INVARIANT PROJECTIONS AND CONVOLUTION OPERATORS

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ABSTRACT. We prove the existence of invariant projections $P$ from the Banach space $PM_p(G)$ of $p$-pseudomeasures onto $PM_p(H)$ with supp $P(T) \subseteq$ supp $T$ for $H$ closed neutral subgroup of a locally compact group $G$. As a main application we obtain that every closed neutral subgroup is a set of $p$-synthesis in $G$ and in fact locally $p$-Ditkin in $G$. We also obtain an extension theorem concerning the Fourier algebra.

1. Introduction

For a locally compact group $G$, let $CV_p(G)$ be the Banach algebra of all convolution operators of $L^p(G)$ where $1 < p < \infty$. In 1974 Lohoué [10] proved, for an amenable closed subgroup $H$ of $G$ (and $G$ $\sigma$-compact), the existence of a projection of $CV_p(G)$ onto $CV_p(H)$. We obtain the existence of a projection $P$ for the class of closed neutral subgroups. This class includes the following situations: (i) the normalizer of $H$ in $G$ is open in $G$; (ii) $G \in [SIN]_H$. We write $G \in [SIN]_H$ if there is a fundamental system of neighborhoods $U$ of $e$ in $G$ such that $hUh^{-1} = U$ for every $h \in H$. The class $[SIN]_H$ has been thoroughly investigated by Henrichs [5].

The existence of such a $P$ for $H$ normal in $G$ is already in [1]. In fact we are now able to show that supp $P(T) \subseteq$ supp $T$ and $P(uT) = (\text{Res}_H u)P(T)$ for $u \in A_p(G)$ and $T \in CV_p(G)$. The existence of a $P$ with these properties is new even for $G$ abelian and $p \neq 2$; if $p = 2$ and $G$ is abelian the result is due to C. Herz [6]. As a main application we prove that every closed neutral subgroup is a set of $p$-synthesis in $G$ and locally $p$-Ditkin in $G$. A closed subset $F$ of $G$ is locally $p$-Ditkin in $G$ (see [8], p. 102) if for every $\varepsilon > 0$ and every $u \in A_p(G) \cap C_0(G)$ with Res$_F u = 0$ there is $v \in A_p(G) \cap C_0(G)$ with supp $v \cap F = \emptyset$ and $\|u - uv\|_{A_p(G)} < \varepsilon$. $F$ is $p$-Ditkin in $G$ if for every $\varepsilon > 0$ and every $u \in A_p(G)$ with Res$_F u = 0$ there is $v \in A_p(G) \cap C_0(G)$ with supp $v \cap F = \emptyset$ and $\|u - uv\|_{A_p(G)} < \varepsilon$. The method used in the construction of $P$ gives the following extension theorem concerning the Figa-Talamanca Herz algebra of $G$: given $u \in A_p(H) \cap C_0(H)$, $\varepsilon > 0$, and an open subset $\Omega$ of $G$ with supp $u \subseteq \Omega$, there exists $v \in A_p(G) \cap C_0(G)$ with Res$_H v = u$, $\|v\|_{A_p(G)} \leq \|u\|_{A_p(H)} + \varepsilon$ and supp $v \subseteq \Omega$.
We have the pleasure to thank Professor Lohoué for stimulating conversations and Professor Roelke for the communication of unpublished results on neutral subgroups.

2. Definitions and preliminaries

For the precise definitions of $CV_p(G)$, $PM_p(G)$, $A_p(G)$ and all the unexplained notations we refer to [1], p. 38. Let $H$ be an arbitrary closed subgroup of $G$: for $T \in CV_p(H)$, $i(T)$ denotes the image of $T$ under the inclusion $i$ of $H$ in $G$, as defined in [2], p. 76. We assume that $G/H$ admits a $G$-invariant measure. For $k, \ell \in C_{00}(G)$ the relation

$$ \langle \Lambda_k, \ell(T) \varphi, \psi \rangle_{L^p(H), L^p(H')} = \langle T \tau_p(k * H \tau_p \varphi), \tau_p(\ell * H \tau_p \psi) \rangle_{L^p(G), L^p(G')} $$

defines a linear continuous map $\Lambda_k, \ell$ from $CV_p(G)$ into $CV_p(H)$. We have $||\Lambda_k, \ell|| \leq ||T||_p ||k||_p ||\ell||_p$ and $\Lambda_k, \ell(PM_p(G)) \subset PM_p(H)$ ([1], pp. 38–39).

The following unpublished result is due to Roelke. Let $H$ be a closed neutral subgroup of $G$ ([2]). Then $e$ admits a fundamental system of neighborhoods $V$ with $HV = VH$. Indeed let $U$ be an open neighborhood of $e$ in $G$. There is an open $W$ with $e \in W$, $W = W^{-1}$ and $HW \subset UH$. Then for $V = (HWH) \cap U$ one has precisely $HV = VH$.

A one-sided version of the following lemma is already in [2], p. 71.

**Lemma 1.** Let $G$ be a locally compact group, $H$ an arbitrary closed subgroup, $U$ a neighborhood of $e$ in $G$, and $W$ a neighborhood of $e$ in $H$. Then there exists $k \in C_{00}^+$ with $k \subset U$, $(\text{supp } k) \cap H \subset W$ and

$$ \int_H k(h)dh = 1, \int_H k(xh)dh \leq 1, \int_H \Delta_H(h^{-1})k(hx)dh \leq 1 $$

for every $x \in G$.

**Proof.** There is an open neighborhood $U_1$ of $e$ in $G$ with $U_1 \cap H \subset W$ and an open neighborhood $U_2$ of $e$ in $G$ with $U_2 = U_2^{-1}$ and $U_2 \subset U \cap U_1$. Let $K$ be a compact neighborhood of $e$ in $G$ with $K = K^{-1}$ and $K \subset U_2$. We choose $\varphi' \in C_{00}^+(G)$ with $\varphi'(K) = \{1\}$ and $\text{supp } \varphi \subset U_2$. Consider also $\psi' \in C_{00}(G/H)$ with $\psi' \leq 1_{G/H}$, $\psi'(H) = 1$ and $\text{supp } \psi' \subset \omega(K)$. ($\omega$ is the canonical map from $G$ to $G/H$.) Let

$$ \varphi = \frac{\varphi' + \varphi'^\vee}{2} \quad (\varphi'^\vee(x) = \varphi'(x^{-1})) \quad \text{and} \quad k'(x) = \frac{\varphi(x)\psi'(\omega(x))}{\int_H \varphi(xh)dh} $$

for $x \in AH$,

where $A = \{y \in G \mid \varphi(y) > 0\}$ and $k'(x) = 0$ if $x \in G \setminus AH$. Taking into account that $G = AH \cup (G \setminus HK)$, we obtain that $k' \in C_{00}^+(G)$, $\text{supp } k' \subset \text{supp } \varphi$ and $\int_H k'(xh)dh \leq 1$ for every $x \in G$.

Now let $\omega'$ be the canonical map from $G$ onto $H/G$. We similarly choose $\psi'' \in C_{00}^+(H/G)$ with $\psi'' \leq 1_{H/G}$, $\psi''(H) = 1$, $\text{supp } \psi'' \subset \omega'(K)$ and

$$ k''(x) = \frac{\varphi(x)\psi''(\omega'(x))}{\int_H \Delta_H(h^{-1})\varphi(hx)dh} $$

for $x \in HA$,

$k''(x) = 0$ for $x \in G \setminus HA$. We obtain that $k = \min\{k', k''\}$ satisfies all the required properties.
3. PROJECTIONS OF CVₚ(G) ONTO CVₚ(H)

According to Leischner and Roelcke [11], H is said to be locally neutral in G if there is a compact neighborhood U₀ of e such that for every neighborhood U of e there is a neighborhood V of e with (HV'H) ∩ U₀ ⊂ U'H.

**Proposition 2.** Let G be a locally compact group and H a closed subgroup locally neutral in G. We assume that G/H admits an invariant measure. Let (r⁽ⁿ⁾ₙ)∞₁ be m sequences of Lᵖ(H) and (s⁽ⁿ⁾ₙ)∞₁ be in sequences of Lᵖ⁺(H). We assume that for every 1 ≤ j ≤ m, ∑ᵢ₌₁ⁿ inf ||r⁽ⁿ⁾ᵢ||ₚ ||s⁽ⁿ⁾ᵢ||ₚ' < ∞. Then, for every ε > 0 and every open neighborhood U, there is k, ℓ ∈ C₀⁺(G) with supp k, supp ℓ ⊂ U, ||Ak,ℓ|| ≤ 1 and

\[
\sum_{n,1}^{∞} |(Ak,ℓ)(i(S))r⁽ⁿ⁾ₙ, s⁽ⁿ⁾ₙ|_{Lᵖ(H),Lᵖ⁺(H)} - \langle Sr⁽ⁿ⁾ₙ, s⁽ⁿ⁾ₙ,n₁,H',Lᵖ⁺₁(H'),H₁ \rangle_1 ≤ ε||S||ₚ
\]

for every S ∈ CVₚ(H) and every 1 ≤ j ≤ m.

**Proof.** To avoid unessential technical difficulties, we suppose m = 1 and r⁽ⁿ⁾₁, s⁽ⁿ⁾₁ ∈ C₀⁺(H) for every n ∈ N. We put r⁽ⁿ⁾₁ = r⁽ⁿ⁾₁ and s⁽ⁿ⁾₁ = s⁽ⁿ⁾₁. There is N ∈ N such that ∑ᵢ₌₁ⁿ⁺1⁻N inf ||r⁽ⁿ⁾ᵢ||ₚ ||s⁽ⁿ⁾ᵢ||ₚ' < \frac{ε₁}{8} where 0 < ε₁ < min{1, ε}. We can find a relatively compact neighborhood V of e in H with ||r⁽ⁿ⁾₁ - (r⁽ⁿ⁾₁)₁Δ₁||ₚ, ||s⁽ⁿ⁾₁ - (s⁽ⁿ⁾₁)₁Δ₁||ₚ' < \frac{ε₂}{2} for every h ∈ V and every 1 ≤ n ≤ N. We have chosen ε₂ such that

\[
0 < ε₂ < min \left\{ \frac{ε₁}{2r₂ + 2(1 + ||rᵢ||ₚ + ||sᵢ||ₚ')} : 1 ≤ j ≤ N \right\}.
\]

Let U₀ be a compact neighborhood of e in G with U₀ = U₀⁻¹ which guarantees the local neutrality of H in G: for every neighborhood U' of e in G, there is a neighborhood V' of e in G with U₀ ∩ (H'V'H) ⊂ U'H.

Let

\[
0 < ε₃ < \frac{ε₁}{40(1 + ∑ᵢ₌₁ⁿ⁺1 inf ||r⁽ⁿ⁾ᵢ||ₚ ||s⁽ⁿ⁾ᵢ||ₚ')}.
\]

There is a compact neighborhood U₁ of e in G with Δ₁ < 1 + ε₃ for every x ∈ U₁. We choose an open neighborhood U₂ of e in G with U₂² ⊂ U₁. According to Lemma 1, there is k' ∈ C₀⁺(G) with \( \int_H k'(h)dh = 1, \int_H k'(hx)Δ₁(h⁻¹)dh ≤ 1 \) for every x ∈ G, supp k' ⊂ U₀ ∩ U₂ ∩ U⁻¹ and supp k' ∩ H ⊂ V. This implies, for every 1 ≤ n ≤ N, ||r⁽ⁿ⁾₁ - Res₁(r⁽ⁿ⁾₁k')||ₚ ≤ \frac{ε₂}{2} and ||s⁽ⁿ⁾₁ - Res₁(s⁽ⁿ⁾₁k')||ₚ < \frac{ε₂}{2}.

There is a relatively compact open neighborhood U₃ of e in G such that, for every x ∈ U₃ and every 1 ≤ n ≤ N, ||(r⁽ⁿ⁾₁k')ₓ,H - Res₁(r⁽ⁿ⁾₁k')||ₚ ≤ \frac{ε₂}{2} and ||(s⁽ⁿ⁾₁k')ₓ,H - Res₁(s⁽ⁿ⁾₁k')||ₚ < \frac{ε₂}{2}. (We recall that, for a function f on G and for x ∈ G, fₓ,H denotes the function defined on H by fₓ,H(h) = f(xh).)
Now let $A$ be an arbitrary open neighborhood of $e$ in $G$ with $A \subset U_3$. Then, for every $S \in CV_p(H)$ and every $1 \leq n \leq N$, we have

\[
\frac{\langle (S)1_{AH}(r_n \ast H k'), 1_{AH}(s_n \ast H k') \rangle_{L^p(G), L^{p'}(G)}}{\hat{m}(\omega(A))} = \frac{1}{\hat{m}(\omega(A))} \int_G 1_{AH}(x) \beta(x) \left(\langle (S(r_n \ast H k')_{x,H}, (s_n \ast H k')_{x,H})_{L^p(H), L^{p'}(H)} \rangle - \langle S(r_n, s_n)_{L^p(H), L^{p'}(H)} \rangle \right) dx,
\]

where $\hat{m}(\omega(A)) = \int_{\omega(A)} dx$ and $\beta$ is a Bruhat function for $H, G$.

Taking into account that we have, for $x = uh$ with $u \in A$, $h \in H$,

\[
(S(r_n \ast H k')_{x,H}, (s_n \ast H k')_{x,H})_{L^p(H), L^{p'}(H)} = \langle S(r_n \ast H k')_{u,H}, (s_n \ast H k')_{u,H} \rangle_{L^p(H), L^{p'}(H)},
\]

we obtain

\[
\left| \frac{\langle (S)1_{AH}(r_n \ast H k'), 1_{AH}(s_n \ast H k') \rangle_{L^p(G), L^{p'}(G)}}{\hat{m}(\omega(A))} - \langle S(r_n, s_n)_{L^p(H), L^{p'}(H)} \rangle \right| \leq \|\|S\|_p \varepsilon_2 (1 + \|r_n\|_p + \|s_n\|_{p'})
\]

for every $1 \leq n \leq N$.

Let $K$ be a compact subset of $H$ with $e \in K$ and supp $r_n \cup$ supp $s_n \subset K$ for every $1 \leq n \leq N$. There is an open neighborhood $U_4$ of $e$ in $G$ such that $U_4^{-1} = U_4$ and $U_4 \subset U_2 \cap U_3$. There is an open neighborhood $U_5$ of $e$ in $G$ such that $kU_5k^{-1} \subset U_4$ for every $k \in K$. By assumption we can find an open neighborhood $U_6$ of $e$ in $G$ such that $(HU_6H) \cap U_6 \subset U_3H$. This implies $(HU_6H) \cap KU_6H \subset U_4H$. Consider $U_8 = (HU_7H) \cap U_4$ where $U_7$ is an open neighborhood of $e$ in $G$ with $U_7^{-1} = U_7$ and $U_7 \subset U_6$. Roelcke’s argument gives $(KU_0K^{-1}) \cap U_8 = (KU_0K^{-1}) \cap HU_8$. If we take into account that supp$(r_n \ast H k')$, supp$(s_n \ast H k') \subset KU_0K^{-1}$ (1 $\leq n \leq N$), we obtain, for $1 \leq n \leq N$,

\[
1_{U_nH}(r_n \ast H k') = 1_{HU_n}(r_n \ast H k') \quad \text{and} \quad 1_{U_nH}(s_n \ast H k') = 1_{HU_n}(s_n \ast H k').
\]

Let

\[
k'' = \frac{\tau_p(1_{HU_nk'})}{\hat{m}(\omega(U_8))^{1/p}} \quad \text{and} \quad \ell'' = \frac{\tau_{p'}(1_{HU_nk'})}{\hat{m}(\omega(U_8))^{1/p'}}.
\]

Then, for $1 \leq n \leq N$,

\[
\langle \Lambda_{k''}, \ell''(\langle (S) \rangle r_n, s_n \rangle_{L^p(H), L^{p'}(H)} \rangle \nabla_{U_nH}(r_n \ast H k'), 1_{U_nH}(s_n \ast H k') \rangle_{L^p(G), L^{p'}(G)} \rangle_{L^p(G), L^{p'}(G)}.
\]

\[
\nabla_{U_nH}(r_n \ast H k'), 1_{U_nH}(s_n \ast H k') \rangle_{L^p(G), L^{p'}(G)}
\]
From the inequality

$$\sum_{n=1}^{\infty} \left| \langle S_{r_n}, s_n \rangle_{L^p(H), L^{p'}(H)} - \langle \Lambda_{k^{''}, \ell^{''}}(i(S)) r_n, s_n \rangle_{L^p(H), L^{p'}(H)} \right|$$

$$\leq \sum_{n=1}^{N} \left\| S \left\|_p \varepsilon_2 (1 + \| r_n \|_p + \| s_n \|_{p'}) + \sum_{n=N+1}^{\infty} \left| \langle S_{r_n}, s_n \rangle_{L^p(H), L^{p'}(H)} \right|$$

$$+ \sum_{n=N+1}^{\infty} \left| \langle \Lambda_{k^{''}, \ell^{''}}(i(S)) r_n, s_n \rangle_{L^p(H), L^{p'}(H)} \right|,$$

we get

$$\sum_{n=1}^{\infty} \left| \langle S_{r_n}, s_n \rangle_{L^p(H), L^{p'}(H)} - \langle \Lambda_{k^{''}, \ell^{''}}(i(S)) r_n, s_n \rangle_{L^p(H), L^{p'}(H)} \right|$$

$$\leq \frac{3\varepsilon_1 \left\| S \right\|_p}{8} + \frac{\varepsilon_1 \left\| S \right\|_p \left\| T_Hk^{''} \right\|_p \left\| T_H\ell^{''} \right\|_{p'}}{8}.$$ 

To estimate $$\left\| T_Hk^{''} \right\|_p \left\| T_H\ell^{''} \right\|_{p'},$$ observe that, for $$x = uh$$ with $$u \in U_S, h' \in H,$$

$$\int_H \tau_p k'(x)dh = \int_H k'(hu^{-1}) \Delta_G(hu^{-1})^{1/p} \Delta_H(h^{-1})dh.$$ 

But $$u^{-1} \in U_1$$ implies

$$\int_H \tau_p k'(x)dh \leq (1 + \varepsilon_3)^{1/p} \int_H k'(hu^{-1}) \Delta_G(h)^{1/p} \Delta_H(h^{-1})dh.$$ 

From $$\int_H k'(hu^{-1}) \Delta_G(h)^{1/p} \Delta_H(h^{-1})dh = \int_{U_1 \cap H} k'(hu^{-1}) \Delta_G(h)^{1/p} \Delta_H(h^{-1})dh,$$ we therefore get $$\left\| T_Hk^{''} \right\|_p \leq \left( 1 + \varepsilon_3 \right)^{2/p}$$ and similarly $$\left\| T_H\ell^{''} \right\|_{p'} \leq \left( 1 + \varepsilon_3 \right)^{2/p'}.$$ This gives, for every $$S \in CV(H),$$

$$\sum_{n=1}^{\infty} \left| \langle S_{r_n}, s_n \rangle_{L^p(H), L^{p'}(H)} - \langle \Lambda_{k^{''}, \ell^{''}}(i(S)) r_n, s_n \rangle_{L^p(H), L^{p'}(H)} \right| \leq \frac{7\varepsilon_1}{8} \left\| S \right\|_p.$$ 

Consider now $$f, g \in C_{00}(G/H)$$ with $$\left\| f - \frac{1_{\omega(U_S)}}{m(\omega(U_S))^{1/p}} \right\|_p, \left\| g - \frac{1_{\omega(U_S)}}{m(\omega(U_S))^{1/p'}} \right\|_{p'}$$ both smaller than

$$\frac{\varepsilon_4}{(1 + \left\| T_H\tau_p k' \right\|_{\infty})(1 + \left\| T_H\tau_p \ell' \right\|_{\infty})},$$

where

$$0 < \varepsilon_4 < \frac{\varepsilon_3}{(1 + 2^{2/p} + 2^{2/p'}) \left( 1 + \sum_{n=1}^{\infty} \| r_n \|_p \| s_n \|_{p'} \right)}.$$ 

Putting $$k^{'''} = (f \circ \omega) \tau_p k'$$ and $$\ell^{'''} = (g \circ \omega) \tau_p \ell'',$$ we have successively

$$\left\| T_H(k^{'''} - k'') \right\|_p, \left\| T_H(\ell^{'''} - \ell'') \right\|_{p'} < \varepsilon_4$$

$$\left\| T_Hk^{'''} \right\|_p < \varepsilon_4 + (1 + \varepsilon_3)^{2/p} \quad \text{and} \quad \left\| T_H\ell^{'''} \right\|_{p'} < \varepsilon_4 + (1 + \varepsilon_3)^{2/p'}. \quad \text{Finally it suffices to choose} \quad k = \frac{\varepsilon_4 + (1 + \varepsilon_3)^{2/p}}{\varepsilon_4 + (1 + \varepsilon_3)^{2/p'}} \quad \text{and} \quad \ell = \frac{\varepsilon_4 + (1 + \varepsilon_3)^{2/p'}}{\varepsilon_4 + (1 + \varepsilon_3)^{2/p'}} \quad \text{to obtain}$$

$$\sum_{n=1}^{\infty} \left| \langle S_{r_n}, s_n \rangle_{L^p(H), L^{p'}(H)} - \langle \Lambda_{k, \ell}(i(S)) r_n, s_n \rangle_{L^p(H), L^{p'}(H)} \right| \leq \varepsilon \left\| S \right\|_p$$

with $$\| \Lambda_{k, \ell} \| \leq 1$$ and supp $$k, \text{ supp } \ell \subset U.$$
We are now ready to prove our first result concerning projections of $CV_p(G)$ onto $CV_p(H)$.

**Theorem 3.** Let $G$ be a locally compact group and $H$ a closed subgroup, locally neutral in $G$. We assume that $G/H$ admits an invariant measure. Then there is a linear contraction $Q$ from $L(L^p(G))$ (the Banach space of all bounded operators of $L^p(G)$) into $L(L^p(H))$ such that:

1. $Q(T) \in CV_p(H)$ for every $T \in CV_p(G)$,
2. $\text{supp} Q(T) \subseteq \text{supp} T$ for every $T \in CV_p(G)$,
3. $Q(\iota(S)) = S$ for every $S \in CV_p(H)$,
4. $Q(T) \in PM_p(G)$ for every $T \in PM_p(G)$.

**Proof.** Let $A$ be the set of all pairs $(\{r_n\}_{n=1}^{\infty}, (s_n)_{n=1}^{\infty})$ where $(r_n)_{n=1}^{\infty}$ is a sequence of $L^p(H)$ and $(s_n)_{n=1}^{\infty}$ is a sequence of $L^p(H)$ with $\sum_{n=1}^{\infty} ||r_n||_p ||s_n||_{p'} < \infty$. We denote by $E$ the set of all maps $F$ from $L(L^p(G)) \times L^p(H) \times L^p(H)$ to $C$, linear in the first two variables, conjugate linear in the third one, and for which there is a positive real number $C$ with $|F(T, \varphi, \psi)| \leq C||T||_p ||\varphi||_p ||\psi||_{p'}$. For $F \in E$ we put $|F| = \sup\{|F(T, \varphi, \psi)| : \text{supp } T \leq 1, ||\varphi||_p \leq 1, ||\psi||_{p'} \leq 1\}$. For $k, \ell \in C_0(G), F_{k, \ell}(T, \varphi, \psi) = \langle \Lambda_k, \iota(T) \varphi, \psi \rangle_{L^p(H), L^p(H)}$ is an element of $E$ with $|F_{k, \ell}| \leq |T_H| ||k||_p ||T_H||_p ||\ell||_p$.

Let $A$ be a finite subset of $A$, $B$ a finite subset of $CV_p(H), U$ an open neighborhood of $e$ in $G$ and $\varepsilon > 0$. Proposition 2 implies precisely that the set

$$K_{A, B, U, \varepsilon} = \left\{ F_{k, \ell} \mid k, \ell \in C_0^+(G), ||F_{k, \ell}|| \leq 1, \text{supp } k, \text{supp } \ell \subseteq U, \sum_{n=1}^{\infty} |F_{k, \ell}(\iota(S), r_n, s_n) - \langle S r_n, s_n \rangle_{L^p(H), L^p(H)}| < \varepsilon \right\}$$

for every $\{(r_n)_{n=1}^{\infty}, (s_n)_{n=1}^{\infty}\} \in A$ and every $S \in B$ is nonempty. Let $\overline{K}_{A, B, U, \varepsilon}$ be the closure of $K_{A, B, U, \varepsilon}$ with respect to the topology $\sigma(E, L(L^p(G)) \times L^p(H) \times L^p(H))$. The set $\cap \{\overline{K}_{A, B, U, \varepsilon} \mid A$ finite subset of $A, B$ finite subset of $CV_p(H), 0 < \varepsilon < 1, U$ open neighborhood of $e$ in $G\}$ is not empty. Choose $J$ in this set. There is a linear map $Q$ from $L(L^p(G))$ to $L(L^p(H))$ with $J(T, \varphi, \psi) = \langle Q(T) \varphi, \psi \rangle_{L^p(H), L^p(H)}$ for $T \in L(L^p(G)), \varphi \in L^p(H), \psi \in L^p(H)$. Clearly $Q$ satisfies conditions (1) to (4).

Let $H$ be a closed subgroup of $G$ for which there is a linear map $Q$ from $CV_p(G)$ onto $CV_p(H)$ satisfying conditions (3) and (4) of Theorem 3. Then $H$ is a set of $p$-synthesis in $G$. Indeed let $T \in PM_p(G)$ with $\text{supp } T \subseteq H$. From $Q(\iota(S)) \in PM_p(H)$ we deduce that $S \in PM_p(H)$ and therefore $\langle u, T \rangle_{A_p(G), PM_p(G)} = \langle \text{Res}_H u, S \rangle_{A_p(H), PM_p(H)} = 0$.

**Corollary 4.** Let $G$ be a locally compact group and $H$ a closed subgroup, locally neutral in $G$, for which $G/H$ admits an invariant measure. Then $H$ is a set of $p$-synthesis of $G$.

The following extension theorem was proved by C. Herz for $G$ second countable and $H$ normal in $G$ ([2], p. 115).
Corollary 5. Let $G$ be a locally compact group and $H$ a closed subgroup as in Theorem 3. Given $u \in A_p(H) \cap C^0_0(H)$, $\varepsilon > 0$ and an open subset $\Omega$ of $G$ with $\text{supp} \ u \subset \Omega$, there exists $v \in A_p(G) \cap C^0_0(G)$ with $\text{Res}_H v = u$, $||v||_{A_p(G)} \leq ||u||_{A_p(H)} + \varepsilon$ and $\text{supp} \ v \subset \Omega$.

Proof. According to [7], p. 115, it suffices to find $v \in A_p(G) \cap C^0_0(G)$ with $\text{supp} \ v \subset \Omega$, $||v||_{A_p(G)} \leq ||u||_{A_p(H)}$ and $||u - \text{Res}_H v||_{A_p(H)} < \varepsilon$.

There is $(r_n)_{n=1}^\infty$ and $(s_n)_{n=1}^\infty$, two sequences of $C^0_0(H)$, such that $u = \sum_{n=1}^\infty r_n * s_n$ and $\sum_{n=1}^\infty ||r_n||_{L^p(H)}||s_n||_{L^{p'}(H)} < \infty$. There also exists an open neighborhood $U$ of $e$ in $G$ such that $U \text{supp} \ u U^{-1} \subset \Omega$. There is $k, \ell \in C^0_0(G)$ with $\text{supp} \ k, \text{supp} \ \ell \subset U$, $||A_{k,\ell}|| \leq 1$ and

$$\sum_{n=1}^\infty |(A_{k,\ell}(i(S))r_{p,n}, \tau_{p'}s_n)_{L^p(G),L^{p'}(G)} - (S\tau_{p}r_{n}, \tau_{p'}s_n)_{L^p(H),L^{p'}(H)}| \leq \varepsilon ||S||_p$$

for every $S \in CV_p(H)$. There exists a unique $v \in A_p(G)$ such that

$$\langle u, i(S) \rangle_{A_p(G),PM_p(G)} = \langle u, A_{k,\ell}(S) \rangle_{A_p(H),PM_p(H)}$$

for every $S \in PM_p(G)$. From

$$||u - \text{Res}_H v||_{A_p(H)} \leq \varepsilon \text{ with } \text{supp} \ v \subset \text{supp} \ k \text{ supp} \ u (\text{supp} \ \ell)^{-1}.$$  

Remark. Suppose $G$ is abelian. According to J. Inoue [8] for every neighborhood $U$ of $e$ in $G$ there is a linear isometric map $\Omega$ of $A_2(H)$ into $A_2(G)$ with $\text{Res}_H \circ \Omega = \text{id}_{A_2(H)}$ (such a map is called a linear lifting) and $\text{supp} \Omega(u) \subset (\text{supp} u)U$. By duality we easily derive the existence of a projection of $PM_2(G)$ onto $PM_2(H)$ as in Theorem 3. On the other hand B. Forrest [4] has shown that for $G$ amenable and $H$ closed abelian normal subgroup of $G$ a linear lifting does not always exist. Consequently the map $Q$ of Theorem 3 can be considered as a substitute, for $G$ nonabelian, to the nonexistence of linear liftings of $A_2(H)$ into $A_2(G)$.

4. INVARIANT PROJECTIONS

In [1] we proved for $H$ normal in $G$ the existence of a projection $P$ of $CV_p(G)$ onto $CV_p(H)$ satisfying the condition $P(uT) = (\text{Res}_H u)P(T)$; however condition (2) of Theorem 3 was out of our reach. A projection $P$ with $P(uT) = (\text{Res}_H u)P(T)$ will be called an invariant projection.

Theorem 6. Let $G$ be a locally compact group and $H$ a closed neutral subgroup of $G$. There is an invariant projection $P$ of $CV_p(G)$ onto $CV_p(H)$ which also satisfies all conditions of Theorem 3.

Proof. Let $X$ be the set of all maps $f$ from $L^p(G) \times L^p(G) \times L^{p'}(G)$ to $C$ which are linear in the first two variables and conjugate linear in the third one and for which there is a positive real number $C$ with $|f(T, \varphi, \psi)| \leq C||T||_p||\varphi||_p||\psi||_p'$. 

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For \(a, b \in S(G/H)\) (\(S(G/H)\) is the set of all bounded measurable functions on \(G/H\) with compact support), \(T \in \mathcal{L}(L^p(G))\), \(\varphi, \psi \in C_0(G)\) we define

\[g_{a,b}(T, \varphi, \psi) = \int_{G/H} \langle TB(\varphi, a)(t), B(\psi, b)(t) \rangle_{L^p(G), L^{p'}(G)} \, dt\]

with \(B(\varphi, a)(t)(x) = \varphi(x) a(x^{-1}t)\) for \(x, t \in G\). Let \(K\) be a compact subset of \(G\) with \(K \cap H = \emptyset\) and \(\varepsilon > 0\). At first we show that the set \(D_{K,\varepsilon} = \left\{ g_{a,b} \mid a, b \in S(G/H), a, b \geq 0, \|a\|_p \|b\|_{p'} < 1 + \varepsilon \right\}\) is nonempty. We indeed choose an open neighborhood \(U_1\) of \(e\) in \(G\) with \(U_1 = U_2^{-1} = KU_1 \cap HU_2 = \emptyset\). There is an open neighborhood \(U_2\) of \(e\) in \(G\) with \(U_2^{-1} = U_2, U_2 \subset U_1\) and \(U_2H = HU_2\). Let \(U_3\) be an open neighborhood of \(e\) in \(G\) relatively compact with \(U_3 \subset U_2\) and \(U_3H = HU_3\). This implies that \(U_3H = HU_3\). There is an open subset \(U_4\) of \(G/H\) with \(U_4 \supset \omega(U_3) \) and \(m(U_4 - \omega(U_3)) < ((1 + \varepsilon)^{p'} - 1)m(\omega(U_3))\). Let \(U_5 = \omega^{-1}(U_4) \cap U_2\). We consider an open neighborhood \(U_6\) of \(e\) with \(U_6^{-1} = U_6\) and \(U_6U_3 \subset U_5\). We define \(v(x) = \int_{G/H} a(x^{-1}t)b(t) \, dt\) with \(a = \frac{1}{m(\omega(U_5))}\) and \(b = 1\).

We have

\[\|a\|_p \|b\|_{p'} < \left( \frac{m(\omega(U_5))}{m(\omega(U_3))} \right)^\frac{1}{p'} < 1 + \varepsilon.\]

Suppose \(v(x) \neq 0\); there is \(\hat{y} \in G/H\) with \(x\hat{y} \in \omega(U_3)\) and \(\hat{y} \in \omega(U_6U_3)\). This implies \(x \in U_6U_3H^{-1} = \emptyset\). Let \(x \in HU_6\) for every \(y \in \omega(U_3)\) with \(x^{-1}y \in \omega(U_6H)\) but \(\omega(U_6H)U_3 = \emptyset\); this implies precisely \(v(x) = 1\).

Let \(g\) be an element of \(\cap (\mathcal{D}_{K,\varepsilon} \mid K\) compact subset of \(G\) with \(K \cap H = \emptyset, 0 < \varepsilon < 1\})\) where \(\mathcal{D}_{K,\varepsilon}\) is the closure of \(D_{K,\varepsilon}\) in \(X\) with respect to the topology \(\sigma(X, \mathcal{L}(L^p(G)) \times L^{p'}(G), \times L^{p'}(G))\). Let \(P\) be the corresponding map of \(\mathcal{L}(L^p(G))\) to itself. It suffices to consider \(Q \circ P\) where \(Q\) is the map of Theorem 3.

**Corollary 7.** Let \(G\) be a locally compact group, \(H\) a closed neutral subgroup of \(G\) and \(F\) a closed subset of \(H\). If \(F\) is locally \(p\)-Ditkin (\(1 < p < \infty\)) with respect to \(H\), then \(F\) is locally \(p\)-Ditkin with respect to \(G\).

**Proof.** Let \(u \in A_p(G), T \in CV_p(G)\) with \(\text{Res}_F u = 0\) and \(\text{supp}(uT) \subset F\). It suffices to show that \(uT = 0\). Let \(S \in CV_p(H)\) such that \(uT = i(S)\). Let \(P\) be the invariant projection of Theorem 6. From \(\text{supp} P(uT) \subset \text{supp} uT\) and \(P(uT) = (\text{Res}_H u)P(T) = S\) we deduce that \(S = 0\) and therefore \(uT = 0\).

**Remark.** For \(H\) normal in \(G\), this corollary is already in [3], p. 103. Our proof there was completely different: it was based on the use of \(A_p(G/H)\). The present approach is not only more conceptual but permits us to treat the case of certain interesting nonnormal subgroups: \(H\) compact or \(G \in [SIN]_H\).

**References**


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