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ON ABSORBING EXTENSIONS

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Abstract. Building on the work of Kasparov we show that there always exists a trivial absorbing extension of A by $B \otimes \mathcal{K}$, provided only that A and B are separable. If A is unital there is a unital trivial extension which is unitally absorbing.

1. Introduction

Absorbing trivial extensions play an important role in the theory of extensions of C^*-algebras; cf. 15.12 in [Bl]. Recently the interest in such extensions has been renewed because of the way KK-theory comes into the classification program. In this connection, as well as in the proper theory of C^*-extensions, it is slightly disturbing that the existence of an absorbing trivial extension has only been established in the case where at least one of the C^*-algebras involved is nuclear; cf. Theorem 5 of [K]. The purpose of the present note is to show that such extensions always exist when both C^*-algebras are separable. The argument for this is a modification of Kasparov's approach from [K]. The absorbing trivial extensions were constructed, in [K] as well as before Kasparov's work, by taking the infinite direct sum of the same copy of a faithful unital representation of the separable C^*-algebra A (for the moment assumed to be unital) which plays the role of the quotient in the extensions. The resulting representation $\pi : A \to \mathcal{B}(\mathcal{H})$ was then composed with the natural imbedding $\mathcal{B}(\mathcal{H}) \subseteq \mathcal{M}(B \otimes \mathcal{K})$, where $B \otimes \mathcal{K}$ is the stable C^*-algebra which features as the ideal in the extensions. So in practice this means that the absorbing extension was constructed by taking a weak* dense sequence of states of A, repeating all states in the sequence infinitely often, and then adding the corresponding GNS-representations. This procedure has nothing to do with the C^*-algebra B, and it is a highly non-trivial task to show that it often results in an absorbing extension when prolonged to a map $A \to \mathcal{M}(B \otimes \mathcal{K})$; cf. [K]. The observation we offer here is that if one instead takes a sequence $s_n : A \to B \otimes \mathcal{K}$ of completely positive contractions which is dense for the topology of pointwise norm-convergence among all completely positive contractions (such a sequence exists when both A and B are separable), repeats each s_n infinitely often and add up the unital representations

$$\pi_n \ : \ A \to \mathcal{M}(B \otimes \mathcal{K}) \ , \ n \in \mathbb{N} \ ,$$

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coming from the Kasparov-Stinespring decompositions

$$s_n(\cdot) \;=\; W_n^* \pi_n(\cdot) W_n \ ,$$

the resulting representation $A \to \mathcal{M}(B \otimes \mathcal{K})$ will be a unitally absorbing trivial extension. The general trivial absorbing extensions can then be obtained (for a not necessarily unital C^*-algebra A) by taking a unitally absorbing representation $\pi : A^+ \to \mathcal{M}(B \otimes \mathcal{K})$ and restricting it to A.

In order to illustrate how the absorbing $*$-homomorphisms constructed here can be used to extend known results we prove a general version of the Paschke-Valette-Skandalis duality which realizes the group $KK(A, B)$ as the K_1-group of a C^*-algebra D_π built out of A and B by using an absorbing $*$-homomorphism $\pi : A \to \mathcal{M}(B)$; cf. [P], [V], [S], [H].

2. Absorbing $*$-homomorphisms

Given Hilbert B-modules E and F, we let $\mathcal{L}_B(E, F)$ denote the Banach space of adjoinable operators from E to F. The ideal of 'compact' operators from E to F is denoted by $\mathcal{K}_B(E, F)$. When $E = F$ we write $\mathcal{L}_B(E)$ and $\mathcal{K}_B(E)$ instead of $\mathcal{L}_B(E, E)$ and $\mathcal{K}_B(E, E)$, respectively. In the special case where $E = B$ there are well-known identifications $\mathcal{L}_B(B) = \mathcal{M}(B) =$ the multiplier algebra of B and $\mathcal{K}_B(B) = B$, which we shall use freely.

Theorem 2.1. *Let A and B be separable C^*-algebras with A unital and B stable. Let $\pi : A \to \mathcal{M}(B)$ be a unital $*$-homomorphism. Then the following conditions are equivalent:*

1) *For any completely positive contraction $\varphi : A \to B$ there is a sequence $\{W_n\} \subseteq \mathcal{M}(B)$ such that*
   1a) $\lim_{n\to\infty} \|\varphi(a) - W_n^* \pi(a) W_n\| = 0$ *for all* $a \in A$,
   1b) $\lim_{n\to\infty} \|W_n^* b\| = 0$ *for all* $b \in B$.
2) *For any completely positive unital map $\varphi : A \to \mathcal{M}(B)$ there is a sequence $\{V_n\}$ of isometries in $\mathcal{M}(B)$ such that*
   2a) $V_n^* \pi(a) V_n - \varphi(a) \in B \ , \ n \in \mathbb{N} \ , \ a \in A \ ,$
   2b) $\lim_{n\to\infty} \|V_n^* \pi(a) V_n - \varphi(a)\| = 0 \ , \ a \in A.$
3) *For any unital $*$-homomorphism $\varphi : A \to \mathcal{M}(B)$ there is a sequence $\{U_n\}$ of unitaries $U_n \in \mathcal{L}_B(B \oplus B, B)$ such that*
   3a) $U_n \begin{pmatrix} \pi(a) & 0 \\ 0 & \varphi(a) \end{pmatrix} U_n^* - \pi(a) \in B \ , \ n \in \mathbb{N} \ , \ a \in A,$
   3b) $\lim_{n\to\infty} \|U_n \begin{pmatrix} \pi(a) & 0 \\ 0 & \varphi(a) \end{pmatrix} U_n^* - \pi(a)\| = 0 \ , \ a \in A \ .$
4) *For any unital $*$-homomorphism $\varphi : A \to \mathcal{M}(B)$ there is a sequence $\{U_n\}$ of unitaries $U_n \in \mathcal{L}_B(B \oplus B, B)$ such that*

$$\lim_{n\to\infty} \|U_n \begin{pmatrix} \pi(a) & 0 \\ 0 & \varphi(a) \end{pmatrix} U_n^* - \pi(a)\| = 0 \ , \ a \in A \ .$$

Proof. 1) $\Rightarrow$ 2): Let $F \subseteq A$ be a finite set containing 1 and $\epsilon > 0$. Let $\varphi : A \to \mathcal{M}(B)$ be a completely positive unital map. It suffices to find an element $V \in \mathcal{M}(B)$ such that

$$V^* \pi(a) V - \varphi(a) \in B \tag{2.1}$$

for all $a \in A$ and

$$\|V^*\pi(x)V - \varphi(x)\| < 3\epsilon \tag{2.2}$$

for all $x \in F$. If namely ϵ is small enough this will imply that $W = V[V^*V]^{-\frac{1}{2}}$ is an isometry close to V such that $V - W \in B$, and we can then work with W instead of V. We repeat Kasparov's arguments: Let X be a compact subset of A containing F and with dense span in A. By Lemma 10 of [K] there is a sequence $\psi_k : A \to B,\ k \in \mathbb{N}$, of completely positive contractions such that $\psi(a) = \sum_{k=1}^{\infty} \psi_k(a)$ converges in the strict topology, $\varphi(a) - \psi(a) \in B$ for all $a \in A$, and $\|\varphi(x) - \psi(x)\| < \epsilon$ for all $x \in X$. Let $\{b_i\}$ be a countable approximate unit for B. It follows from 1) that we can find a sequence $\{m_i\} \subseteq B$ such that

1) $\|\psi_i(x) - m_i^*\pi(x)m_i\| \leq \epsilon 2^{-i}$, $x \in X,\ i \in \mathbb{N}$,
2) $\|m_i^*\pi(x)m_j\| \leq \epsilon 2^{-i-j}$, $x \in X,\ i, j \in \mathbb{N},\ i \neq j$,
3) $\sum_{i=1}^{\infty} \|m_i^* b_k\| < \infty$ for all $k \in \mathbb{N}$.

The argument from the proof of Theorem 5 in [K] shows that $\sum_{i=1}^{\infty} m_i$ converges in the strict topology to an element $V \in \mathcal{M}(B)$ satisfying (2.1) and (2.2).

2) $\Rightarrow$ 3): This follows from the arguments of Arveson given on pp. 338-339 of [A] by substituting Hilbert spaces with Hilbert B-modules. We leave this to the reader.

3) $\Rightarrow$ 4) is trivial.

4) $\Rightarrow$ 1): Let $\varphi : A \to B$ be a completely positive contraction. Let $F \subseteq A$ and $G \subseteq B$ be finite sets and $\epsilon > 0$. Since A and B are separable it suffices to find an element $L \in \mathcal{M}(B)$ such that $\|\varphi(a) - L^*\pi(a)L\| < \epsilon,\ a \in F$, and $\|Lb\| < \epsilon$ for all $b \in B$. By Kasparov's Stinespring theorem (Theorem 3 of [K]), there is a unital $*$-homomorphism $\chi : A \to \mathcal{M}(B)$ and an element $W \in \mathcal{M}(B)$ such that $\varphi(\cdot) = W^*\chi(\cdot)W$. Let $S_i,\ i = 1, 2, 3, \cdots$, be a sequence of isometries in $\mathcal{M}(B)$ such that $S_i^* S_j = 0,\ i \neq j$, and $\sum_{i=1}^{\infty} S_i S_i^* = 1$ in the strict topology, and set $\chi^{\infty}(a) = \sum_{i=1}^{\infty} S_i \chi(a) S_i^*$. It follows from 4) that there is a sequence $\{U_n\}$ of unitaries in $\mathcal{L}_B(B \oplus B, B)$ such that

$$\lim_{n\to\infty} \|U_n \begin{pmatrix} \pi(a) & 0 \\ 0 & \chi^{\infty}(a) \end{pmatrix} U_n^* - \pi(a)\| = 0\ ,\ a \in A\ .$$

Define $T_i : B \to B \oplus B$ by $T_i b = (0, S_i b)$. Then $\chi(a) = T_i^* \begin{pmatrix} \pi(a) & 0 \\ 0 & \chi^{\infty}(a) \end{pmatrix} T_i$ and $\varphi(a) = W^* T_i^* \begin{pmatrix} \pi(a) & 0 \\ 0 & \chi^{\infty}(a) \end{pmatrix} T_i W$ for all a and i. Choose n so large that

$$\| \begin{pmatrix} \pi(a) & 0 \\ 0 & \chi^{\infty}(a) \end{pmatrix} - U_n^* \pi(a) U_n \| < \frac{\epsilon}{1 + \|W\|^2}\ ,\ a \in F\ .$$

Then

$$\|\varphi(a) - W^* T_i^* U_n^* \pi(a) U_n T_i W\| < \epsilon\ ,\ a \in F,$$

for all i. Since $\lim_{i\to\infty} \|T_i^* x\| = 0$ for all $x \in B \oplus B$, we can choose i so large that $\|W^* T_i^* U_n^* b\| < \epsilon$ for all $b \in G$. Set $L = U_n T_i W$. □

Definition 2.2. Let A and B be separable C^*-algebras with A unital and B stable. A unital $*$-homomorphism $\pi : A \to \mathcal{M}(B)$ which satisfies the four equivalent conditions in Theorem 2.1 is called *unitally absorbing* (for (A, B)).

The following lemma is surely known, but it is so crucial for us here that we include a proof.

Lemma 2.3. *Let A and B be separable C^*-algebras. There is then a countable set X of completely positive contractions $A \to B$ such that for any completely positive contraction $\mu : A \to B$, any finite set $F \subseteq A$ and any $\epsilon > 0$ there is an element $l \in X$ such that*

$$\|\mu(f) - l(f)\| \leq \epsilon , \ f \in F .$$

Proof. Let $\{a_1, a_2, a_3, \cdots\}$ be a dense sequence in the unit ball of A and set $F_n = span\{a_1, a_2, \cdots, a_n\}$. Let ω be a faithful state of A and let (π_ω, H_ω) be the GNS-representation coming from ω. We can then consider A as a subspace of H_ω. The orthogonal projection $P_n : H_\omega \to F_n$ gives us then by restriction a continuous idempotent map $P_n : A \to F_n$. Let $1 < m_1 < m_2 < m_3 < \cdots$ be a sequence of numbers such that $\|P_n\| \leq m_n$ for all n. We can then define a metric d on the space $\mathcal{B}(A, B)$ of continuous linear maps $L : A \to B$ by

$$d(L_1, L_2) = \sum_{i=1}^{\infty} \frac{2^{-i}}{m_i} \|L_1(a_i) - L_2(a_i)\| .$$

Choose a linear basis $\{x_1, x_2, \cdots, x_{n_0}\}$ for F_n. For each n_0-tuple $\underline{b} = (b_1, b_2, \cdots, b_{n_0}) \in B^{n_0}$ there is a linear map $L_{\underline{b}} : F_n \to B$ such that $L_{\underline{b}}(x_i) = b_i, \ i = 1, 2, \cdots, n$. By using that B^{n_0} is separable this construction gives us a countable set $\mathcal{M}$ of linear maps $F_n \to B$ which is dense in the strong topology of $\mathcal{B}(F_n, B)$. Now let $0 < \epsilon < 1$ and let a finite set $D \subseteq F_n$ be given. Let $\mu \in \mathcal{B}(F_n, B)$ be a contraction. There is a finite subset G of F_n such that every $x \in F_n$ with $\|x\| \leq 1 - \epsilon$ is a convex combination of elements from G. Choose $l \in \mathcal{M}$ such that

$$\|\mu(z) - l(z)\| < \epsilon , \ z \in D \cup G . \tag{2.3}$$

Then $\|\mu(x) - l(x)\| \leq \epsilon$ for all $x \in F_n$ with $\|x\| \leq 1 - \epsilon$, and hence $\|l\| \leq \frac{1+\epsilon}{1-\epsilon}$. Let q be a positive rational number in $]\frac{1-2\epsilon}{1+\epsilon}, \frac{1-\epsilon}{1+\epsilon}[$. Then $ql \in \mathbb{Q}_+\mathcal{M}$ is a contraction and we find that

$$\begin{aligned} \|\mu(z) - ql(z)\| &\leq \|\mu(z) - l(z)\| + \|l(z) - ql(z)\| \\ &\leq \epsilon + |1 - q|\|l\| \sup\{\|z\| : z \in D\} \\ &< \frac{2\epsilon + 2\epsilon^2}{1 - \epsilon^2} \sup\{\|z\| : z \in D\} + \epsilon \end{aligned}$$

for all $z \in D$. It follows that we can find a countable set $\mathcal{Y}_n \subseteq \mathbb{Q}_+\mathcal{M}$ of linear contractions which is strongly dense among all contractions in $\mathcal{B}(F_n, B)$. Set

$$\mathcal{Y} = \bigcup_{n=1}^{\infty} \{l \circ P_n : l \in \mathcal{Y}_n\} .$$

Let $\mu : A \to B$ be a linear contraction and let $\epsilon > 0$. Choose n so large that $2\sum_{i \geq n+1} 2^{-i} < \frac{\epsilon}{2}$. From what we have just proved there is an element $l \in \mathcal{Y}_n$ such that

$$\|\mu(a_i) - l(a_i)\| < \frac{\epsilon}{2}, \ i = 1, 2, \cdots, n.$$

Then $l \circ P_n \in \mathcal{Y}$ and

$$d(\mu, l \circ P_n) \ \leq \ \sum_{i=1}^{n} \frac{2^{-i}}{m_i}\frac{\epsilon}{2} \ + \sum_{i \geq n+1} \frac{2^{-i}}{m_i}(1 + \|P_n\|)$$
$$\leq \ \frac{\epsilon}{2} + \sum_{i \geq n+1} \frac{2^{-i}}{m_i}(1 + m_i) \ \leq \ \epsilon \ .$$

It follows that $\mathcal{Y}$ is a countable set in $\mathcal{B}(A, B)$ with the property that for any linear contraction $\mu : A \to B$ and any $\epsilon > 0$ there is an element $l \in \mathcal{Y}$ such that $d(\mu, l) < \epsilon$. For each $l \in \mathcal{Y}$ choose a completely positive contraction $l' : A \to B$ such that

$$d(l, l') \leq 2 \inf\{d(l, L) \ : \ L \in \mathcal{B}(A, B) \text{ is a completely positive contraction}\}.$$

Then $\mathcal{Y}' = \{l' : l \in \mathcal{Y}\}$ is a countable set of completely positive contractions in $\mathcal{B}(A, B)$ with the property that for any completely positive linear contraction $\mu : A \to B$ and any $\epsilon > 0$ there is an element $l \in \mathcal{Y}'$ such that $d(\mu, l) < \epsilon$. □

Theorem 2.4. *Let A and B be separable C^*-algebras. Assume that B is stable and A unital. Then there exists a unitally absorbing $*$-homomorphism $\pi : A \to \mathcal{M}(B)$ for (A, B).*

Proof. By Lemma 2.3 there is a dense sequence $\{s_n\}$ in the set of completely positive contractions from A to B. We may assume that each s_n is repeated infinitely often in this sequence. By Kasparov's Stinespring Theorem (Theorem 3 of [K]), there are elements $V_n \in \mathcal{M}(B)$ and unital $*$-homomorphisms $\pi_n : A \to \mathcal{M}(B)$ such that

$$s_n(\cdot) \ = \ V_n^* \pi_n(\cdot) V_n$$

for all n. Note that $\|V_n\|^2 = \|V_n^* V_n\| = \|s_n(1)\| \leq 1$ for all n. Define a unital $*$-homomorphism $\pi_\infty : A \to \mathcal{L}_B(l_2(B))$ by

$$\pi_\infty(a)(b_1, b_2, b_3, \cdots) \ = \ (\pi_1(a)b_1, \pi_2(a)b_2, \pi_3(a)b_3, \cdots).$$

Define $L_n \in \mathcal{L}_B(B, l_2(B))$ by

$$L_n b = (0, 0, \cdots, 0, V_n b, 0, 0, \cdots),$$

where the non-trivial entry occurs at the n'th coordinate. Since we repeated the s_n's infinitely often there is, for each n, a sequence $k_1 < k_2 < k_3 < \cdots$ in $\mathbb{N}$ such that

$$s_n(a) \ = \ L_{k_i}^* \pi_\infty(a) L_{k_i} \tag{2.4}$$

for all $a \in A, \ i \in \mathbb{N}$, and

$$\lim_{i \to \infty} \|L_{k_i}^* \psi\| = 0 \ , \qquad \psi \in l_2(B) \ . \tag{2.5}$$

By Lemma 1.3.2 of [KJT] there is an isomorphism $S : l_2(B) \to B$ of Hilbert B-modules. Set $T_n = SL_n \in \mathcal{M}(B)$ and $\pi(\cdot) \ = \ S\pi_\infty(\cdot)S^*$. We assert that π satisfies condition 1) of Theorem 2.1, and to prove it we let $\varphi : A \to B$ be a completely positive contraction. In order to construct a sequence $\{W_n\}$ in $\mathcal{M}(B)$ such that 1a) and 1b) of Theorem 2.1 hold it suffices, because A and B are separable, to pick $\epsilon > 0$ and finite subsets $F_1 \subseteq A$ and $F_2 \subseteq B$ and find an element $W \in \mathcal{M}(B)$ such that $\|\varphi(a) - W^*\pi(a)W\| < \epsilon, \ a \in F_1$, and $\|W^* b\| < \epsilon, \ b \in F_2$. Choose first an $n \in \mathbb{N}$ such that $\|\varphi(a) - s_n(a)\| < \epsilon, \ a \in F_1$. If we then choose $k_1 < k_2 < k_3 < \cdots$

such that (2.4) and (2.5) hold we have that $T_{k_i}^* \pi(a) T_{k_i} = s_n(a)$ for all $a \in F_1$ and $\|T_{k_i}^* b\| < \epsilon$ for all $b \in F_2$, provided only that i is large enough. We can then set $W = T_{k_i}$ for such an i. □

We now turn to the case of a not neccesarily unital C^*-algebra A and the general notion of absorbing $*$-homomorphisms. Given a C^*-algebra A we denote in the following by A^+ the C^*-algebra obtained by adding a unit to A. Let B be another C^*-algebra. Any linear completely positive contraction $\varphi : A \to \mathcal{M}(B)$ admits a unique linear extension $\varphi^+ : A^+ \to \mathcal{M}(B)$ such that $\varphi^+(1) = 1$. φ^+ is automatically a completely positive contraction (cf. e.g. Lemma 3.2.8 of [KJT]), and is automatically a $*$-homomorphism when φ is. The following theorem is therefore an immediate consequence of Theorem 2.1.

Theorem 2.5. *Let A and B be separable C^*-algebras with B stable. Let $\pi : A \to \mathcal{M}(B)$ be a $*$-homomorphism. Then the following conditions are equivalent:*

1) $\pi^+ : A^+ \to \mathcal{M}(B)$ *is unitally absorbing for* (A^+, B).
2) *For any completely positive contraction $\varphi : A \to \mathcal{M}(B)$ there is a sequence $\{V_n\}$ of isometries in $\mathcal{M}(B)$ such that*
   2a) $V_n^* \pi(a) V_n - \varphi(a) \in B$, $n \in \mathbb{N}$, $a \in A$,
   2b) $\lim_{n\to\infty} \|V_n^* \pi(a) V_n - \varphi(a)\| = 0$, $a \in A$.
3) *For any $*$-homomorphism $\varphi : A \to \mathcal{M}(B)$ there is a sequence $\{U_n\}$ of unitaries $U_n \in \mathcal{L}_B(B \oplus B, B)$ such that*
   3a) $U_n \begin{pmatrix} \pi(a) & 0 \\ 0 & \varphi(a) \end{pmatrix} U_n^* - \pi(a) \in B$, $n \in \mathbb{N}$, $a \in A$,
   3b) $\lim_{n\to\infty} \|U_n \begin{pmatrix} \pi(a) & 0 \\ 0 & \varphi(a) \end{pmatrix} U_n^* - \pi(a)\| = 0$, $a \in A$.
4) *For any $*$-homomorphism $\varphi : A \to \mathcal{M}(B)$ there is a sequence $\{U_n\}$ of unitaries $U_n \in \mathcal{L}_B(B \oplus B, B)$ such that*
$$\lim_{n\to\infty} \|U_n \begin{pmatrix} \pi(a) & 0 \\ 0 & \varphi(a) \end{pmatrix} U_n^* - \pi(a)\| = 0, \ a \in A.$$

Definition 2.6. Let A and B be separable C^*-algebras with B stable. A $*$-homomorphism $\pi : A \to \mathcal{M}(B)$ is *absorbing* (for (A, B)) when it satisfies the four equivalent conditions of Theorem 2.5.

Theorem 2.7. *Let A and B be separable C^*-algebras with B stable. There exists an absorbing $*$-homomorphism $\pi : A \to \mathcal{M}(B)$ for (A, B).*

Proof. Combine Theorem 2.5 and Theorem 2.4. □

An absorbing $*$-homomorphism is clearly unique in the following sense: Given two absorbing $*$-homomorphisms $\pi_1, \pi_2 : A \to \mathcal{M}(B)$ there is a sequence $\{U_n\} \subseteq \mathcal{M}(B)$ of unitaries such that $U_n \pi_1(a) U_n^* - \pi_2(a) \in B$, $a \in A$, $n \in \mathbb{N}$, and $\lim_{n\to\infty} U_n \pi_1(a) U_n^* - \pi_2(a) = 0$, $a \in A$.

3. Duality in KK-theory

Throughout this section A and B will be separable C^*-algebras and B will be stable. A $*$-homomorphism $\pi : A \to \mathcal{M}(B)$ is *of infinite multiplicity* when π is unitarily equivalent to π^∞, where $\pi^\infty : A \to \mathcal{M}(B)$ is the $*$-homomorphism given by $\pi^\infty(a) = \sum_{i=1}^\infty S_i \pi(a) S_i^*$, for some sequence S_i, $i \in \mathbb{N}$, of isometries in $\mathcal{M}(B)$ such that $S_i^* S_j = 0$, $i \neq j$, and $\sum_{i=1}^\infty S_i S_i^* = 1$ in the strict topology.

Lemma 3.1. *Let $\pi : A \to \mathcal{M}(B)$ be a $*$-homomorphism of infinite multiplicity and set*

$$E \;=\; \{m \in \mathcal{M}(B) \;:\; m\pi(a) = \pi(a)m \;\forall a \in A\;\} \;.$$

Then $K_*(E) = \{0\}$.

Proof. Since π has infinite multiplicity,

$$E \;\simeq\; \{m \in \mathcal{L}_B(l_2(B)) \;:\; m\mu(a) = \mu(a)m \;\forall a \in A\}$$

where $\mu : A \to \mathcal{L}_B(l_2(B))$ is given by

$$\mu(a)(b_1, b_2, b_3, \cdots) \;=\; (\pi(a)b_1, \pi(a)b_2, \pi(a)b_3, \cdots).$$

The usual proof that $K_*(\mathcal{L}_B(l_2(B))) = 0$ works to show that $K_*(E) = 0$; cf. e.g. Proposition 12.2.1 of [Bl]. ☐

Given an absorbing $*$-homomorphism $\pi : A \to \mathcal{M}(B)$ we set

$$C_\pi \;=\; \{x \in \mathcal{M}(B) \;:\; x\pi(a) - \pi(a)x \in B \;,\; a \in A\}$$

and

$$A_\pi \;=\; \{x \in C_\pi \;:\; x\pi(A) \subseteq B\} \;.$$

Then A_π is a closed two-sided ideal in C_π and we set $D_\pi \;=\; C_\pi/A_\pi$. The quotient map $C_\pi \to D_\pi$ will be denoted by q. If $\tau : A \to \mathcal{M}(B)$ is another absorbing $*$-homomorphism there is a unitary $w \in \mathcal{M}(B)$ such that $\operatorname{Ad} w \circ \pi(a) - \tau(a) \in B$ for all $a \in A$ and then $x \mapsto wxw^*$ defines a $*$-isomorphism of C_π onto C_τ which takes A_π onto A_τ. In particular, $D_\pi \;\simeq\; D_\tau$.

Let u be a unitary in $M_n(D_\pi)$. Choose $v \in M_n(C_\pi)$ such that $\mathrm{id}_{M_n} \otimes q(v) \;=\; u$. Define $\pi^n : A \to \mathcal{L}_B(B^n)$ by $\pi^n(a)(b_1, b_2, \cdots, b_n) \;=\; (\pi(a)b_1, \pi(a)b_2, \cdots, \pi(a)b_n)$. Let $B^n \oplus B^n$ be graded by $(x, y) \mapsto (x, -y)$. Then

$$\left(B^n \oplus B^n \;,\; \begin{pmatrix} \pi^n & \\ & \pi^n \end{pmatrix} \;,\; \begin{pmatrix} & v \\ v^* & \end{pmatrix}\right)$$

is a Kasparov $A - B$-module. We leave it to the reader to check that the class of this module in $KK(A, B)$ only depends on the class of u in $K_1(D_\pi)$, and that the construction gives rise to a group homomorphism $\Theta : K_1(D_\pi) \to KK(A, B)$.

Theorem 3.2. *Assume that $\pi : A \to \mathcal{M}(B)$ is an absorbing $*$-homomorphism. Then $\Theta \;:\; K_1(D_\pi) \to KK(A, B)$ is an isomorphism.*

Proof. When τ is another absorbing $*$-homomorphism there is a commuting diagram

(3.1)
$$\begin{array}{ccc} K_1(D_\pi) & \xrightarrow{\;\Theta\;} & KK(A,B) \\ \big\downarrow & \nearrow & \\ K_1(D_\tau) & & \end{array}$$

where $K_1(D_\pi) \to K_1(D_\tau)$ is induced by the isomorphism $D_\pi \to D_\tau$ described above, and $K_1(D_\tau) \to KK(A, B)$ is the map obtained by using τ instead of π in the definition of Θ. Indeed if one considers a specific unitary in $M_n(D_\pi)$, the Kasparov $A - B$-module which results by going down and up in the diagram differs from the one which arises by going across by an isomorphism and a compact perturbation. Thus if we prove that $\Theta : K_1(A_\pi) \to KK(A, B)$ is an isomorphism for one absorbing

$*$-homomorphism π it will follow that it is an isomorphism for any other. Hence by working with π^∞ instead of π we may assume that π is of infinite multiplicity.

Θ is injective: Let $u \in M_n(D_\pi)$ be a unitary and choose $v \in M_n(C_\pi)$ such that $\mathrm{id}_{M_n} \otimes q(v) = u$. Assume that $[B^n \oplus B^n,\ \begin{pmatrix} \pi^n & \\ & \pi^n \end{pmatrix},\ \begin{pmatrix} & v \\ v^* & \end{pmatrix}] = 0$ in $KK(A,B)$. This means that there are degenerate Kasparov $A-B$-modules $\mathcal{D}_1$ and $\mathcal{D}_2$ such that $(B^n \oplus B^n,\ \begin{pmatrix} \pi^n & \\ & \pi^n \end{pmatrix},\ \begin{pmatrix} & v \\ v^* & \end{pmatrix}) \oplus \mathcal{D}_1$ is operator homotopic to $(B^n \oplus B^n,\ \begin{pmatrix} \pi^n & \\ & \pi^n \end{pmatrix},\ \begin{pmatrix} & 1 \\ 1 & \end{pmatrix})$ $\oplus\, \mathcal{D}_2$. Since $\mathcal{D}_1$ and $\mathcal{D}_2$ are degenerate we can define a new degenerate Kasparov $A-B$-module $\mathcal{D}$ by $\mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \cdots$. Then $\mathcal{D}_1 \oplus \mathcal{D}$ and $\mathcal{D}_2 \oplus \mathcal{D}$ are both isomorphic to $\mathcal{D}$ and hence $(B^n \oplus B^n,\ \begin{pmatrix} \pi^n & \\ & \pi^n \end{pmatrix},\ \begin{pmatrix} & v \\ v^* & \end{pmatrix}) \oplus \mathcal{D}$ is operator homotopic to $(B^n \oplus B^n,\ \begin{pmatrix} \pi^n & \\ & \pi^n \end{pmatrix},\ \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}) \oplus \mathcal{D}$. By combining Kasparov's stabilization theorem (Theorem 2.12 of [KJT]) with Lemma 1.3.2 of [KJT] we may assume that $a = w$ and $b = w^*$ for some unitary $w \in \mathcal{M}(B)$. Finally, by applying the unitary of the Hilbert B-module $B \oplus B$ given by $(x,y) \mapsto (x, wy)$, we see that we can assume that $w = 1$. So all in all we have that

$$(B^n \oplus B^n\ ,\ \begin{pmatrix} \pi^n & \\ & \pi^n \end{pmatrix}\ ,\ \begin{pmatrix} & v \\ v^* & \end{pmatrix})\ \oplus\ (B \oplus B\ ,\ \begin{pmatrix} \lambda_+ & \\ & \lambda_- \end{pmatrix}\ ,\ \begin{pmatrix} & 1 \\ 1 & \end{pmatrix})$$

is operator homotopic to

$$(B^n \oplus B^n\ ,\ \begin{pmatrix} \pi^n & \\ & \pi^n \end{pmatrix}\ ,\ \begin{pmatrix} & 1 \\ 1 & \end{pmatrix})\ \oplus\ (B \oplus B\ ,\ \begin{pmatrix} \lambda_+ & \\ & \lambda_- \end{pmatrix}\ ,\ \begin{pmatrix} & 1 \\ 1 & \end{pmatrix})\ .$$

Note that $\lambda_+ = \lambda_-$ since $(B \oplus B,\ \begin{pmatrix} \lambda_+ & \\ & \lambda_- \end{pmatrix},\ \begin{pmatrix} & 1 \\ 1 & \end{pmatrix})$ is degenerate. Finally, by adding on an infinite number of copies of $(B \oplus B,\ \begin{pmatrix} \lambda_+ & \\ & \lambda_- \end{pmatrix},\ \begin{pmatrix} & 1 \\ 1 & \end{pmatrix})$ we find that there is a $*$-homomorphism of infinite multiplicity $\lambda : A \to \mathcal{M}(B)$ such that

$$(B^n \oplus B^n\ ,\ \begin{pmatrix} \pi^n & \\ & \pi^n \end{pmatrix}\ ,\ \begin{pmatrix} & v \\ v^* & \end{pmatrix})\ \oplus\ (B \oplus B\ ,\ \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix}\ ,\ \begin{pmatrix} & 1 \\ 1 & \end{pmatrix})$$

is operator homotopic to

$$(B^n \oplus B^n\ ,\ \begin{pmatrix} \pi^n & \\ & \pi^n \end{pmatrix}\ ,\ \begin{pmatrix} & 1 \\ 1 & \end{pmatrix})\ \oplus\ (B \oplus B\ ,\ \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix}\ ,\ \begin{pmatrix} & 1 \\ 1 & \end{pmatrix})\ .$$

Furthermore, by adding on $(B \oplus B,\ \begin{pmatrix} \pi & \\ & \pi \end{pmatrix},\ \begin{pmatrix} & 1 \\ 1 & \end{pmatrix})$ we may assume that there is a unitary $w \in \mathcal{M}(B)$ such that $w\lambda(a)w^* - \pi(a) \in B,\ a \in A$. The operator homotopy consists of an isomorphism of Kasparov $A-B$ modules and a norm-continuous path of operators. The isomorphism gives us a unitary $S \in M_{n+1}(\mathcal{M}(B))$ such that $S \begin{pmatrix} \pi^n(a) & \\ & \lambda(a) \end{pmatrix} = \begin{pmatrix} \pi^n(a) & \\ & \lambda(a) \end{pmatrix} S$ for all $a \in A$, and in addition we have a norm-continuous path $F_t,\ t \in [0,1]$, in $M_{n+1}(\mathcal{M}(B))$ such that $F_0 = S,\ F_1 = \begin{pmatrix} v & \\ & 1 \end{pmatrix}$, and $(F_tF_t^* - 1_{n+1}) \begin{pmatrix} \pi^n(a) & \\ & \lambda(a) \end{pmatrix},\ (F_t^*F_t - 1_{n+1}) \begin{pmatrix} \pi^n(a) & \\ & \lambda(a) \end{pmatrix},\ F_t \begin{pmatrix} \pi^n(a) & \\ & \lambda(a) \end{pmatrix} - \begin{pmatrix} \pi^n(a) & \\ & \lambda(a) \end{pmatrix} F_t$ are in $M_{n+1}(B)$ for all t and a. Here and in the following we let 1_k denote the unit of $M_k(\mathcal{M}(B))$. Note that $\nu = \begin{pmatrix} \pi^n & \\ & \lambda \end{pmatrix}$ is of infinity multiplicity, as a $*$-homomorphism $A \to \mathcal{M}(M_{n+1}(B))$, since π and λ both are of infinite multiplicity. By Lemma 3.1 we can therefore find an $m \in \mathbb{N}$ and a norm-continuous path of unitaries in $\{x \in M_{m(n+1)}(\mathcal{M}(B)) \ :\ x\nu^m(a) = \nu^m(a)x,\ a \in A\ \}$ connecting $\begin{pmatrix} S & \\ & 1_{(m-1)(n+1)} \end{pmatrix}$ to $1_{m(n+1)}$. In combination with F this gives us a norm-continuous path H_t, $t \in [0,1]$, in $M_{m(n+1)}(\mathcal{M}(B))$ such that $H_0 = 1_{m(n+1)}, H_1 = \begin{pmatrix} v & \\ & 1_{m(n+1)-n} \end{pmatrix}$, $(H_tH_t^* - 1 - 1_{m(n+1)})\nu^m(a)$, $(H_t^*H_t - 1_{m(n+1)})\nu^m(a)$, $H_t\nu^m(a) - \nu^m(a)H_t$ are in $M_{m(n+1)}(B)$ for all t and a. Set

$$W\ =\ \mathrm{diag}(\underbrace{1_n, w, 1_n, w, \cdots, 1_n, w}_{m \text{ times}})\ \in\ M_{m(n+1)}(\mathcal{M}(B))$$

and $G_t = WH_tW^*$. Then G_t is a norm-continuous path in $M_{m(n+1)}(\mathcal{M}(B))$ such that $G_0 = 1_{m(n+1)}$, $G_1 = \begin{pmatrix} v & \\ & 1_{m(n+1)-n} \end{pmatrix}$ and $(G_tG_t^* - 1_{m(n+1)})\pi^{m(n+1)}(a)$, $(G_t^*G_t - 1_{m(n+1)})\pi^{m(n+1)}(a)$, $G_t\pi^{m(n+1)}(a) - \pi^{m(n+1)}(a)G_t$ are in $M_{m(n+1)}(B)$ for all t and a. Thus $(\mathrm{id}_{M_{m(n+1)}} \otimes q)(G_t)$ is a path of unitaries in $M_{m(n+1)}(D_\pi)$ connecting $\begin{pmatrix} u & \\ & 1_{m(n+1)-n} \end{pmatrix}$ to $1_{m(n+1)}$.

Θ is surjective: Let (E, ψ, F) be a Kasparov $A-B$-module. The constructions on pages 125-126 of [KJT] show that $[E, \psi, F] \in KK(A, B)$ is also represented by a Kasparov $A-B$-module of the form $(B \oplus B, \begin{pmatrix} \varphi_+ & \\ & \varphi_- \end{pmatrix}, \begin{pmatrix} & v \\ v^* & \end{pmatrix})$ for some $*$-homomorphisms $\varphi_\pm : A \to \mathcal{M}(B)$ and some unitary $v \in \mathcal{M}(B)$. Using the trick from p. 354 of [H] we may assume that $\varphi_- = \varphi_+ = \varphi$. By adding on $(B \oplus B, \begin{pmatrix} \pi & \\ & \pi \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix})$ and using that π is absorbing we may assume that there is a unitary $u \in \mathcal{M}(B)$ such that $u\varphi(a)u^* - \pi(a) \in B$ for all $a \in A$. Then $(B \oplus B, \begin{pmatrix} \varphi & \\ & \varphi \end{pmatrix}, \begin{pmatrix} & v \\ v^* & \end{pmatrix})$ is isomorphic to

$$\left(B \oplus B,\ \begin{pmatrix} \mathrm{Ad}\, u \circ \varphi & \\ & \mathrm{Ad}\, u \circ \varphi \end{pmatrix},\ \begin{pmatrix} & uvu^* \\ uv^*u^* & \end{pmatrix}\right)$$

which in turn is a compact perturbation of $(B \oplus B, \begin{pmatrix} \pi & \\ & \pi \end{pmatrix}, \begin{pmatrix} & uvu^* \\ uv^*u^* & \end{pmatrix})$. Then uvu^* is a unitary C_π such that $\Theta([q(uvu^*)]) = [E, \psi, F]$ in $KK(A, B)$. $\square$

Of course there is also an isomorphism

$$K_0(D_\pi) \simeq \mathrm{Ext}^{-1}(A, B)$$

which can be proved in basically the same way.

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