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HOLOMORPHIC PERTURBATION OF FOURIER COEFFICIENTS

THOMAS VILS PEDERSEN

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ABSTRACT. Let $\mathbb{T}$ be the unit circle, let $\mathcal{B}$ be a Banach space continuously embedded in $L^1(\mathbb{T})$ and suppose that $\mathcal{B}$ is a Banach $L^1(\mathbb{T})$-module under convolution. We show that if $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in \mathcal{B}$ and F is holomorphic in a neighbourhood U of 0 with $F(0) = 0$ and $a_n \in U$ $(n \in \mathbb{Z})$, then $\sum_{n=-\infty}^{\infty} F(a_n) z^n \in \mathcal{B}$.

Recently, Render ([3]) proved the result stated above for the case where $\mathcal{B}$ is the space $H^\infty(\mathbb{D})$ of bounded analytic functions on the open unit disc $\mathbb{D}$. In this note, we show that Render's result follows easily from the theory of so-called abstract Segal algebras, and that this approach allows us to generalize his result to a large class of spaces of functions. We also give a direct proof of this result.

For $f, g \in L^1(\mathbb{T})$, we define the convolution product $f * g$ by

$$(f * g)(e^{it}) = \frac{1}{2\pi} \int_{\mathbb{T}} f(e^{is}) g(e^{i(t-s)})\, ds \qquad (e^{it} \in \mathbb{T}).$$

Then $\widehat{f * g}(n) = \widehat{f}(n)\widehat{g}(n)$, where $\widehat{f}(n)$ $(n \in \mathbb{Z})$ are the Fourier coefficients of f.

Proposition 1. *Let $\mathcal{B}$ be a Banach space continuously embedded in $L^1(\mathbb{T})$ and suppose that $\mathcal{B}$ is a Banach $L^1(\mathbb{T})$-module under convolution. Let $E = \{n \in \mathbb{Z} :$ there exists $f \in \mathcal{B}$ with $\widehat{f}(n) \neq 0\}$. Considered as a Banach algebra under convolution, every character on $\mathcal{B}$ is of the form*

$$f \mapsto \widehat{f}(n) \qquad (f \in \mathcal{B})$$

for some $n \in E$.

Proof. Let $L^1_E(\mathbb{T}) = \{f \in L^1(\mathbb{T}) : \widehat{f}(n) = 0$ for $n \in \mathbb{Z} \setminus E\}$. For $n \in E$, choose $f \in \mathcal{B}$ with $\widehat{f}(n) \neq 0$. Then $e^{int} = (e^{int} * f)/\widehat{f}(n) \in \mathcal{B}$, so $\mathcal{B}$ contains every trigonometric polynomial in $L^1_E(\mathbb{T})$, and in particular $\mathcal{B}$ is dense in $L^1_E(\mathbb{T})$. Hence $\mathcal{B}$ is an abstract Segal algebra with respect to $L^1_E(\mathbb{T})$ in the sense of [1, Definition 1.1], so the result follows from [1, Theorem 2.1] (see [2, Theorem 6.2.4] for the case $E = \mathbb{Z}$). □

Remark 2. Let $\mathcal{B}$ be a Banach space continuously embedded in $L^1(\mathbb{T})$ and for $f \in \mathcal{B}$ and $e^{is} \in \mathbb{T}$, let $f_s(e^{it}) = f(e^{i(t-s)})$ $(e^{it} \in \mathbb{T})$. If $f_s \in \mathcal{B}$ with $\|f_s\|_{\mathcal{B}} = \|f\|_{\mathcal{B}}$ and

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$f_s \to f$ in $\mathcal{B}$ as $e^{is} \to 1$, or if $\mathcal{B}$ is a dual space (like $H^\infty(\mathbb{D})$) and $f_s \to f$ weak star in $\mathcal{B}$ as $e^{is} \to 1$, then

$$g * f = \frac{1}{2\pi} \int_{\mathbb{T}} g(e^{is}) f_s \, ds \in \mathcal{B}$$

and $\|g * f\|_{\mathcal{B}} \le \|g\|_{L^1(\mathbb{T})} \|f\|_{\mathcal{B}}$ for $g \in L^1(\mathbb{T})$, so $\mathcal{B}$ is a Banach $L^1(\mathbb{T})$-module under convolution. This provides a large class of spaces of functions to which our results apply.

With $\mathcal{B}$ as above, let $\mathcal{A} = \mathcal{B} \oplus \mathbb{C}\delta_1$ be the Banach algebra obtained by adjoining the identity δ_1 (the Dirac measure at $z = 1$) to $\mathcal{B}$.

Corollary 3. *For* $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in \mathcal{B}$, *we have* $\sigma_{\mathcal{A}}(f) = \{a_n : n \in E\} \cup \{0\}$.

Our main result now follows from the holomorphic functional calculus for Banach algebras (see, for instance, [4, Theorem 10.2.7]).

Theorem 4. *Let* $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in \mathcal{B}$ *and let* F *be holomorphic in a neighbourhood* U *of* 0 *with* $F(0) = 0$ *and* $a_n \in U$ $(n \in \mathbb{Z})$. *Then* $\sum_{n=-\infty}^{\infty} F(a_n) z^n \in \mathcal{B}$.

Remark 5. Writing $F(z) = zG(z)$ $(z \in U)$, the conclusion of the theorem takes the form $\sum_{n=-\infty}^{\infty} G(a_n) a_n z^n \in \mathcal{B}$; that is, the sequence $(G(a_n))$ acts as a "local coefficient multiplier" on f.

We wish to point out that it is possible to give a proof of Corollary 3 (and thus of Theorem 4) which does not depend on the theory of abstract Segal algebras: Let $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in \mathcal{B}$ and suppose that $\lambda \notin \{a_n : n \in E\} \cup \{0\}$. It is well known that every character on $L^1(\mathbb{T})$ is of the form $g \mapsto \widehat{g}(n)$ $(g \in L^1(\mathbb{T}))$ for some $n \in \mathbb{Z}$, so it follows that $\lambda \notin \sigma_{L^1(\mathbb{T}) \oplus \mathbb{C}\delta_1}(f)$. Since the Fourier series of δ_1 is $\sum_{n=-\infty}^{\infty} z^n$, we thus have

$$(\lambda\delta_1 - f)^{-1} = \sum_{n=-\infty}^{\infty} \frac{1}{\lambda - a_n} z^n = \frac{1}{\lambda} \sum_{n=-\infty}^{\infty} \frac{a_n}{\lambda - a_n} z^n + \frac{1}{\lambda}\delta_1 \in L^1(\mathbb{T}) \oplus \mathbb{C}\delta_1.$$

Moreover, since $\mathcal{B}$ is a Banach $L^1(\mathbb{T})$-module under convolution, we have

$$\sum_{n=-\infty}^{\infty} \frac{a_n}{\lambda - a_n} z^n = (\lambda\delta_1 - f)^{-1} * f \in \mathcal{B},$$

so $(\lambda\delta_1 - f)^{-1} \in \mathcal{A}$, and we deduce that $\lambda \notin \sigma_{\mathcal{A}}(f)$.

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Laboratoire de Mathématiques Pures, Université Bordeaux 1, 351, cours de la Libération, F-33405 Talence cédex, France

E-mail address: vils@math.u-bordeaux.fr