105** (1987), 285–302. MR **87m**:13029
- [8] M. Hochster and C. Huneke, *Tight closure, invariant theory, and the Briançon-Skoda theorem*, J. Am. Math. Soc. **3** (1990), 31–116. MR **91g**:13010
- [9] E. Hyry, *Blow-up algebras and rational singularities*, Manus. Math. **98** (1999), 377–390. CMP 2000:03

- [10] ———, *Necessary and sufficient conditions for the Cohen-Macaulayness of the form ring*, J. Algebra **212** (1999), 17–27. CMP 99:07
- [11] E. Hyry, M. Herrmann, and J. Ribbe, *On multi-Rees algebras (with an appendix by N. V. Trung)*, Math. Ann. **301** (1995), 249–279. MR **96c**:13005
- [12] M. Johnson and B. Ulrich, *Artin-Nagata properties and Cohen-Macaulay associated graded rings*, Compositio Math. **103** (1996), 7–29. MR **97f**:13006
- [13] J. Lipman, *Rational singularities, with applications to algebraic surfaces and unique factorization*, Publ. Math. Inst. Hautes Etudes Sci. **36** (1969), 195–280. MR **43**:1986
- [14] ———, *Desingularization of two-dimensional schemes*, Ann. of Math. **107** (1978), 151–207. MR **58**:10924
- [15] ———, *Adjoints of ideals in regular local rings*, Math. Res. Letters **1** (1994), 739–755. MR **95k**:13028
- [16] ———, *Cohen-Macaulayness in graded algebras*, Math. Res. Letters **1** (1994), 149–157. MR **95d**:13006
- [17] J. Lipman and A. Sathaye, *Jacobian ideals and a theorem of Briançon-Skoda*, Michigan Math. J. **28** (1981), 199–222. MR **83m**:13001
- [18] J. Lipman and B. Teissier, *Pseudo-rational local rings and a theorem of Briançon-Skoda*, Michigan Math. J. **28** (1981), 97–116. MR **82f**:14004
- [19] A. Simis, B. Ulrich, and W. V. Vasconcelos, *Cohen-Macaulay Rees algebras and degrees of polynomial relations*, Math. Ann. **301** (1995), 421–444. MR **96a**:13005
- [20] N. V. Trung, D.Q. Viêt, and S. Zarzuela, *When is the Rees algebra Gorenstein?*, J. Algebra **175** (1995), 137–156. MR **96h**:13007
- [21] W. V. Vasconcelos, *Aritmetic of blowup algebras*, LMS Lect. Note Ser., vol. 195, Cambridge University Press, Cambridge, 1994. MR **95g**:13005
- [22] ———, *Lecture notes on Cohen-Macaulay blowup algebras and their equations*, Lecture notes for the Workshop on Commutative Algebra and its Relation to Combinatorics and Computer Algebra in Trieste, 1994.
- [23] O. Zariski and P. Samuel, *Commutative algebra, vol II*, Springer Graduate Texts in Mathematics, vol. 29, Springer-Verlag, Berlin-Heidelberg-New York, 1975. MR **52**:10706

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