

## THE EXPECTED $L_p$ NORM OF RANDOM POLYNOMIALS

PETER BORWEIN AND RICHARD LOCKHART

(Communicated by Albert Baernstein II)

ABSTRACT. The results of this paper concern the expected  $L_p$  norm of random polynomials on the boundary of the unit disc (equivalently of random trigonometric polynomials on the interval  $[0, 2\pi]$ ). Specifically, for a random polynomial

$$q_n(\theta) = \sum_0^{n-1} X_k e^{ik\theta}$$

let

$$\|q_n\|_p^p = \int_0^{2\pi} |q_n(\theta)|^p d\theta / (2\pi).$$

Assume the random variables  $X_k, k \geq 0$ , are independent and identically distributed, have mean 0, variance equal to 1 and, if  $p > 2$ , a finite  $p^{\text{th}}$  moment  $E(|X_k|^p)$ . Then

$$\frac{E(\|q_n\|_p^p)}{n^{p/2}} \rightarrow \Gamma(1 + p/2)$$

and

$$\frac{E(\|q_n^{(r)}\|_p^p)}{n^{(2r+1)p/2}} \rightarrow (2r + 1)^{-p/2} \Gamma(1 + p/2)$$

as  $n \rightarrow \infty$ .

In particular if the polynomials in question have coefficients in the set  $\{+1, -1\}$  (a much studied class of polynomials), then we can compute the expected  $L_p$  norms of the polynomials and their derivatives

$$\frac{E(\|q_n\|_p)}{n^{1/2}} \rightarrow (\Gamma(1 + p/2))^{1/p}$$

and

$$\frac{E(\|q_n^{(r)}\|_p)}{n^{(2r+1)/2}} \rightarrow (2r + 1)^{-1/2} (\Gamma(1 + p/2))^{1/p}.$$

This complements results of Fielding in the  $p := 0$  case, Newman and Byrnes in the  $p := 4$  case, and Littlewood et al. in the  $p = \infty$  case.

### 1. INTRODUCTION

There are a number of difficult old conjectures that concern the possible rates of growth of polynomials with all coefficients in the set  $\{+1, -1\}$ . Since many of these were raised by Littlewood we denote the set of such polynomials by  $\mathcal{L}_n$  and

---

Received by the editors December 18, 1998 and, in revised form, August 31, 1999.

1991 *Mathematics Subject Classification*. Primary 26D05.

*Key words and phrases*. Random polynomial, Littlewood polynomial, expected  $L_p$  norm.

Research supported in part by the NSERC of Canada.

refer to them as Littlewood polynomials. Specifically

$$\mathcal{L}_n := \left\{ p : p(x) = \sum_{j=0}^n a_j x^j, \quad a_j \in \{-1, 1\} \right\}.$$

In [Li-66] Littlewood conjectures that it is possible to find  $p_n \in \mathcal{L}_n$  so that

$$C_1 \sqrt{n+1} \leq |p_n(z)| \leq C_2 \sqrt{n+1}$$

for all complex  $z$  of modulus 1. Such polynomials are often called “flat”. Because the  $L_2$  norm of a polynomial from  $\mathcal{L}_n$  is exactly  $\sqrt{n+1}$  the constants must satisfy  $C_1 \leq 1$  and  $C_2 \geq 1$ . This is discussed in some detail in problem 19 of Littlewood’s delightful monograph [Li-68]. A sequence of polynomials that satisfies just the upper bound is given by the Rudin-Shapiro polynomials. No sequence is known that satisfies the lower bound.

This conjecture is complemented by a conjecture of Erdős [Er-62] that the constant  $C_2$  is bounded away from 1 (independently of  $n$ ). This is also still open. Although a remarkable result of Kahane [Kah-85, Kah-80] shows that if the polynomials are allowed to have complex coefficients of modulus 1, then “flat” polynomials exist and indeed that it is possible to make  $C_1$  and  $C_2$  asymptotically arbitrarily close to 1. Equally remarkable is a result of Beck [Bec-91] who proves that “flat” polynomials exist from the class of polynomials of degree  $n$  whose coefficients are 400th roots of unity.

The relationship between these problems and Barker polynomials is discussed in [Sa-90]. The most famous problem concerning polynomials with coefficients in the set  $\{0, -1, +1\}$  is the celebrated, now resolved, Littlewood conjecture [Ko-80]. It asserts that the  $L_1$  norm of a polynomial  $\sum_{j=0}^n \pm x^{k_j}$  must grow at least like  $\log(n)$ . Here and in what follows the  $L_p$  norms are on the boundary of the unit disk in the complex plane.

In [Sa-54] it is shown that for all but  $o(2^n)$  Littlewood polynomials the supremum on the unit disc lies between  $c_1 \sqrt{n \log n}$  and  $c_2 \sqrt{n \log n}$ . In fact, Halász [Ha-73] shows that  $\lim \|q_n / \sqrt{n \log n}\|_\infty = 1$  almost surely. See also [An-83].

The expected  $L_4$  to the fourth power of a Littlewood polynomial of degree  $n$  is computed by Newman and Byrnes [Ne-90]. They show that

$$E(\|p\|_4^4) = 2(n+1)^2 - (n+1)$$

where  $p$  is a random element of  $\mathcal{L}_n$ .

In the  $L_0$  case Fielding [Fi-70] computes the expected norm (which in this case is the Mahler Measure) over the polynomials with *complex* coefficients of modulus 1. He proves that

$$E(\|p\|_0) \geq \exp(-\gamma/2) \sqrt{n} (1 + O(n^{-1/2+\delta}))$$

where  $\gamma$  is Euler’s constant. See also [Ul-88].

Our principal aim in this paper is to compute the expected  $L_p$  norms of Random Littlewood polynomials. The complete results are stated in the next sections. For random Littlewood polynomials,  $q_n \in \mathcal{L}_n$ , we have

$$\frac{E(\|q_n\|_p)}{n^{1/2}} \rightarrow (\Gamma(1 + p/2))^{1/p}$$

and for their derivatives

$$\frac{E(\|q_n^{(r)}\|_p)}{n^{(2r+1)/2}} \rightarrow (2r + 1)^{-1/2}(\Gamma(1 + p/2))^{1/p}.$$

From this and the inequality [Bor-95, p. 406]

$$\frac{\|q'_n\|_p}{n\|q_n\|_p} \leq 1$$

we can deduce an expected Bernstein Inequality for Littlewood polynomials, namely

$$E\left(\frac{\|q'_n\|_p}{n\|q_n\|_p}\right) \rightarrow \frac{1}{\sqrt{3}}.$$

This should be compared to interesting results of Nazarov and Queffélec and Saffari [Qe-96] which say that

$$\max_{q_n \in \mathcal{L}_n} \frac{\|q'_n\|_p}{n\|q_n\|_p} \rightarrow 1$$

for all  $p > 1$  except  $p = 2$  where the lim sup is  $1/\sqrt{3}$ .

Of course, because of the monotonicity of the  $L_p$  norms it is relevant to rephrase Littlewood’s conjecture in other norms. It has been conjectured that

$$\|p\|_4^4 \geq (7 - \delta)n^2/6$$

for  $p \in \mathcal{L}_n$  and  $n$  sufficiently large. This would be best possible and would imply Erdős’ conjecture above. See [Bor-98] for a discussion of this.

Random polynomials have been much looked at, particularly the location of their roots. See for example [Bh-86], [Bri-75] and [Kac-48].

## 2. RESULTS

Consider a random polynomial

$$q_n(\theta) = \sum_0^{n-1} X_k e^{ik\theta}$$

for  $0 \leq \theta \leq 2\pi$ . We study the  $p^{th}$  power of the  $L_p$  norm of  $q_n$ ; that is,

$$\|q_n\|_p^p = \int_0^{2\pi} |q_n(\theta)|^p d\theta/2\pi.$$

**Theorem 1.** Fix  $0 < p < \infty$ . Assume that the random variables  $X_k, k \geq 0$ , are independent and identically distributed, have mean 0, variance equal to 1 and, if  $p > 2$ , a finite  $p^{th}$  moment  $E(|X_k|^p)$ . Then

$$\frac{E(\|q_n\|_p^p)}{n^{p/2}} \rightarrow \Gamma(1 + p/2)$$

as  $n \rightarrow \infty$ . If, in addition,  $E(|X_k|^{2p}) < \infty$ , then

$$\frac{\|q_n\|_p}{n^{1/2}} \rightarrow \Gamma(1 + p/2)^{1/p}$$

in probability and

$$\frac{E(\|q_n\|_p)}{n^{1/2}} \rightarrow (\Gamma(1 + p/2))^{1/p}.$$

For  $\pm 1$  coefficients randomly chosen the moment conditions are trivially satisfied for all finite  $p$ . Numerical confirmation of the principle results based on computations up to degree 24 is presented in Figures 1 and 2. The calculations are courtesy of Lesley Robinson [Ro-97]. Figure 1 shows the average  $L_3$  and  $L_4$  norms (normalized by division by  $\sqrt{n+1}$ ) for the Littlewood polynomials up to degree 24. Figure 2 is a similar graph for  $L_3^3$  and  $L_4^4$  (normalized by division by  $(n+1)^{3/2}$  and  $(n+1)^2$  respectively).

*Proof.* In what follows all unlabelled sums run from 0 through  $n-1$ . Define

$$\sigma_{n,c}^2(\theta) = \sum \cos^2(j\theta)$$

and

$$\sigma_{n,s}^2(\theta) = \sum \sin^2(j\theta)$$

and write

$$a_{k,n}(\theta) = \frac{\cos(k\theta)}{\sigma_{n,c}(\theta)}$$

and, for  $\theta$  not a multiple of  $\pi$ ,

$$b_{k,n}(\theta) = \frac{\sin(k\theta)}{\sigma_{n,s}(\theta)}.$$

Then

**Lemma 2.1.** *There is a constant  $M$ , free of  $n$ ,  $k$  and  $\theta$  such that*

$$|a_{k,n}(\theta)| + |b_{k,n}(\theta)| \leq \frac{M}{\sqrt{n}}.$$

Postpone the proof of the lemma and consider an arbitrary sequence  $\theta_n$ . The lemma permits application of the Lindeberg central limit theorem to show that for an arbitrary sequence  $\theta_n$

$$C_n(\theta_n) \equiv \frac{\sum_0^{n-1} X_k \cos(k\theta_n)}{\sigma_{n,c}(\theta_n)} = \sum_0^{n-1} a_{k,n}(\theta_n) X_k$$

converges in distribution to standard normal and, provided no  $\theta_n$  is an integer multiple of  $\pi$ ,

$$S_n(\theta_n) \equiv \frac{\sum_0^{n-1} X_k \sin(k\theta_n)}{\sigma_{n,s}(\theta_n)}$$

converges in distribution to standard normal. Moreover, for any fixed  $\theta$  not an integer multiple of  $\pi$ , the elementary convergences  $\sigma_{n,c}^2/n \rightarrow 1/2$ ,  $\sigma_{n,s}^2/n \rightarrow 1/2$  and  $\sum \cos(k\theta) \sin(k\theta)/n \rightarrow 0$  show that the covariance matrix of  $(C_n(\theta), S_n(\theta))$  converges to the 2-by-2 identity so that  $(C_n(\theta), S_n(\theta))$  converges in distribution to  $(Z_1, Z_2)$  where the  $Z_i$  are independent standard normals. It follows for such  $\theta$  that  $|q_n(\theta)|^2/n$  converges in distribution to  $(Z_1^2 + Z_2^2)/2$  which has a standard exponential distribution.

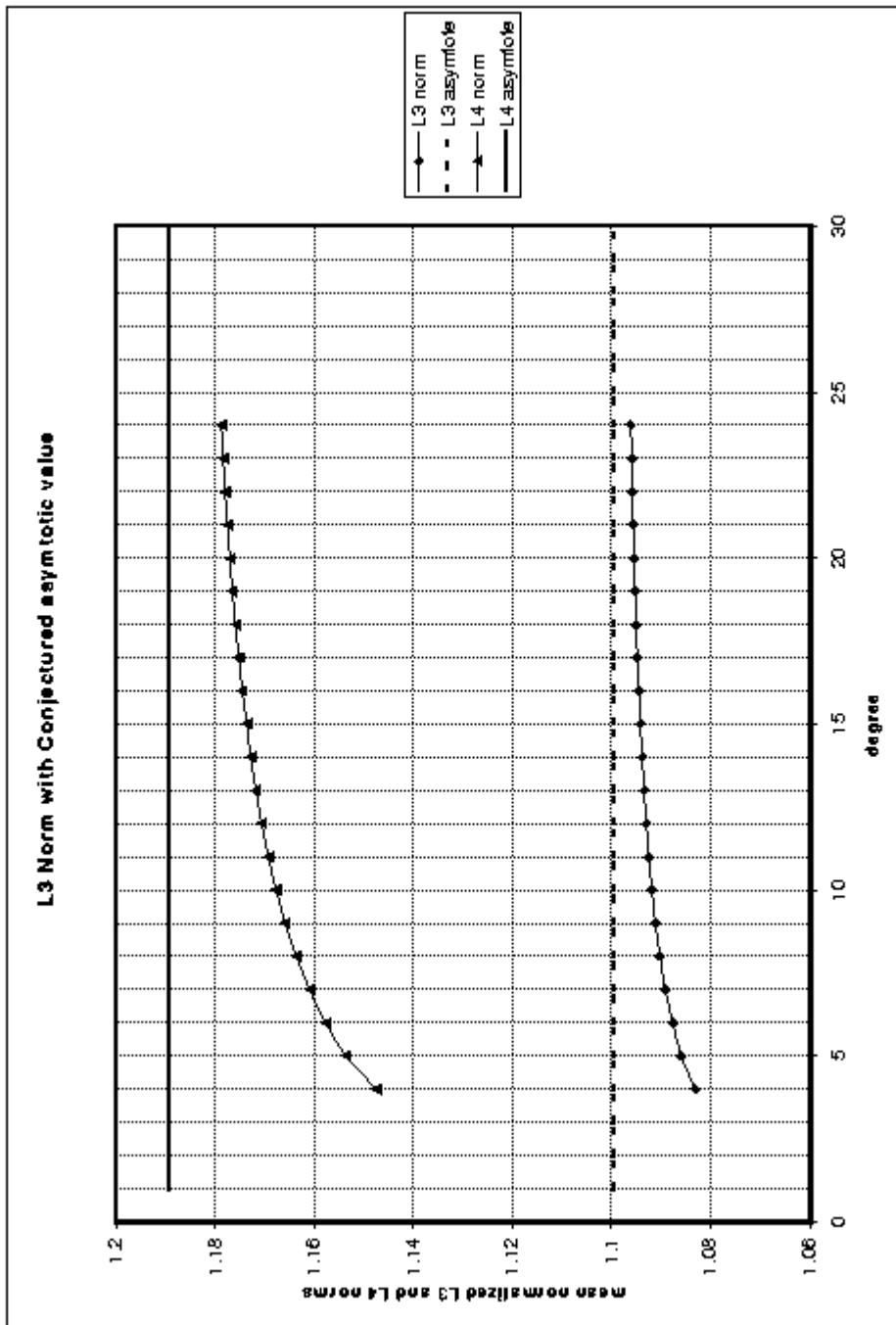


FIGURE 1.

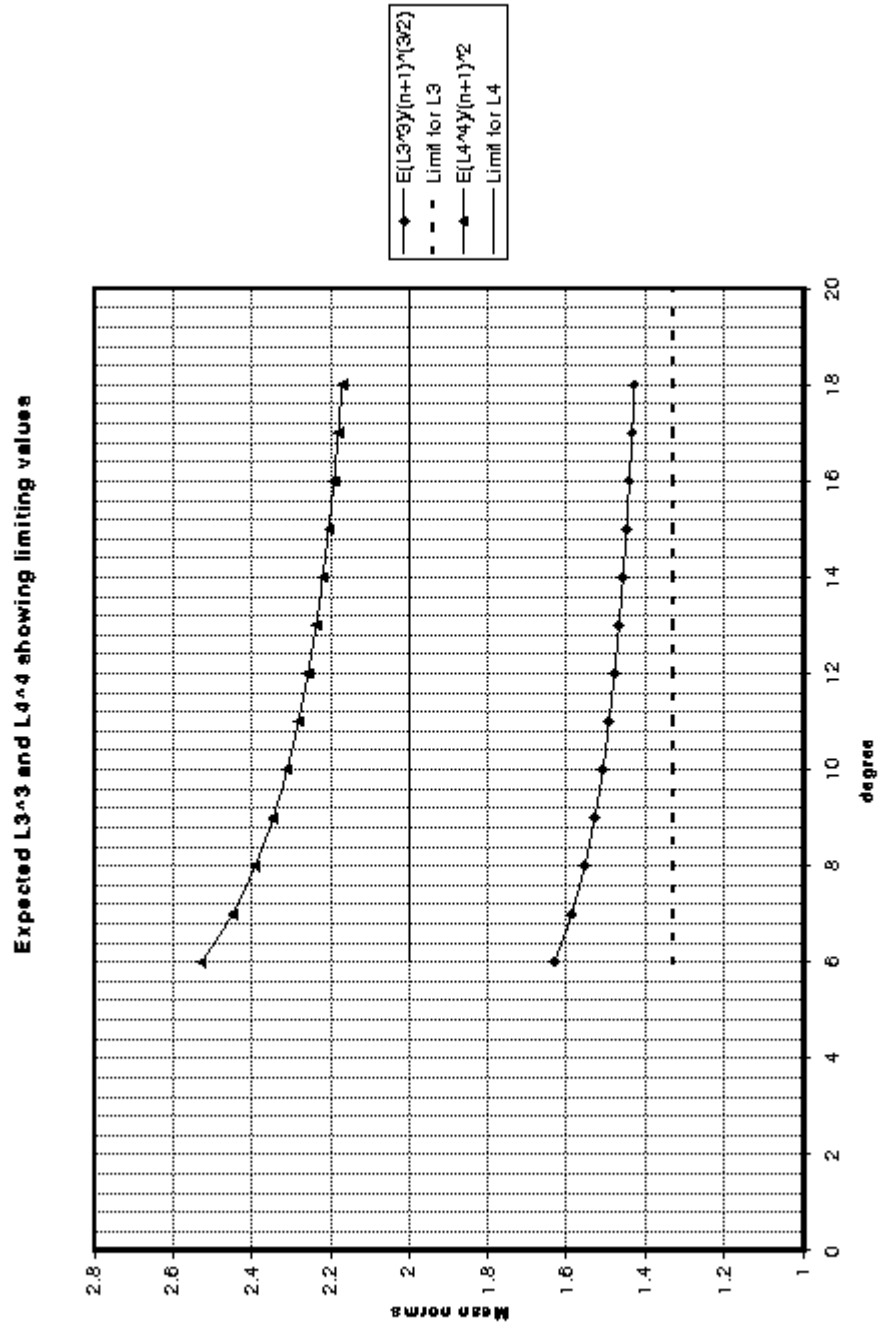


FIGURE 2.

For  $p > 2$ , a theorem of Bernstein ([Be-39], [Br-70]) asserts that, in the central limit theorem,  $p^{th}$  moments converge to the  $p^{th}$  moment of the normal distribution provided an analogue of the Lindeberg condition holds. In particular,

$$E(|C_n(\theta_n)|^p) \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |u|^p \exp(-u^2/2) du$$

provided

$$\sum E(|a_{k,n}|^p |X_k|^p 1(a_{k,n}^2 X_k^2 > \eta)) \rightarrow 0$$

for each positive  $\eta$  (and analogously for  $S_n$  replacing  $a_{k,n}$  by  $b_{k,n}$ ). (For  $p \leq 2$  we could proceed straight from the central limit theorem.) The quantity in question is bounded by

$$\sum |a_{k,n}|^p E(|X_0|^p 1(X_0^2 > n\eta/M^2)) \leq \sum |a_{k,n}|^p E(|X_0|^p 1(X_0^2 > n\eta/M^2)).$$

But  $\sum |a_{k,n}|^p$  is, for  $p \geq 2$ , bounded by  $\sum a_{k,n}^2 = 1$  and  $X_0$  has a finite  $p^{th}$  moment so the bound converges to 0.

Now note

$$\begin{aligned} |q_n(\theta)|^p / n^{p/2} &= \left( \frac{\sigma_{n,c}^2(\theta) C_n^2(\theta) + \sigma_{n,s}^2(\theta) S_n^2(\theta)}{n} \right)^{p/2} \\ &\leq 2^{p/2-1} \left[ \left( \frac{\sigma_{n,c}^2(\theta)}{n} \right)^{p/2} |C_n(\theta)|^p + \left( \frac{\sigma_{n,s}^2(\theta)}{n} \right)^{p/2} |S_n(\theta)|^p \right] \\ &\leq 2^{p/2-1} [|C_n(\theta)|^p + |S_n(\theta)|^p]. \end{aligned}$$

Putting  $g_n(\theta) = E(|q_n(\theta)|^p / n^{p/2})$  and  $h_n(\theta) = 2^{p/2-1} E(|C_n(\theta)|^p + |S_n(\theta)|^p)$  we see that  $0 \leq g_n \leq h_n$  almost everywhere and that  $h_n$  converges uniformly on  $(0, \pi) \cup (\pi, 2\pi)$  to  $2^{p/2} E(|Z|^p)$  where  $Z$  is standard normal. If we establish that  $g_n$  converges almost everywhere to  $\Gamma(1 + p/2)$ , then the theorem will follow by dominated convergence.

To do so we can apply the following result. Suppose  $U_n$  and  $V_n$  are random variables with  $U_n$  converging in distribution to some  $U$ ,  $V_n$  converging in distribution to some  $V$  and  $0 \leq U_n \leq V_n$ . If  $E(V_n) \rightarrow E(V) < \infty$ , then  $E(U_n) \rightarrow E(U)$ . (This is the Dominated Convergence Theorem for convergence in distribution.)

Use the result with  $U_n = |q_n(\theta)|^p / n^{p/2}$ , which converges in distribution, for  $\theta$  not an integer multiple of  $\pi$ , to  $U = ((Z_1^2 + Z_2^2)/2)^{p/2}$ , and

$$V_n = 2^{p/2-1} [|C_n(\theta_n)|^p + |S_n(\theta_n)|^p]$$

which converges in distribution to  $V = 2^{p/2-1} (|Z_1|^p + |Z_2|^p)$ . Since  $E(U) = \Gamma(1 + p/2)$  we are done.

To establish the convergence in probability we compute the variance of  $\|q_n\|_p^p / n^{p/2}$  and show that this converges to zero. It suffices to show that

$$\frac{E(\|q_n\|_p^{2p})}{n^p} \rightarrow \Gamma^2(1 + p/2).$$

However,

$$\frac{E(\|q_n\|_p^{2p})}{n^p} = \int_0^{2\pi} \int_0^{2\pi} g_n^{(*)}(\theta_1, \theta_2) d\theta_1 d\theta_2,$$

where  $g_n^{(*)}(\theta_1, \theta_2) = E(|q_n(\theta_1)q_n(\theta_2)|^p)/n^p$ . For fixed  $\theta_1$  and  $\theta_2$ , neither an integer multiple of  $\pi$ , it is easily checked that the variance covariance matrix of  $(C_n(\theta_1), S_n(\theta_1), C_n(\theta_2), S_n(\theta_2))$  converges to the 4-by-4 identity matrix. In view of the Lindeberg condition already checked the random vector converges in distribution to the standard normal and  $|q_n(\theta_1)q_n(\theta_2)|^p/n^p$  converges in distribution to  $(Z_1^2 + Z_2^2)^p(Z_3^2 + Z_4^2)^p/2^{2p}$  where  $Z_1, \dots, Z_4$  are independent standard normal. Since

$$g_n^{(*)}(\theta_1, \theta_2) \leq h_n^{(*)}(\theta_1, \theta_2) \equiv \left( \frac{E(|q_n(\theta_1)|^{2p})}{n^p} + \frac{E(|q_n(\theta_2)|^{2p})}{n^p} \right) / 2$$

we may apply the first part of the theorem and the dominated convergence theorem to conclude that  $g_n^{(*)}$  converges almost everywhere to  $\Gamma^2(1 + p/2)$ . Moreover we have already checked that  $h_n^{(2)}$  converges almost everywhere to  $\Gamma(1 + p)$  and that

$$\int_0^{2\pi} \int_0^{2\pi} h_n^{(*)}(\theta_1, \theta_2) d\theta_1 d\theta_2 / (4\pi^2) \rightarrow \Gamma(1 + p).$$

Hence

$$\int_0^{2\pi} \int_0^{2\pi} g_n^{(*)}(\theta_1, \theta_2) d\theta_1 d\theta_2 / (4\pi^2) \rightarrow \Gamma^2(1 + p/2)$$

by dominated convergence. This establishes the convergence in probability.

The final statement of the theorem is simply dominated convergence applied via the elementary inequality  $\|q_n/n^{1/2}\|_p \leq 1 + \|q_n/n^{1/2}\|_p^p$ . □

*Proof of Lemma.* For  $\theta \leq \pi/(2n)$  and  $1 \leq j \leq n - 1$  we have  $2j\theta/\pi \leq \sin(j\theta) \leq j\theta$  whence

$$|b_{k,n}| \leq \frac{\pi k \theta}{2\sqrt{\sum_{j=1}^{n-1} j^2 \theta^2}} = \frac{\sqrt{6}\pi k}{2\sqrt{(n-1)n(2n-1)}} \leq \frac{\sqrt{6}\pi/2}{\sqrt{n}}.$$

Moreover, on  $[0, \pi/(2n)]$  the function  $\cos(k\theta)$  is monotone decreasing for all  $k \leq n$ . Hence

$$\begin{aligned} |a_{k,n}(\theta)| &\leq \frac{1}{\sigma_{n,c}(\theta)} \\ &\leq \frac{1}{\sigma_{n,c}(\pi/(2n))} \\ &= \sqrt{\frac{2}{n+1}} \\ &\leq \frac{\sqrt{2}}{\sqrt{n}}. \end{aligned}$$

On the other hand if  $\theta > \pi/(2n)$ , then

$$\sum \{ \cos^2(k\theta) - \sin^2(k\theta) \} = \sin^2(n\theta) + \sin(n\theta) \cos(\theta) \cos(n\theta) / \sin(\theta).$$

Since

$$\frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta) \geq \theta$$

we see that for  $\pi/(2n) \leq \theta \leq \pi/2$  we have

$$\left| \sum \{ \cos^2(k\theta) - \sin^2(k\theta) \} \right| \leq 1 + 1/\tan(\theta) \leq 1 + 2n/\pi \leq \delta_0 n$$

for a  $\delta_0 < 1$  and all  $n \geq 3$ . It follows that

$$\sum \cos^2(k\theta) \geq \frac{1 - \delta_0}{2}n$$

and

$$\sum \sin^2(k\theta) \geq \frac{1 - \delta_0}{2}n$$

from which the lemma follows for  $0 \leq \theta \leq \pi/2$ . Use easy symmetries of the weights  $a_{k,n}$  and  $b_{k,n}$  to get all values of  $\theta$ .  $\square$

Similar techniques permit the extension of our main result to the derivative  $q_n^{(r)}$  of order  $r$ .

**Theorem 2.** Fix  $0 < p < \infty$ . Assume that the random variables  $X_k, k \geq 0$ , are independent and identically distributed, have mean 0, variance equal to 1 and, if  $p > 2$ , a finite  $p^{\text{th}}$  moment  $E(|X_k|^p)$ . Then

$$\frac{E(\|q_n^{(r)}\|_p^p)}{n^{(2r+1)p/2}} \rightarrow (2r+1)^{-p/2}\Gamma(1+p/2)$$

as  $n \rightarrow \infty$ . If, in addition,  $E(|X_k|^{2p}) < \infty$ , then

$$\frac{\|q_n^{(r)}\|_p}{n^{(2r+1)/2}} \rightarrow (2r+1)^{-1/2}(\Gamma(1+p/2))^{1/p}$$

in probability and

$$\frac{E(\|q_n^{(r)}\|_p)}{n^{(2r+1)/2}} \rightarrow (2r+1)^{-1/2}(\Gamma(1+p/2))^{1/p}.$$

#### REFERENCES

- [An-83] Hong-Zhi An, Zhao-Guo Chen and E.J. Hannan, *The Maximum of the Periodogram*, Jour. Mult. Anal. **13** (1983), 383–400. MR **85e**:60033
- [Bec-91] J. Beck, *Flat polynomials on the unit circle – note on a problem of Littlewood*, Bull. London Math. Soc. **23** (1991), 269–277. MR **93b**:42002
- [Be-39] S. Bernstein, *Quelques remarques sur le théorème limite Liapounoff*, Doklad. Akad. Nauk. SSSR **24** (1939), 3–8.
- [Bh-86] A.T. Bharucha-Reid and M. Sambandham, *Random Polynomials*, Academic Press, Orlando, 1986. MR **87m**:60118
- [Bor-98] P. Borwein, *Some Old Problems on Polynomials with Integer Coefficients*, in Approximation Theory IX, ed. C. Chui and L. Schumacher, Vanderbilt Univ. Press (1998), 51–58. CMP 2000:11
- [Bor-95] P. Borwein and T. Erdélyi, *Polynomials and Polynomial Inequalities*, Springer-Verlag, New York, 1995. MR **97e**:41001
- [Boy-97] D. Boyd, *On a problem of Byrnes concerning polynomials with restricted coefficients*, Math. Comput. **66** (1997), 1697–1703. MR **98a**:11033
- [Bri-75] D.R. Brillinger, *Time Series Data Analysis and Theory*, Holt, Rinehart and Winston, New York, 1975.
- [Br-70] B. M. Brown, *Characteristic functions, moments and the central limit theorem*, Ann. Math. Statist. **41** (1970), 658–664. MR **41**:6285
- [Ca-77] F.W. Carrol, D. Eustice and T. Figiel, *The minimum modulus of polynomials with coefficients of modulus one*, Jour. London Math. Soc. **16** (1977), 76–82. MR **58**:1102
- [Cl-59] J. Clunie, *On the minimum modulus of a polynomial on the unit circle*, Quart. J. Math. **10** (1959), 95–98.
- [Er-62] P. Erdős, *An inequality for the maximum of trigonometric polynomials*, Annales Polonica Math. **12** (1962), 151–154. MR **25**:5330

- [Fi-70] G.T. Fielding, *The expected value of the integral around the unit circle of a certain class of polynomials*, Bull. London Math. Soc. **2** (1970), 301–306. MR **43**:6408
- [Ha-73] G. Halász, *On a result of Salem and Zygmund concerning random polynomials*, Studia Sci. Math. Hungar. **8** (1973), 369–377. MR **51**:3787
- [Kac-48] M. Kac, *On the average number of real roots of a random algebraic equation, II*, Proc. London Math. Soc. **50** (1948), 390–408. MR **11**:40e
- [Kah-80] J-P. Kahane, *Sur les polynômes à coefficients unimodulaires*, Bull. London Math. Soc. **12** (1980), 321–342. MR 82a:30003
- [Kah-85] J-P. Kahane, *Some Random Series of Functions*, vol. 5, Cambridge Studies in Advanced Mathematics, Cambridge, 1985; Second Edition. MR **87m**:60119
- [Ko-80] S. Konjagin, *On a problem of Littlewood*, Izv. A. N. SSSR, ser. mat. **45**, **2** (1981), 243–265.
- [Kö-80] T.W. Körner, *On a polynomial of J.S. Byrnes*, Bull. London Math. Soc. **12** (1980), 219–224. MR **81i**:30009
- [Li-61] J.E. Littlewood, *On the mean value of certain trigonometric polynomials*, Jour. London Math. Soc. **36** (1961), 307–334. MR **25**:5331a
- [Li-66] J.E. Littlewood, *On polynomials  $\sum^n \pm z^m$  and  $\sum^n e^{\alpha m i} z^m$ ,  $z = e^{\theta i}$* , Jour. London Math. Soc. **41** (1966), 367–376. MR **33**:4237
- [Li-68] J.E. Littlewood, *Some Problems in Real and Complex Analysis*, Heath Mathematical Monographs, Lexington, Massachusetts, 1968. MR **39**:5777
- [Ma-63] K. Mahler, *On two extremal properties of polynomials*, Illinois J. Math. **7** (1963), 681–701. MR **28**:192
- [Ne-90] D.J. Newman and J. S. Byrnes, *The  $L^4$  norm of a polynomial with coefficients  $\pm 1$* , MAA Monthly **97** (1990), 42–45. MR **91d**:30006
- [Od-93] A. Odlyzko and B. Poonen, *Zeros of polynomials with 0,1 coefficients*, Ens. Math. **39** (1993), 317–348. MR **95b**:11026
- [Qe-96] H. Queffélec and B.Saffari, *On Bernstein's inequality and Kahane's ultraflat polynomials*, J. Fourier Anal. Appl. **2** (1996), 519–582. MR **97h**:41027
- [Ro-97] L. Robinson, *Polynomials With  $\pm 1$  Coefficients: Growth Properties on the Unit Circle*, Simon Fraser University, Masters Thesis (1997). MR
- [Sa-90] B. Saffari, *Barker sequences and Littlewood's "two sided conjectures" on polynomials with  $\pm 1$  coefficients*, Séminaire d'Analyse Harmonique, 1989/90, Univ. Paris XI, Orsay, 1990, 139–151.
- [Sa-54] R. Salem and A. Zygmund, *Some properties of trigonometric series whose terms have random signs*, Acta Math **91** (1954), 254–301. MR **16**:467b
- [Ul-88] D. Ullrich, *An extension of the Kahane-Khinchine inequality*, Bull. Amer. Math. Soc. **18** (1988), 52–54. MR **88k**:60090

DEPARTMENT OF MATHEMATICS AND STATISTICS, SIMON FRASER UNIVERSITY, BURNABY,  
BRITISH COLUMBIA, CANADA V5A 1S6

*E-mail address:* pborwein@cecm.sfu.ca

DEPARTMENT OF MATHEMATICS AND STATISTICS, SIMON FRASER UNIVERSITY, BURNABY,  
BRITISH COLUMBIA, CANADA V5A 1S6

*E-mail address:* lockhart@sfu.ca