HOLOMORPHIC PERTURBATION
OF FOURIER COEFFICIENTS

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ABSTRACT. Let $\mathbb{T}$ be the unit circle, let $\mathcal{B}$ be a Banach space continuously embedded in $L^1(\mathbb{T})$ and suppose that $\mathcal{B}$ is a Banach $L^1(\mathbb{T})$-module under convolution. We show that if $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in \mathcal{B}$ and $F$ is holomorphic in a neighbourhood $U$ of 0 with $F(0) = 0$ and $a_n \in U$ ($n \in \mathbb{Z}$), then $\sum_{n=-\infty}^{\infty} F(a_n)z^n \in \mathcal{B}$.

Recently, Render ([3]) proved the result stated above for the case where $\mathcal{B}$ is the space $H^\infty(\mathbb{D})$ of bounded analytic functions on the open unit disc $\mathbb{D}$. In this note, we show that Render’s result follows easily from the theory of so-called abstract Segal algebras, and that this approach allows us to generalize his result to a large class of spaces of functions. We also give a direct proof of this result.

For $f, g \in L^1(\mathbb{T})$, we define the convolution product $f \ast g$ by

$$(f \ast g)(e^{it}) = \frac{1}{2\pi} \int_{\mathbb{T}} f(e^{is})g(e^{i(t-s)}) \, ds \quad (e^{it} \in \mathbb{T}).$$

Then $\hat{f \ast g}(n) = \hat{f}(n)\hat{g}(n)$, where $\hat{f}(n)$ ($n \in \mathbb{Z}$) are the Fourier coefficients of $f$.

**Proposition 1.** Let $\mathcal{B}$ be a Banach space continuously embedded in $L^1(\mathbb{T})$ and suppose that $\mathcal{B}$ is a Banach $L^1(\mathbb{T})$-module under convolution. Let $E = \{n \in \mathbb{Z} : \text{there exists } f \in \mathcal{B} \text{ with } \hat{f}(n) \neq 0\}$. Considered as a Banach algebra under convolution, every character on $\mathcal{B}$ is of the form

$$f \mapsto \hat{f}(n) \quad (f \in \mathcal{B})$$

for some $n \in E$.

**Proof.** Let $L^1_E(\mathbb{T}) = \{f \in L^1(\mathbb{T}) : \hat{f}(n) = 0 \text{ for } n \in \mathbb{Z} \setminus E\}$. For $n \in E$, choose $f \in \mathcal{B}$ with $\hat{f}(n) \neq 0$. Then $e^{int} = (e^{int} \ast f)/\hat{f}(n) \in \mathcal{B}$, so $\mathcal{B}$ contains every trigonometric polynomial in $L^1_E(\mathbb{T})$, and in particular $\mathcal{B}$ is dense in $L^1_E(\mathbb{T})$. Hence $\mathcal{B}$ is an abstract Segal algebra with respect to $L^1_E(\mathbb{T})$ in the sense of [1] Definition 1.1, so the result follows from [1] Theorem 2.1] (see [2] Theorem 6.2.4 for the case $E = \mathbb{Z}$).

**Remark 2.** Let $\mathcal{B}$ be a Banach space continuously embedded in $L^1(\mathbb{T})$ and for $f \in \mathcal{B}$ and $e^{is} \in \mathbb{T}$, let $f_s(e^{it}) = f(e^{it-s})$ ($e^{it} \in \mathbb{T}$). If $f_s \in \mathcal{B}$ with $\|f_s\|_{\mathcal{B}} = \|f\|_{\mathcal{B}}$ and

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Moreover, since $B$ is a dual space (like $H^\infty(\mathbb{D})$) and $f_s \to f$ weak star in $B$ as $e^{ix} \to 1$, then

$$
g \ast f = \frac{1}{2\pi} \int_T g(e^{ix}) f_s \, ds \in B$$

and $\|g \ast f\|_B \leq \|g\|_{L^1(T)} \|f\|_B$ for $g \in L^1(T)$, so $B$ is a Banach $L^1(T)$-module under convolution. This provides a large class of spaces of functions to which our results apply.

With $B$ as above, let $A = B \oplus \mathbb{C} \delta_1$ be the Banach algebra obtained by adjoining the identity $\delta_1$ (the Dirac measure at $z = 1$) to $B$.

**Corollary 3.** For $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in B$, we have $\sigma_A(f) = \{a_n : n \in E\} \cup \{0\}$.

Our main result now follows from the holomorphic functional calculus for Banach algebras (see, for instance, [4, Theorem 10.2.7]).

**Theorem 4.** Let $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in B$ and let $F$ be holomorphic in a neighbourhood $U$ of 0 with $F(0) = 0$ and $a_n \in U \quad (n \in \mathbb{Z})$. Then $\sum_{n=-\infty}^{\infty} F(a_n) z^n \in B$.

**Remark 5.** Writing $F(z) = zG(z) \quad (z \in U)$, the conclusion of the theorem takes the form $\sum_{n=-\infty}^{\infty} G(a_n) a_n z^n \in B$; that is, the sequence $(G(a_n))$ acts as a “local coefficient multiplier” on $f$.

We wish to point out that it is possible to give a proof of Corollary 3 (and thus of Theorem 4) which does not depend on the theory of abstract Segal algebras: Let $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in B$ and suppose that $\lambda \notin \{a_n : n \in E\} \cup \{0\}$. It is well known that every character on $L^1(T)$ is of the form $g \mapsto \hat{g}(n) \quad (g \in L^1(T))$ for some $n \in \mathbb{Z}$, so it follows that $\lambda \notin \sigma_{L^1(T)\oplus \mathbb{C} \delta_1}(f)$. Since the Fourier series of $\delta_1$ is $\sum_{n=-\infty}^{\infty} z^n$, we thus have

$$
(\lambda \delta_1 - f)^{-1} = \sum_{n=-\infty}^{\infty} \frac{1}{\lambda - a_n} z^n = \frac{1}{\lambda} \sum_{n=-\infty}^{\infty} \frac{a_n}{\lambda - a_n} z^n + \frac{1}{\lambda} \delta_1 \in L^1(T) \oplus \mathbb{C} \delta_1.
$$

Moreover, since $B$ is a Banach $L^1(T)$-module under convolution, we have

$$
\sum_{n=-\infty}^{\infty} \frac{a_n}{\lambda - a_n} z^n = (\lambda \delta_1 - f)^{-1} \ast f \in B,
$$

so $(\lambda \delta_1 - f)^{-1} \in A$, and we deduce that $\lambda \notin \sigma_A(f)$.

**REFERENCES**


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