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LOCAL COMPLETENESS AND DUAL LOCAL QUASI-COMPLETENESS

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ABSTRACT. It is proved that l^q-completeness ($1 < q < \infty$) is equivalent to l^1-completeness (defined by Saxon and Sánchez Ruiz), and becomes a new characteristic condition for local completeness. The relationship between dual local completeness, dual local quasi-completeness and the Banach-Mackey property is investigated. For a quasi-Mackey space, dual local quasi-completeness, c_0-quasi-barrelledness, Ruess' property (quasi-L) and C-quasi-barrelledness are equivalent to each other.

1. LOCAL COMPLETENESS

In this paper, every space will be assumed a Hausdorff locally convex space over the scalar field of real or complex numbers. Let (E,t) be a space; then $(E,t)'$, or briefly E', denotes the topological dual of (E,t) and $E^{\#}$ denotes the algebraic dual of E. Let E'' denote $(E',\beta(E',E))'$. Recall that a space E is locally complete if and only if every bounded closed absolutely convex subset of E is a Banach disk [4, Proposition 5.1.6]. It is easy to see that local completeness is duality invariant. That is to say, if two spaces (E,t) and (E,s) have the same topological dual and one is locally complete, then so is the other. P. Dierolf ([2] or [4, Theorem 5.1.11]) proved that the following conditions on a space (E,t) are equivalent:

(I) (E,t) is locally complete.

(II) The closed absolutely convex hull of every locally null sequence in (E,t) is compact.

(III) The closed absolutely convex hull of every null sequence in $(E,\sigma(E,E'))$ is compact in $(E,\sigma(E,E'))$.

(IV) The closed absolutely convex hull of every null sequence in (E,t) is compact.

From [4, the proof of Proposition 3.2.12], we also see that (E,t) is locally complete if and only if for each null sequence (x_n) in (E,t) and each scalar sequence $(\lambda_n)\in l^1$, the series $\sum_{n=1}^{\infty}\lambda_n x_n$ converges in (E,t). Recently Saxon and Sánchez Ruiz [6] proved that a space E is locally complete if and only if it is l^1-complete. Reminiscently of De Wilde [1, Proposition III.1.4 and V.3.2], they defined a space E to be l^1-complete if, for each bounded sequence $(x_n)\subset E$ and each $(\lambda_n)\in l^1$, the series $\sum_{n=1}^{\infty}\lambda_n x_n$ converges in E. We shall extend the concept to l^q-completeness

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$(1 \le q \le \infty)$. If $1 \le p \le \infty$, let $l^p(E)$ denote all sequences (x_n) in E such that $(\rho(x_n)) \in l^p$ for each continuous seminorm ρ on E (see [4, Definition 4.8.1 and 4.8.2]). Thus, for example, $l^\infty(E)$ denotes all bounded sequences in E.

Definition 1. A space E is said to be l^q-complete $(1 \le q \le \infty)$ if, for each $(\lambda_n) \in l^q$ and each $(x_n) \in l^p(E)$, the series $\sum_{n=1}^\infty \lambda_n x_n$ converges in E, where $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 1. When $q = 1$ we have l^1-completeness defined just as in [6]. When $q = \infty$ we have l^∞-completeness of E, clearly equivalent to the following condition: for every $(x_n) \in l^1(E)$, the series $\sum_{n=1}^\infty x_n$ converges in E; i.e., every absolutely Cauchy series converges in E. If every unconditionally Cauchy series converges, E is said to be $\sum$-complete [4, 5.3]. Here, $\sum_{n=1}^\infty x_n$ unconditionally Cauchy means that each $\sum_{n\in\sigma} x_n$ is arbitrarily close to the origin whenever σ is a finite set of sufficiently large positive integers. Clearly, $\sum$-complete $\Rightarrow$ l^∞-complete. The converse fails, as shown by the following Example 1.

Example 1. For any $x = (\xi_n) \in c_0$, define $\|x\| = \sup_n |\xi_n|$. Then $(c_0, \| \ \|)' = l^1$ and $(l^1, \beta(l^1, c_0)) = (l^1, \| \ \|_1)$, where for any $y = (\eta_n) \in l^1$, $\|y\|_1 = \sum_{n=1}^\infty |\eta_n|$. As is well known, a subset K of $(l^1, \| \ \|_1)$ is relatively compact if and only if the following conditions are satisfied:

(I) there is $M > 0$ such that $\sum_{n=1}^\infty |\eta_n| \le M$ for any $y = (\eta_n) \in K$;
(II) for any $\epsilon > 0$, there is $m_0 \in N$ such that when $m \ge m_0$, $\sum_{n=m}^\infty |\eta_n| < \epsilon$ for every $y = (\eta_n) \in K$.

For any relatively compact subset K of $(l^1, \| \ \|_1)$, we define a seminorm p_K on c_0 as follows:

$$p_K(x) = \sup\left\{ \left| \sum_{n=1}^\infty \xi_n \eta_n \right| : \ y = (\eta_n) \in K \right\},$$

for any $x = (\xi_n) \in c_0$. Let t denote the topology on c_0 generated by all seminorms p_K as above. Since $(l^1, \| \ \|_1)$ is complete and in $(l^1, \| \ \|_1)$ the weakly compact sets and strong compact sets coincide, we may conclude that $(c_0, t) = (c_0, \tau(l^\infty, l^1)|c_0)$. Thus $(E, t) := (c_0, t)$ illustrates what we defined in the next section as a quasi-Mackey space. Consider a series $\sum_{n=1}^\infty x_n$ in E. If each x_n has 1 at the nth coordinate and 0's elsewhere, (II) ensures that $p_K(\sum_{n\in\sigma} x_n) < \epsilon$ whenever each member m of the finite set σ satisfies $m \ge m_0$. Thus the series is unconditionally Cauchy but not even weakly convergent, proving E is not $\sum$-complete. On the other hand, given any series in E with $\sum_{n=1}^\infty \|x_n\| = \infty$, we may choose a sequence (b_n) of posive scalars tending to 0 so slowly that $\sum_{n=1}^\infty b_n \|x_n\| = \infty$ still holds. We then choose each $y_n \in l^1$ with $\|y_n\| = b_n$ and $\sum_{i=1}^\infty x_n(i) \cdot y_n(i) = \|x_n\| \cdot \|y_n\|_1 = b_n \|x_n\|$. The null sequence (y_n) is a relatively compact set K in $(l^1, \| \ \|_1)$ such that each $p_K(x_n) \ge \|x_n\| b_n$, which implies that $\sum_{n=1}^\infty p_K(x_n) = \infty$. This proves that any absolutely Cauchy series in (E, t) is also absolutely Cauchy and then convergent in the Banach space $(c_0, \| \ \|)$, and therefore convergent in the weaker topology t. We conclude that the non-$\sum$-complete (E, t) is l^∞-complete.

Remark 2. l^∞-completeness certainly implies l^1-completeness, equivalently, local completeness [6], but the converse is false. Indeed, Pérez Carreras and Bonet [4, Example 5.1.12] noted that. By duality invariance, $E := (c_0, \sigma(c_0, l^1))$ is locally complete but is not l^∞-complete since the canonical unit vectors are not summable in E. (They put "sequentially complete" in place of "l^∞-complete".)

It is somewhat surprising that l^q-completeness $(1 < q < \infty)$ is equivalent to l^1-completeness. Thus l^q-completeness $(1 < q < \infty)$ becomes a new characteristic condition for local completeness.

Theorem 1. *For any space E the following statements are equivalent:*
(I) *E is locally complete.*
(II) (Saxon and Sánchez Ruiz) *E is l^1-complete.*
(III) *E is l^q-complete* $(1 < q < \infty)$.

Proof. The equivalence of (I) and (II) is Theorem 2.1 of [6].

(III) $\Rightarrow$ (II): If $(\lambda_n) \in l^1$ and $(x_n) \in l^\infty(E)$, then for $1 < q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$\sum_{n=1}^{\infty} \lambda_n x_n = \sum_{n=1}^{\infty} |\lambda_n|^{\frac{1}{q}} \, |\lambda_n|^{\frac{1}{p}} \, (sgn\, \lambda_n) \, x_n$$

converges in E by (III), since $(|\lambda_n|^{\frac{1}{q}}) \in l^q$ and $(|\lambda_n|^{\frac{1}{p}} \, (sgn\, \lambda_n) \, x_n) \in l^p$.

(II) $\Rightarrow$ (III): Suppose $(\lambda_n) \in l^q$ and $(x_n) \in l^p(E)$, with p and q as above. Since $\sum_{n=1}^{\infty} |\lambda_n|^q < \infty$, we may inductively find $1 = m_1 < m_2 < \cdots$ such that, defining $\sigma_k = \{n \in N : \, m_k \le n < m_{k+1}\}$ for $k = 1, 2, \cdots$, we have

$$\sum_{n \in \sigma_k} |\lambda_n|^q \le 2^{-kq} \; for \; k = 2, 3, \cdots .$$

We must use (II) to show that $\sum_{n=1}^{\infty} \lambda_n x_n$ converges in E. The series is absolutely Cauchy by the Hölder inequality, so we need only show that its sequence of partial sums has a convergent subsequence. Define $y_k = 2^k \sum_{n \in \sigma_k} \lambda_n x_n$ for $k \ge 1$. Given a continuous seminorm ρ, the sum $(\sum_{n=1}^{\infty} [\rho(x_n)]^p)^{\frac{1}{p}} := M$ is finite. Thus for $k \ge 2$ the triangle and Hölder inequalities imply that

$$\rho(y_k) \, \le \, 2^k \left(\sum_{n \in \sigma_k} |\lambda_n|^q \right)^{\frac{1}{q}} \left(\sum_{n \in \sigma_k} [\rho(x_n)]^p \right)^{\frac{1}{p}} \, \le \, 2^k \, (2^{-kq})^{\frac{1}{q}} \, M \, = \, M.$$

Hence $(y_k) \in l^\infty(E)$. Now $(2^{-k}) \in l^1$ and (II) implies that $\sum_{k=1}^{\infty} 2^{-k} y_k$ converges in E, so its sequence of partial sums is the desired convergent subsequence. □

We end this section with a brief discussion of the relationship between convex compactness (cc) and various notions of completeness. Recall that a space E is said to have the convex compactness property if the closed absolutely convex hull of every compact subset of E is still compact [10, Definition 9-2-8]. In [4, 5.3], E is said to be p-complete (in [10, Problem 6-5-107], N-complete; see [3, §23, 9(1)]) if every precompact subset of E is relatively compact; i.e. every closed totally bounded set is compact. Surely, quasi-complete $\Rightarrow$ p-complete, Wheeler denied the converse [10, Table 4], p-complete $\Rightarrow$ cc [10, Problem 9-2-111], and cc $\Rightarrow$ locally complete by (iv) of Dierolf's Theorem. An amplified Pérez Carreras/Bonet scheme (see [4, 5.3]) emerges:

$$\begin{array}{ccccccccccccc}
\text{complete} & \overset{\Rightarrow}{\not\Leftarrow} & \text{q-c} & \overset{\Rightarrow}{\not\Leftarrow} & \text{p-c} & \overset{\Rightarrow}{\not\Leftarrow} & \text{s-c} & \overset{\Rightarrow}{\not\Leftarrow} & \sum\text{-c} & \overset{\Rightarrow}{\not\Leftarrow} & l^\infty\text{-c} & \overset{\Rightarrow}{\not\Leftarrow} & l^q\text{-c} \quad (1 \le q < \infty) \Leftrightarrow lc \\
 & & & & \Downarrow\not\Uparrow & & & & & & & & \not\Downarrow\Uparrow \\
 & & & & \text{convex} & & & & \text{compactness} & & & & \text{property}
\end{array}$$

Here, q-c, p-c, s-c, $\sum$-c, l^∞-c, l^q-c and lc respectively denote "quasi-complete", "p-complete", "sequentially complete", "$\sum$-complete", "l^∞-complete", "l^q-complete" and "locally complete". The convex compactness property cannot be better placed, since s-c $\not\Rightarrow$ cc $\not\Rightarrow$ l^∞-c. Indeed, Ostling and Wilansky found an s-c space without the cc property [10, Problem 9-2-301], and $(c_0, \sigma(c_0, l^1))$ has the cc property [10, Theorem 14-2-4] but is not l^∞-c (Remark 2). All eight notions of completeness coincide under metrizability [4, Corollary 5.1.9].

2. Dual local quasi-completeness

Recall that a space E is dual locally complete if $(E', \sigma(E', E))$ is locally complete (cf. [6]). Saxon and Sánchez Ruiz [6] investigated dual local completeness and obtained a number of interesting results. Each characterization of local completeness gives one of dual local completeness and vice versa, since [E is locally complete] $\Leftrightarrow$ [$(E, \sigma(E, E'))$ is locally complete] $\Leftrightarrow$ [$(E', \sigma(E', E))$ is dual locally complete]. Thus, for example, the Ruess characterization in [6, Theorem 2.3(4)] may be written as follows:

(Ruess) A space E is locally complete if and only if, given any $h \in E'^{\#}$ and any absolutely convex absorbing set A in E' such that $h|_A$ is relatively $\sigma(E', E)$-continuous, it must be so that h is $\sigma(E', E)$-continuous; i.e., there exists $x_0 \in E$ such that $h = \hat{x}_0$, or $h(f) = f(x_0)$ for every $f \in E'$.

Following Tsirulnikov [9], a space E is called dual locally quasi-complete if its strong dual $(E', \beta(E', E))$ is locally complete. Tsirulnikov, Ruess, et al. surely knew that dual locally complete (dlc) $\Rightarrow$ dual locally quasi-complete (dlqc); indeed, this follows from [4, Proposition 5.1.6(iv)], or we may observe that if $\sum_{n=1}^\infty \lambda_n f_n$ converges to f in $(E', \sigma(E', E))$ with $(\lambda_n) \in l^1$, and if (f_n) is $\beta(E', E)$-bounded, then, routinely, $\sum_{n=1}^\infty \lambda_n f_n$ converges to f uniformly on bounded sets of E, i.e. in $(E', \beta(E', E))$.

Let (E, t) be a space. An increasing sequence $\sigma = \{A_n : n \in N\}$ of absolutely convex subsets of E is said to be absorbing (bornivorous) if for every x in E (every bounded subset B of E) there is a positive integer m such that A_m absorbs x (absorbs B); see [4, Definition 8.1.15]. We denote by t_σ the finest locally convex topology on E that induces the same topology as t on each A_n. The topology t_σ is defined by the family of those seminorms whose restrictions to the sets A_n are continuous for the topology induced on A_n by t (cf. [6]). Absorbing and bornivorous sequences of absolutely convex sets were considered by Valdivia, De Wilde, Houet, Garling, Roelcke, Ruess, et al. (cf. [4, 8.9]). Ruess [5] defined a space (E, t) to have property ([*quasi*-]L) if $t = t_\sigma$ holds for each absorbing [bornivorous] sequence σ, and to have the weaker property ([*quasi*-]LC) if each t_σ is compatible with the dual pair (E, E'). We remark that in the above definitions of Ruess' four properties ([*quasi*-]L) and ([*quasi*-]LC), it makes no difference whether the absorbing [bornivorous] sequences are required to be closed or not, by [4, Proposition 8.1.17(i)].

Ruess' characterizations of dual local [quasi-]completeness also make it clear that dlc $\Rightarrow$ dlqc. Please refer to [4, Proposition 8.1.29] and [6, Theorem 2.3].

Theorem 2 (Ruess). *For any space E the following statements are equivalent:*

(I) *E is dual locally* [*quasi-*]*complete.*

(II) *E has property* ([*quasi-*]LC).

(III) *If $f \in E^{\#}$ such that $f|_A$ is continuous, where A is a* [*bornivorous*] *barrel in E, then $f \in E'$.*

A modern statement of the Banach-Mackey Theorem [6] is that every dlc space E has the Banach-Mackey property; i.e., the $\sigma(E', E)$-bounded sets are $\beta(E', E)$-bounded, or, equivalently, barrels in E are bornivores. The converse is evidently true when E is dlqc, and this observation transforms the Banach-Mackey theorem into another characterization of dual local completeness that augments, along with dual l^q-completeness $(1 < q < \infty)$, the collection in [6].

Theorem 3 (Banach-Mackey). *A space E is dual locally complete if and only if it is dual locally quasi-complete and has the Banach-Mackey property. That is to say, a dual locally quasi-complete space is dual locally complete if and only if it has the Banach-Mackey property.*

When E is a Mazur space [10, Definition 8-6-3], then $(E', \beta(E', E))$ is complete [10, Corollary 8-6-6], and is certainly locally complete; i.e., Mazur $\Rightarrow$ dlqc. By Theorem 3, then, a Mazur space is dlc if and only if it has the Banach-Mackey property. This is just [6, Theorem 2.6], viewed now as a corollary to the Banach-Mackey Theorem. Note that E need not be a Mazur space even though $(E', \beta(E', E))$ is complete; e.g., take $E = (l^1, \tau(l^1, c_0))$ [10, Problem 8-6-119]. Hence, Theorem 3 is a useful characterization of dual local completeness which properly extends [6, Theorem 2.6].

The observations that

(I) E is dlc $\Rightarrow$ E is dlqc $\Leftrightarrow$ $(E'', \sigma(E'', E'))$ is dlc, and

(II) each $\sigma(E', E)$-bounded set is $\beta(E', E'')$-bounded $\Leftrightarrow$ both E, $(E'', \sigma(E'', E'))$ have the Banach-Mackey property

combine with Theorem 3 as follows.

Corollary 1. *In a dlc space E, each $\sigma(E', E)$-bounded set is $\beta(E', E'')$-bounded.*

Remark 3. Banach-Mackey $\overset{\not\Rightarrow}{\not\Leftarrow}$ dlqc: Trivially, any non-barrelled normed space is dlqc but is without the Banach-Mackey property. Conversely, let $(X, \|\ \|)$ be any barrelled normed space which is not complete [10, Problem 3-1-4], equivalently, not lc, and put $E := (X', \tau(X', X))$. Then $(E', \beta(E', E)) = (X, \beta(X, X')) = (X, \|\ \|)$ is not locally complete; i.e. E is not dlqc. But E does have the Banach-Mackey property, since $\sigma(E', E)$-bounded $\Leftrightarrow$ $\sigma(X, X')$-bounded $\Leftrightarrow$ $\beta(X, X')$-bounded $\Leftrightarrow$ $\beta(E', E)$-bounded.

Remark 4. Note in the above that $E = E''$. Therefore, [each $\sigma(E', E)$-bounded set is $\beta(E', E'')$-bounded] $\not\Rightarrow$ [E is dlqc].

Mazon defined a space E to be C-[*quasi-*]*barrelled* [4, Definition 8.2.6] if $U := \bigcap_{n=1}^{\infty} U_n$ is a neighborhood of 0 whenever (U_n) is a sequence of absolutely convex closed neighborhoods of 0 such that any given singleton [bounded] set is contained in U_n for almost all n. According to Webb, (E, t) is c_0-[*quasi-*]*barrelled* if each $\sigma(E', E)$-null [$\beta(E', E)$-null] sequence is t-equicontinuous. It is known [4, Proposition 8.1.29, Observation 8.2.7 and 8.2.23] that C-[quasi-]barrelled implies both c_0-[quasi-]barrelled and Ruess' property ([quasi-]L), either of which implies dual locally [quasi-]complete, respectively. Furthermore, each one of the four quasi-exclusive conditions implies the corresponding quasi-inclusive condition. It has also been known for some time that, in both the quasi-inclusive and -exclusive cases, the latter three notions coincide under the assumption of the Mackey topology (see [4, Proposition 8.1.29 and Observation 8.2.23(c)]). Recently, Saxon and Sánchez Ruiz

[7] showed that [Mackey and dlc] $\Rightarrow$ [C-barrelled], so that, in the quasi-exclusive case, all four notions coincide for Mackey spaces. Their proof applies equally well to the inclusive case.

Theorem 4 (Saxon and Sánchez Ruiz). *For a Mackey space E the following are respectively equivalent:*
(a) *E is dual locally* [*quasi-*]*complete.*
(b) *E is c_0-*[*quasi-*]*barrelled.*
(c) *E has property* ([*quasi-*]*L*).
(d) *E is C-*[*quasi-*]*barrelled.*

Thus a space E with dual local quasi-completeness, the weakest of the four "quasi" properties, suddenly enjoys C-quasi-barrelledness, the strongest of the four, when it is endowed with its Mackey topology. Let us identify another such compatible topology.

Definition 2. A space E is called a *quasi-Mackey space* if it has its *quasi-Mackey topology;* i.e., the topology induced by $(E'', \tau(E'', E'))$.

Köthe [3, §23,4.(6)] noted that the quasi-Mackey topology is compatible with the pairing (E, E') and may be strictly coarser than the Mackey topology $\tau(E, E')$, e.g., when $E = c_0$ with $E' = l^1$. This permits a new parallel to Theorem 4's quasi-inclusive case.

Theorem 5. *For a quasi-Mackey space E the following statements are equivalent:*
(a) *E is dual locally quasi-complete.*
(b) *E is c_0-quasi-barrelled.*
(c) *E has property* (*quasi-L*).
(d) *E is C-quasi-barrelled.*

Proof. It suffices to show that (a) $\Rightarrow$ (d): Suppose $U := \bigcap_{n=1}^{\infty} U_n$ is given as in the definition of C-quasibarrelled. For $A \subset E'$, let A° and $A^{\bullet}$ denote the polar of A in E and E'', respectively. Since E is quasi-Mackey, for each U_n there is an absolutely convex $\sigma(E', E'')$-compact set C_n such that $U_n \supset C_n^{\bullet} \cap E = C_n^{\circ}$. As C_n is also $\sigma(E', E)$-compact and hence $\sigma(E', E)$-closed, we have $C_n = C_n^{\circ\circ} \supset U_n^{\circ}$. Therefore $C_n^{\bullet} \subset U_n^{\circ\bullet}$, and the latter is a 0-neighborhood in $(E'', \tau(E'', E'))$. Given $z \in E''$, there exists a bounded set $B \subset E$ such that $z \in B^{\circ\bullet}$, and by definition, $B \subset U_n$ for almost all n. Therefore $z \in U_n^{\circ\bullet}$ for almost all n. Duality invariance and (a) imply that $(E'', \tau(E'', E'))$ is dlc; hence C-barrelled, by Theorem 4. It follows that $W := \bigcap_{n=1}^{\infty} U_n^{\circ\bullet}$ is a 0-neighborhood in $(E'', \tau(E'', E'))$, and thus

$$W \bigcap E = \bigcap_{n=1}^{\infty} (U_n^{\circ\bullet} \cap E) = \cap_{n=1}^{\infty} U_n^{\circ\circ} = U$$

is a 0-neighborhood in the quasi-Mackey space E. □

By definition, a c_0-quasi-barrelled space remains so under any finer compatible topology. Dierolf's Theorem and Theorem 5 each implies that if (E, t) is dlqc with t finer than the quasi-Mackey topology, then (E, t) is c_0-quasi-barrelled. This properly extends the quasi-inclusive case of (a) $\Leftrightarrow$ (b), Theorem 4.

Corollary 2. *A space whose topology lies between its Mackey and quasi-Mackey topologies is c_0-quasi-barrelled if and only if it is dual locally quasi-complete.*

We cannot similarly extend the quasi-exclusive case of (a) $\Leftrightarrow$ (b), Theorem 4: If we take $E = c_0$ with $E' = l^1$ and give E its quasi-Mackey topology, as in Example 1, then E is dlc but is not c_0-barrelled, since the canonical unit vectors are not equicontinuous on E. *A fortiori*, E is not C-barrelled. Yet E does have property (L), as noted in [8]. In fact, Theorem 5 provides a general proof: E has property (*quasi*-L), and the Banach-Mackey property says that every closed absorbing sequence is bornivorous [6, Theorem 2.4].

Corollary 3. *A quasi-Mackey space has property* (L) *if and only if it is dual locally complete.*

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References

[1] M. De Wilde, Closed Graph Theorems and Webbed Spaces, Pitman, 1978. MR **81j:**46013
[2] P. Dierolf, Une characterisation des espaces vectoriels complets au sens de Mackey, C. R. Acad. Sci. Paris. **283** (1976), 245-248.
[3] G. Köthe, Topological Vector Spaces I, Springer-Verlag, 1969. MR **40:**1750
[4] P. Pérez Carreras and J. Bonet, Barrelled Locally Convex Spaces, Math. Studies **131**, North Holland, 1987. MR **88j:**46003
[5] W. Ruess, A Grothendieck representation for the completion of cones of continuous seminorms, Math. Ann. **208**(1974), 71-90. MR **50:**14147
[6] S. A. Saxon and L. M. Sánchez Ruiz, Dual local completeness, Proc. Amer. Math. Soc. **125**(1997), 1063-1070. MR **97g:**46004
[7] S. A. Saxon and L. M. Sánchez Ruiz, Mackey weak barrelledness, Proc. Amer. Math. Soc. **126**(1998), 3279-3282. MR **99a:**46006
[8] S. A. Saxon and L. M. Sánchez Ruiz, Reinventing weak barrelledness, preprint.
[9] B. Tsirulnikov, On conservation of barrelledness properties in locally convex spaces, Bull. Soc. Royale Sci. Liège **49**(1980), 5-25. MR **82c:**46005
[10] A. Wilansky, Modern Methods in Topological Vector Spaces, McGraw-Hill, 1978. MR **81d:**46001

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