INARIANT PROJECTIONS AND CONVOLUTION OPERATORS

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ABSTRACT. We prove the existence of invariant projections \( P \) from the Banach space \( PM_p(G) \) of \( p \)-pseudomeasures onto \( PM_p(H) \) with \( \text{supp } P(T) \subset \text{supp } T \) for \( H \) closed neutral subgroup of a locally compact group \( G \). As a main application we obtain that every closed neutral subgroup is a set of \( p \)-synthesis in \( G \) and in fact locally \( p \)-Ditkin in \( G \). We also obtain an extension theorem concerning the Fourier algebra

1. Introduction

For a locally compact group \( G \), let \( CV_p(G) \) be the Banach algebra of all convolution operators of \( L^p(G) \) where \( 1 < p < \infty \). In 1974 Lohoué \([10]\) proved, for an amenable closed subgroup \( H \) of \( G \) (and \( G \) \( \sigma \)-compact), the existence of a projection of \( CV_p(G) \) onto \( CV_p(H) \). We obtain the existence of a projection \( P \) for the class of closed neutral subgroups. This class includes the following situations: (i) \( G \) normalizer of \( H \) in \( G \) is open in \( G \); (ii) \( G \in [SIN]_H \). We write \( G \in [SIN]_H \) if there is a fundamental system of neighborhoods \( U \) of \( e \) in \( G \) such that \( hUh^{-1} = U \) for every \( h \in H \). The class \( [SIN]_H \) has been thoroughly investigated by Henrichs \([8]\).

The existence of such a \( P \) for \( H \) normal in \( G \) is already in \([1]\). In fact we are now able to show that \( \text{supp } P(T) \subset \text{supp } T \) and \( P(uT) = (\text{Res}_H u)P(T) \) for \( u \in A_p(G) \) and \( T \in CV_p(G) \). The existence of a \( P \) with these properties is new even for \( G \) abelian and \( p \neq 2 \); if \( p = 2 \) and \( G \) is abelian the result is due to C. Herz \([9]\). As a main application we prove that every closed neutral subgroup is a set of \( p \)-synthesis in \( G \) and locally \( p \)-Ditkin in \( G \). A closed subset \( F \) of \( G \) is locally \( p \)-Ditkin in \( G \) (see \([3]\), p. 102) if for every \( \varepsilon > 0 \) and every \( u \in A_p(G) \cap C_{00}(G) \) with \( \text{Res}_F u = 0 \) there is \( v \in A_p(G) \cap C_{00}(G) \) with \( \text{supp } v \cap F = \emptyset \) and \( ||u - uv||_{A_p(G)} < \varepsilon \). \( F \) is \( p \)-Ditkin in \( G \) if for every \( \varepsilon > 0 \) and every \( u \in A_p(G) \) with \( \text{Res}_F u = 0 \) there is \( v \in A_p(G) \cap C_{00}(G) \) with \( \text{supp } v \cap F = \emptyset \) and \( ||u - uv||_{A_p(G)} < \varepsilon \). The method used in the construction of \( P \) gives the following extension theorem concerning the Figa-Talamanca Herz algebra of \( G \): given \( u \in A_p(H) \cap C_{00}(H) \), \( \varepsilon > 0 \), and an open subset \( \Omega \) of \( G \) with \( \text{supp } u \subset \Omega \), there exists \( v \in A_p(G) \cap C_{00}(G) \) with \( \text{Res}_H v = u \), \( ||v||_{A_p(G)} \leq ||u||_{A_p(H)} + \varepsilon \) and \( \text{supp } v \subset \Omega \).

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2. Definitions and preliminaries

For the precise definitions of $CV_p(G)$, $PM_p(G)$, $A_p(G)$ and all the unexplained notations we refer to [1], p. 38. Let $H$ be an arbitrary closed subgroup of $G$: for $T \in CV_p(H)$, $i(T)$ denotes the image of $T$ under the inclusion $i$ of $H$ in $G$, as defined in [2], p. 76. We assume that $G/H$ admits a $G$-invariant measure. For $k, \ell \in C_{00}(G)$ the relation

$$
\langle \Lambda_k, \ell (T) \varphi, \psi \rangle_{L^p(G), L^{p'}(G)} = \langle T \tau_p(k * H \tau_p \varphi), \tau_p(\ell * H \tau_p \psi) \rangle_{L^p(G), L^{p'}(G)}
$$

defines a linear continuous map $\Lambda_k, \ell$ from $CV_p(G)$ into $CV_p(H)$. We have $\|\Lambda_k, \ell\| \leq \|T_p \|_p \|T_p(\ell) \|_{p'}$ and $\Lambda_k, \ell(PM_p(G)) \subset PM_p(H)$ ([3], pp. 38–39).

The following unpublished result is due to Roelcke. Let $H$ be a closed neutral subgroup of $G$ ([12]). Then $e$ admits a fundamental system of neighborhoods $V$ with $HV = VH$. Indeed let $U$ be an open neighborhood of $e$ in $G$. There is an open $W$ with $e \in W$, $W = W^{-1}$ and $WHV \subset UH$. Then for $V = (WHV) \cap U$ one has precisely $HV = VH$.

A one-sided version of the following lemma is already in [2], p. 71.

**Lemma 1.** Let $G$ be a locally compact group, $H$ an arbitrary closed subgroup, $U$ a neighborhood of $e$ in $G$, and $W$ a neighborhood of $e$ in $H$. Then there exists $k \in C^+_0(G)$ with $\supp k \subset U$, $(\supp k) \cap H \subset W$ and

$$
\int_H k(h)dh = 1, \int_H k(xh)dh \leq 1, \int_H \Delta_H(h^{-1})k(hx)dh \leq 1
$$

for every $x \in G$.

**Proof.** There is an open neighborhood $U_1$ of $e$ in $G$ with $U_1 \cap H \subset W$ and an open neighborhood $U_2$ of $e$ in $G$ with $U_2 = U_2^{-1}$ and $U_2 \subset U \cap U_1$. Let $K$ be a compact neighborhood of $e$ in $G$ with $K = K^{-1}$ and $K \subset U_2$. We choose $\varphi' \in C^+_0(G)$ with $\varphi'(K) = \{1\}$ and $\supp \varphi \subset U_2$. Consider also $\psi' \in C^+_0(G/H)$ with $\psi' \leq 1_{G/H}$, $\psi'(H) = 1$ and $\supp \psi' \subset \omega(K)$. ($\omega$ is the canonical map from $G$ to $G/H$.) Let

$$
\varphi = \frac{\varphi' + \varphi'^*}{2} \quad (\varphi'^*(x) = \varphi'(x^{-1})) \quad \text{and} \quad k'(x) = \frac{\varphi(x)\psi'(\omega(x))}{\int_H \varphi(xh)dh}
$$

for $x \in AH$, where $A = \{y \in G \mid \varphi(y) > 0\}$ and $k'(x) = 0$ if $x \in G \setminus AH$. Taking into account that $G = AH \cup (G \setminus HK)$, we obtain that $k' \in C^+_0(G)$, $\supp k' \subset \supp \varphi$ and $\int_H k'(xh)dh \leq 1$ for every $x \in G$.

Now let $\omega'$ be the canonical map from $G$ onto $H/G$. We similarly choose $\psi'' \in C^+_0(H/G)$ with $\psi'' \leq 1_{H/G}$, $\psi''(H) = 1$, $\supp \psi'' \subset \omega'(K)$ and

$$
k''(x) = \frac{\varphi(x)\psi''(\omega'(x))}{\int_H \Delta_H(h^{-1})\varphi(hx)dh}
$$

for $x \in HA$, $k''(x) = 0$ for $x \in G \setminus HA$. We obtain that $k = \min\{k', k''\}$ satisfies all the required properties.
3. PROJECTIONS OF $CV_p(G)$ INTO $CV_p(H)$

According to Leischner and Roelcke [11] $H$ is said to be locally neutral in $G$ if there is a compact neighborhood $U_0$ of $e$ such that for every neighborhood $U$ of $e$ there is a neighborhood $V$ of $e$ with $(HVH) \cap U_0 \subset UH$.

**Proposition 2.** Let $G$ be a locally compact group and $H$ a closed subgroup locally neutral in $G$. We assume that $G/H$ admits an invariant measure. Let $(r_n^{(j)})_{n=1}^{\infty}$ be sequences of $L^p(H)$ and $(s_n^{(j)})_{n=1}^{\infty}$ be sequences of $L^{p'}(H)$. We assume that for every $1 \leq j \leq m$, \[
\sum_{n=1}^{\infty} ||r_n^{(j)}||_p ||s_n^{(j)}||_{p'} < \infty. \]
Then, for every $\varepsilon > 0$ and every open neighborhood $U$, there is $k, \ell \in C_{00}^+(G)$ with supp $k$, supp $\ell \subset U$, $||Res_k || \leq 1$ and
\[
\sum_{n=1}^{\infty} |(k,\ell)(i(S))r_n^{(j)}, s_n^{(j)}|_{L^p(H),L^{p'}(H)} - (Sr_n^{(j)}, s_n^{(j)})_{L^p(H),L^{p'}(H)} \leq \varepsilon ||S||_p
\]
for every $S \in CV_p(H)$ and every $1 \leq j \leq m$.

**Proof.** To avoid unessential technical difficulties, we suppose $m = 1$ and $r_n^{(1)}, s_n^{(1)} \in C_{00}(H)$ for every $n \in \mathbb{N}$. We put $r_n = r_n^{(1)}$ and $s_n = s_n^{(1)}$. There is $N \in \mathbb{N}$ such that $\sum_{n=1+1+N}^{\infty} ||r_n||_p ||s_n||_{p'} < \frac{\varepsilon_1}{8}$ where $0 < \varepsilon_1 < \min\{1, \varepsilon\}$. We can find a relatively compact neighborhood $V$ of $e$ in $H$ with $||r_n - (r_n)_{h^{-1}} \Delta_H(h^{-1})||_p$, $||s_n - (s_n)_{h^{-1}} \Delta_H(h^{-1})||_{p'} < \frac{\varepsilon_2}{2}$ for every $h \in V$ and every $1 \leq n \leq N$. We have chosen $\varepsilon_2$ such that
\[
0 < \varepsilon_2 < \min\left\{\frac{\varepsilon_1}{2r+2(1+||r_j||_p + ||s_j||_{p'})} \middle| 1 \leq j \leq N\right\}.
\]
Let $U_0$ be a compact neighborhood of $e$ in $G$ with $U_0 = U_0^{-1}$ which guarantees the local neutrality of $H$ in $G$: for every neighborhood $U'$ of $e$ in $G$, there is a neighborhood $V'$ of $e$ in $G$ with $U_0 \cap (HV'H) \subset U'H$.

Let
\[
0 < \varepsilon_3 < \frac{\varepsilon_1}{40(1 + \sum_{n=1}^{\infty} ||r_n||_p ||s_n||_{p'})}.
\]
There is a compact neighborhood $U_1$ of $e$ in $G$ with $\Delta_G(x) < 1 + \varepsilon_3$ for every $x \in U$. We choose an open neighborhood $U_2$ of $e$ in $G$ with $U_2^2 \subset U_1$. According to Lemma 1, there is $k' \in C_{00}^+(G)$ with $\int_H k'(h) dh = 1$, $\int_H k'(hx) \Delta_H(h^{-1}) dh \leq 1$ for every $x \in G$, supp $k' \subset U_0 \cap U_2 \cap U^{-1}$ and supp $k' \cap H \subset V$. This implies, for every $1 \leq n \leq N$, $||r_n - Res_H(r_n * H k')||_p \leq \frac{\varepsilon_2}{2}$ and $||s_n - Res_H(s_n * H k')||_{p'} \leq \frac{\varepsilon_2}{2}$.

There is a relatively compact open neighborhood $U_3$ of $e$ in $G$ such that, for every $x \in U_3$ and every $1 \leq n \leq N$, $||r_n * H k'||_x, H \leq \frac{\varepsilon_2}{2}$ and $||s_n * H k'||_x, H \leq \frac{\varepsilon_2}{2}$. (We recall that, for a function $f$ on $G$ and for $x \in G$, $f_{x, H}$ denotes the function defined on $H$ by $f_{x, H}(h) = f(xh)$.)
Now let $A$ be an arbitrary open neighborhood of $e$ in $G$ with $A \subset U_3$. Then, for every $S \in CV_p(H)$ and every $1 \leq n \leq N$, we have

\[
\frac{\langle i(S)1_{AH}(r_n * H k'), 1_{AH}(s_n * H k') \rangle_{L^p(G), L^{p'}(G)}}{\hat{m}(\omega(A))} - \langle S r_n, s_n \rangle_{L^p(H), L^{p'}(H)} = \frac{1}{\hat{m}(\omega(A))} \int_G 1_{AH}(x) \beta(x) \left( \langle S(r_n * H k')_{x, H}, (s_n * H k')_{x, H} \rangle_{L^p(H), L^{p'}(H)} - \langle S r_n, s_n \rangle_{L^p(H), L^{p'}(H)} \right) dx,
\]

where $\hat{m}(\omega(A)) = \int_G \omega(A) \, dx$ and $\beta$ is a Bruhat function for $H, G$.

Taking into account that we have, for $x = uh$ with $u \in A, h \in H$,

\[
\langle S(r_n * H k')_{x, H}, (s_n * H k')_{x, H} \rangle_{L^p(H), L^{p'}(H)} = \langle S(r_n * H k')_{u, H}, (s_n * H k')_{u, H} \rangle_{L^p(H), L^{p'}(H)},
\]

we obtain

\[
\frac{\langle i(S)1_{AH}(r_n * H k'), 1_{AH}(s_n * H k') \rangle_{L^p(G), L^{p'}(G)}}{\hat{m}(\omega(A))} - \langle S r_n, s_n \rangle_{L^p(H), L^{p'}(H)} \leq ||S||_p \varepsilon_2 (1 + ||r_n||_p + ||s_n||_{p'})
\]

for every $1 \leq n \leq N$.

Let $K$ be a compact subset of $H$ with $e \in K$ and supp $r_n \cup$ supp $s_n \subset K$ for every $1 \leq n \leq N$. There is an open neighborhood $U_4$ of $e$ in $G$ such that $U_4^{-1} = U_4$ and $U_4 \subset U_2 \cap U_3$. There is an open neighborhood $U_5$ of $e$ in $G$ such that $kU_5k^{-1} \subset U_4$ for every $k \in K$. By assumption we can find an open neighborhood $U_6$ of $e$ in $G$ such that $(HU_6H) \cap U_5 \subset U_5H$. This implies $(HU_6H) \cap KU_0H \subset U_3H$. Consider $U_8 = (HU_6H) \cap U_4$ where $U_7$ is an open neighborhood of $e$ in $G$ with $U_7^{-1} = U_7$ and $U_7 \subset U_6$. Roelcke’s argument gives $(KU_0K^{-1}) \cap U_8H = (KU_0K^{-1}) \cap HU_8$. If we take into account that supp$(r_n * H k')$, supp$(s_n * H k') \subset KU_0K^{-1}$ ($1 \leq n \leq N$), we obtain, for $1 \leq n \leq N$,

\[
1_{U_8H}(r_n * H k') = 1_{HU_8}(r_n * H k') \quad \text{and} \quad 1_{U_8H}(s_n * H k') = 1_{HU_8}(s_n * H k').
\]

Let

\[
k'' = \frac{\tau_p(1_{HU_8k'})}{\hat{m}(\omega(U_8))^{1/p}} \quad \text{and} \quad \ell'' = \frac{\tau_{p'}(1_{HU_8k'})}{\hat{m}(\omega(U_8))^{1/p'}}.
\]

Then, for $1 \leq n \leq N$,

\[
\langle \Lambda_{k'', \ell''}(i(S)) r_n, s_n \rangle_{L^p(H), L^{p'}(H)} = \frac{\langle i(S)1_{U_8H}(r_n * H k'), 1_{U_8H}(s_n * H k') \rangle_{L^p(G), L^{p'}(G)}}{\hat{m}(\omega(U_8))}.
\]
From the inequality
\[
\sum_{n=1}^{\infty} |\langle Sr_n, s_n \rangle_{L^p(H), L^{p'}(H)} - \langle \Lambda_{k', \ell'}(i(S))r_n, s_n \rangle_{L^p(H), L^{p'}(H)}| \\
\leq \sum_{n=1}^{N} |||S|||_p \varepsilon_2 (1 + ||r_n||_p + ||s_n||_{p'}) + \sum_{n=N+1}^{\infty} \left|\langle Sr_n, s_n \rangle_{L^p(H), L^{p'}(H)}\right| \\
+ \sum_{n=N+1}^{\infty} \left|\langle \Lambda_{k', \ell'}(i(S))r_n, s_n \rangle_{L^p(H), L^{p'}(H)}\right|,
\]
we get
\[
\sum_{n=1}^{\infty} |\langle Sr_n, s_n \rangle_{L^p(H), L^{p'}(H)} - \langle \Lambda_{k', \ell'}(i(S))r_n, s_n \rangle_{L^p(H), L^{p'}(H)}| \\
\leq \frac{3\varepsilon_1 |||S|||_p}{8} + \frac{\varepsilon_1 |||S|||_p |||T_H k''|||_p |||T_H \ell''|||_{p'}}{8}.
\]

To estimate $$||T_H k''|||_p |||T_H \ell''|||_{p'}$$, observe that, for $$x = uh$$ with $$u \in U_8$$, $$h' \in H$$,
\[
\int_H (\tau_p k')(x)h)dh = \int_H k'(hu^{-1})\Delta_G(hu^{-1})^{1/p}\Delta_H(h^{-1})dh.
\]
From $$\int_H k'(hu^{-1})\Delta_G(hu^{-1})^{1/p}\Delta_H(h^{-1})dh = \int_{U_1 \cap H} k'(hu^{-1})\Delta_G(h)^{1/p}\Delta_H(h^{-1})dh$$, we therefore get $$||T_H k''|||_p \leq (1 + \varepsilon_3)^2/2$$ and similarly $$||T_H \ell''|||_{p'} \leq (1 + \varepsilon_3)^2/2'$$. This gives, for every $$S \in CV_p(H),$$
\[
\sum_{n=1}^{\infty} |\langle Sr_n, s_n \rangle_{L^p(H), L^{p'}(H)} - \langle \Lambda_{k', \ell'}(i(S))r_n, s_n \rangle_{L^p(H), L^{p'}(H)}| \leq \frac{7\varepsilon_1}{8} |||S|||_p.
\]

Consider now $$f, g \in C_{00}(G/H)$$ with $$\left\| f - \frac{1_{\omega(U_8)}}{m(\omega(U_8))^{1/p}} \right\|_p, \left\| g - \frac{1_{\omega(U_8)}}{m(\omega(U_8))^{1/p'}} \right\|_{p'}$$ both smaller than
\[
\frac{\varepsilon_4}{(1 + |||T_H \tau_p k'|||_\infty)(1 + |||T_H \tau_p \ell'|||_\infty)},
\]
where
\[
0 < \varepsilon_4 < \frac{\varepsilon_3}{(1 + 2^{2/p} + 2^{2/p'}) \left(1 + \sum_{n=1}^{\infty} ||r_n||_p ||s_n||_{p'}\right)}.
\]

Putting $$k'' = (f \circ \omega) \tau_p k'$$ and $$\ell'' = (g \circ \omega) \tau_p \ell'$$, we have successively
\[
||T_H (|k'' - k'|)||_p, ||T_H (|\ell'' - \ell'|)||_{p'} < \varepsilon_4
\]
\[
||T_H k''||_p < \varepsilon_4 + (1 + \varepsilon_3)^2/2, \text{ and } ||T_H \ell''||_p < \varepsilon_4 + (1 + \varepsilon_3)^2/2'.
\]
Finally it suffices to choose $$k = \frac{\varepsilon_4 + (1 + \varepsilon_3)^2/2}{\varepsilon_4 + (1 + \varepsilon_3)^2/2}$$ and $$\ell = \frac{\varepsilon_4}{\varepsilon_4 + (1 + \varepsilon_3)^2/2}$$ to obtain
\[
\sum_{n=1}^{\infty} |\langle Sr_n, s_n \rangle_{L^p(H), L^{p'}(H)} - \langle \Lambda_{k, \ell}(i(S))r_n, s_n \rangle_{L^p(H), L^{p'}(H)}| \leq \varepsilon |||S|||_p
\]
with $$||\Lambda_{k, \ell}|| \leq 1$$ and $$\supp k, \supp \ell \subset U$$. 

We are now ready to prove our first result concerning projections of $CV_p(G)$ onto $CV_p(H)$.

**Theorem 3.** Let $G$ be a locally compact group and $H$ a closed subgroup, locally neutral in $G$. We assume that $G/H$ admits an invariant measure. Then there is a linear contraction $Q$ from $L(L^p(G))$ (the Banach space of all bounded operators of $L^p(G)$) into $L(L^p(H))$ such that:

1. $Q(T) \in CV_p(H)$ for every $T \in CV_p(G)$,
2. $\text{supp } Q(T) \subset \text{supp } T$ for every $T \in CV_p(G)$,
3. $Q(i(S)) = S$ for every $S \in CV_p(H)$,
4. $Q(T) \in PM_p(G)$ for every $T \in PM_p(G)$.

**Proof.** Let $\mathcal{A}$ be the set of all pairs $(r_n)_{n=1}^\infty, (s_n)_{n=1}^\infty$ where $(r_n)_{n=1}^\infty$ is a sequence of $L^p(H)$ and $(s_n)_{n=1}^\infty$ is a sequence of $L^p(H)$ with $\sum_{n=1}^\infty ||r_n||_p ||s_n||_p < \infty$. We denote by $\mathcal{E}$ the set of all maps $F$ from $L(L^p(G)) \times L^p(H) \times L^p(H)$ to $C$, linear in the first two variables, conjugate linear in the third one, and for which there is a positive real number $C$ with $|F(T, \varphi, \psi)| \leq C ||T||_p ||\varphi||_p ||\psi||_p$. For $F \in \mathcal{E}$ we put $||F|| = \sup \{ |F(T, \varphi, \psi)| \mid ||T||_p \leq 1, ||\varphi||_p \leq 1, ||\psi||_p \leq 1 \}$. For $k, \ell \in C_0(G)$, $F_{k, \ell}(T, \varphi, \psi) = \langle \langle A_{k, \ell}(T) \varphi, \psi \rangle_{L^p(H), L^p(H)} \rangle_{L^p(H)}$ is an element of $\mathcal{E}$ with $||F_{k, \ell}|| \leq ||T_H||_p ||T_H||_p ||F_H||_p$.

Let $A$ be a finite subset of $\mathcal{A}$, $B$ a finite subset of $CV_p(H)$, $U$ an open neighborhood of $e$ in $G$ and $\varepsilon > 0$. Proposition 2 implies precisely that the set

$$K_{A, B, U, \varepsilon} = \left\{ F_{k, \ell} \mid k, \ell \in C_0^+(G), ||F_{k, \ell}|| \leq 1, \text{ supp } k, \text{ supp } \ell \subset U, \right.$$

$$\left. \sum_{n=1}^\infty |F_{k, \ell}(i(S), r_n, s_n) - \langle S r_n, s_n \rangle_{L^p(H), L^p(H)}| < \varepsilon \right.$$

for every $(r_n)_{n=1}^\infty, (s_n)_{n=1}^\infty \in A$ and every $S \in B$ \right\}

is nonempty. Let $K_{A, B, U, \varepsilon}$ be the closure of $K_{A, B, U, \varepsilon}$ with respect to the topology $\sigma(\mathcal{E}, L(L^p(G)) \times L^p(H) \times L^p(H))$. The set $\cap \{ K_{A, B, U, \varepsilon} \mid A \text{ finite subset of } \mathcal{A}, B \text{ finite subset of } CV_p(H), 0 < \varepsilon < 1, U \text{ open neighborhood of } e \text{ in } G \}$ is not empty. Choose $J$ in this set. There is a linear map $Q$ from $L(L^p(G))$ to $L(L^p(H))$ with $J(T, \varphi, \psi) = \langle Q(T) \varphi, \psi \rangle_{L^p(H), L^p(H)}$ for $T \in L(L^p(G)), \varphi \in L^p(H), \psi \in L^p(H)$. Clearly $Q$ satisfies conditions (1) to (4).

Let $H$ be a closed subgroup of $G$ for which there is a linear map $Q$ from $CV_p(G)$ onto $CV_p(H)$ satisfying conditions (3) and (4) of Theorem 3. Then $H$ is a set of $p$-synthesis in $G$. Indeed let $T \in PM_p(G)$ with $\text{supp } T \subset H$ and $u \in A_p(G)$ with $\text{Res}_H u = 0$. According to Lohoué ([IL], Théorème 5, p. 190), there is an $S \in CV_p(H)$ with $i(S) = T$. From $Q(i(S)) \in PM_p(H)$ we deduce that $S \in PM_p(H)$ and therefore $\langle u, T \rangle_{A_p(G), PM_p(G)} = (\text{Res}_H u, S)_{A_p(H), PM_p(H)} = 0$.

**Corollary 4.** Let $G$ be a locally compact group and $H$ a closed subgroup, locally neutral in $G$, for which $G/H$ admits an invariant measure. Then $H$ is a set of $p$-synthesis of $G$.

The following extension theorem was proved by C. Herz for $G$ second countable and $H$ normal in $G$ ([IL], p. 115).
Corollary 5. Let $G$ be a locally compact group and $H$ a closed subgroup as in Theorem 3. Given $u \in A_p(H) \cap C^0_0(H), \varepsilon > 0$ and an open subset $\Omega$ of $G$ with $\text{supp} \ u \subset \Omega$, there exists $v \in A_p(G) \cap C^0_0(G)$ with $\text{Res}_H v = u$, $\|v\|_{A_p(G)} \leq \|u\|_{A_p(H)} + \varepsilon$ and $\text{supp} \ v \subset \Omega$.

Proof. According to [7], p. 115, it suffices to find $v \in A_p(G) \cap C^0_0(G)$ with $\text{supp} \ v \subset \Omega$, $\|v\|_{A_p(G)} \leq \|u\|_{A_p(H)}$ and $\|u - \text{Res}_H v\|_{A_p(H)} < \varepsilon$.

There is $(r_n)_{n=1}^\infty$ and $(s_n)_{n=1}^\infty$, two sequences of $C^0_0(H)$, such that $u = \sum_{n=1}^\infty r_n * s_n$ and $\sum_{n=1}^\infty \|r_n\|_{L^p(H)} \|s_n\|_{L^p(H)} < \infty$. There also exists an open neighborhood $U$ of $e$ in $G$ such that $U \text{supp} \ u \subset U^{-1} \subset \Omega$. There is $k, \ell \in C^0_0(G)$ with $\text{supp} \ k$, $\text{supp} \ \ell \subset U$, $\|A_{k,\ell}\| \leq 1$ and

$$\sum_{n=1}^\infty \left| \langle A_{k,\ell}(i(S)) r_n, r_n \rangle_{L^p(G),L^p(G)} - \langle S r_n, s_n \rangle_{L^p(H),L^p(H)} \right| \leq \varepsilon \|S\|_p$$

for every $S \in CV_p(H)$. There exists a unique $v \in A_p(G)$ such that

$$\langle u, i(S) \rangle_{A_p(G),PM_p(G)} = \langle u, A_{k,\ell}(S) \rangle_{A_p(H),PM_p(H)}$$

for every $S \in PM_p(H)$. From

$$\left| \langle u, S \rangle_{A_p(H),PM_p(H)} - \langle \text{Res}_H v, S \rangle_{A_p(H),PM_p(H)} \right|$$

we get $\|u - \text{Res}_H v\|_{A_p(H)} \leq \varepsilon$ with $\text{supp} \ v \subset \text{supp} \ k \text{ supp} \ u \ (\text{supp} \ \ell)^{-1}$.

Remark. Suppose $G$ is abelian. According to J. Inoue [8] for every neighborhood $U$ of $e$ in $G$ there is a linear isometric map $\Omega$ of $A_2(H)$ into $A_2(G)$ with $\text{Res}_H \circ \Omega = \text{id}_{A_2(H)}$ (such a map is called a linear lifting) and $\text{supp} \ \Omega(u) \subset (\text{supp} \ u)U$. By duality we easily derive the existence of a projection of $PM_2(G)$ onto $PM_2(H)$ as in Theorem 3. On the other hand B. Forrest [4] has shown that for $G$ amenable and $H$ closed abelian normal subgroup of $G$ a linear lifting does not always exist. Consequently the map $Q$ of Theorem 3 can be considered as a substitute, for $G$ nonabelian, to the nonexistence of linear liftings of $A_2(H)$ into $A_2(G)$.

4. INVARIANT PROJECTIONS

In [1] we proved for $H$ normal in $G$ the existence of a projection $\mathcal{P}$ of $CV_p(G)$ onto $CV_p(H)$ satisfying the condition $\mathcal{P}(uT) = (\text{Res}_H u)\mathcal{P}(T)$; however condition (2) of Theorem 3 was out of our reach. A projection $\mathcal{P}$ with $\mathcal{P}(uT) = (\text{Res}_H u)\mathcal{P}(T)$ will be called an invariant projection.

Theorem 6. Let $G$ be a locally compact group and $H$ a closed neutral subgroup of $G$. There is an invariant projection $\mathcal{P}$ of $CV_p(G)$ onto $CV_p(H)$ which also satisfies all conditions of Theorem 3.

Proof. Let $X$ be the set of all maps $f$ from $\mathcal{L}(L^p(G)) \times L^p(G) \times L^p(G)$ to $\mathbb{C}$ which are linear in the first two variables and conjugate linear in the third one and for which there is a positive real number $C$ with $| f(T, \varphi, \psi) | \leq C \|T\|_p \|\varphi\|_p \|\psi\|_p$. 
Proof. Let \( H \) and \( T \) be the invariant projection of Theorem 6. From supp \( \mathcal{P}(uT) \subset \supp uT \) and \( \mathcal{P}(uT) = (\text{Res}_H u)\mathcal{P}(T) = S \) we deduce that \( S = 0 \) and therefore \( uT = 0 \).

Remark. For \( H \) normal in \( G \), this corollary is already in [3], p. 103. Our proof there was completely different: it was based on the use of \( A_p(G/H) \). The present approach is not only more conceptual but permits us to treat the case of certain interesting nonnormal subgroups: \( H \) compact or \( G \in [SIN)_H \).

References

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2. A. Derighetti, Relations entre les convoluteurs d’un groupe localement compact et ceux d’un sous-groupe fermé, Bull. Sc. math. 2e Série, 106 (1982), 69–84. \[ MR 84j:43008 \]

For \( a, b \in S(G/H) \) \((S(G/H)\) is the set of all bounded measurable functions on \( G/H \) with compact support), \( T \in \mathcal{L}(L^p(G)) \), \( \varphi, \psi \in C_0(G) \) we define
\[
g_{a,b}(T, \varphi, \psi) = \int_{G/H} \langle TB(\varphi, a)(\hat{t}), B(\psi, b)(\hat{t}) \rangle_{L^p(G), L^{p'}(G)} \, dt
\]
with \( B(\varphi, a)(\hat{t}) = \varphi(x)a(x^{-1}t) \) for \( x, t \in G \). Let \( K \) be a compact subset of \( G \) with \( K \cap H = \emptyset \) and \( \varepsilon > 0 \). At first we show that the set \( D_{K,\varepsilon} = \left\{ g_{a,b} \, | \, a, b \in S(G/H), \, \|a\|_p \|b\|_{p'} < 1 + \varepsilon \right\} \) is nonempty. We indeed choose an open neighborhood \( U_1 \) of \( e \) in \( G \) with \( U_1 = U_1^{-1} \) and \( KU_1 \cap HU_1^{-1} = \emptyset \). There is an open neighborhood \( U_2 \) of \( e \) in \( G \) with \( U_2^{-1} = U_2, \, U_2 \subset U_1 \), and \( U_2H = HU_2 \). Let \( U_3 \) be an open neighborhood of \( e \) in \( G \) relatively compact with \( \overline{U_3} \subset U_2 \) and \( U_3H = HU_3 \). This implies that \( \overline{U_3}H = H \overline{U_3} \). There is an open subset \( U_4 \) of \( G/H \) with \( U_4 \supset \overline{U_3} \) and \( \hat{m}(U_4 - \omega(\overline{U_3})) < (1 + \varepsilon)/p' \). Let \( U_5 = \omega^{-1}(U_4) \cap U_2 \). We consider an open neighborhood \( U_6 \) of \( e \) with \( U_6^{-1} = U_6 \) and \( U_6 \overline{U_3} \subset U_5 \). Let \( v(x) = \int_{G/H} a(xy)b(y) \, dy \) with \( a = \frac{1}{\hat{m}(\omega(\overline{U_3}))} \) and \( b = 1/\omega(U_6) \). We have
\[
\|a\|_p \|b\|_{p'} < \left( \frac{1}{\hat{m}(\omega(\overline{U_3}))} \right)^{1/p'} < 1 + \varepsilon.
\]
Suppose \( v(x) \neq 0 \); there is \( \hat{y} \in G/H \) with \( xy \in \omega(\overline{U_3}) \) and \( \hat{y} \in \omega(U_6 \overline{U_3}) \). This implies \( x \in \overline{U_3}H \) and consequently \( x \in HU_3^{-1} \). Let \( x \in HU_6 \) for every \( \hat{y} \in \omega(\overline{U_3}) \), \( x \in (U_6H \overline{U_3}) \) but \( \omega(U_6H \overline{U_3}) = \omega(U_6 \overline{U_3}) \); this implies precisely \( v(x) = 1 \).

Let \( g \) be an element of \( \cap \{ D_{K,\varepsilon} \, | \, K \text{ compact subset of } G \text{ with } K \cap H = \emptyset, \, 0 < \varepsilon < 1 \} \) where \( \overline{D_{K,\varepsilon}} \) is the closure of \( D_{K,\varepsilon} \) in \( X \) with respect to the topology \( \sigma(X, \mathcal{L}(L^p(G)) \times L^p(G) \times L^{p'}(G)) \). Let \( \mathcal{P} \) be the corresponding map of \( \mathcal{L}(L^p(G)) \) to itself. It suffices to consider \( Q \circ \mathcal{P} \) where \( Q \) is the map of Theorem 3.

Corollary 7. Let \( G \) be a locally compact group, \( H \) a closed neutral subgroup of \( G \) and \( F \) a closed subset of \( H \). If \( F \) is locally \( p \)-Ditkin \((1 < p < \infty)\) with respect to \( H \), then \( F \) is locally \( p \)-Ditkin with respect to \( G \).

Proof. Let \( u \in A_p(G) \), \( T \in \mathcal{CV}_p(G) \) with \( \text{Res}_F u = 0 \) and \( \supp(uT) \subset F \). It suffices to show that \( uT = 0 \). There is \( S \in \mathcal{CV}_p(H) \) such that \( uT = i(S) \). Let \( \mathcal{P} \) be the invariant projection of Theorem 6. From \( \supp \mathcal{P}(uT) \subset \supp uT \) and \( \mathcal{P}(uT) = \text{Res}_H u \mathcal{P}(T) = S \) we deduce that \( S = 0 \) and therefore \( uT = 0 \).


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