

**GENERALIZED LITTLE q -JACOBI POLYNOMIALS
AS EIGENSOLUTIONS OF HIGHER-ORDER
 q -DIFFERENCE OPERATORS**

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ABSTRACT. We consider the polynomials $p_n(x; a, b; M)$ obtained from the little q -Jacobi polynomials $p_n(x; a, b)$ by inserting a discrete mass M at $x = 0$ in the orthogonality measure. We show that for $a = q^j$, $j = 0, 1, 2, \dots$, the polynomials $p_n(x; a, b; M)$ are eigensolutions of a linear q -difference operator of order $2j + 4$ with polynomial coefficients. This provides a q -analog of results recently obtained for the Krall polynomials.

1. q -DERIVATIVE OPERATORS AND THEIR REPRESENTATION COEFFICIENTS

Let T be the q -shift operator that acts on functions according to

$$(1.1) \quad T F(x) = F(qx),$$

with $0 < q < 1$ a real number. Obviously

$$(1.2) \quad T^n F(x) = F(q^n x), \quad n = 0, \pm 1, \pm 2, \dots$$

Introduce the q -derivative operator (see, e.g., [2])

$$(1.3) \quad \mathcal{D}_q F(x) = (x(1 - q))^{-1} (1 - T)F(x).$$

It is called the q -derivative because its action on monomials is

$$(1.4) \quad \mathcal{D}_q x^n = [n] x^{n-1},$$

with

$$(1.5) \quad [n] = (q^n - 1)/(q - 1)$$

the so-called q -number. Moreover $\lim_{q \rightarrow 1} \mathcal{D}_q = D$, where D is the ordinary derivative operator with respect to x : $DF(x) = F'(x)$.

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Consider operators of the form

$$(1.6) \quad L_q = \sum_{k=0}^{2N} a_k(x) \mathcal{D}_q^k,$$

where N is a fixed positive integer and $a_k(x)$ are polynomials in x of degrees not exceeding k :

$$(1.7) \quad a_k(x) = \sum_{s=0}^k \alpha_{ks} x^s, \quad k = 0, 1, \dots, 2N.$$

Let us introduce also the related operators

$$(1.8) \quad \mathcal{L}_q = T^{-N} L_q = \sum_{k=0}^{2N} a_k(q^{-N} x) T^{-N} \mathcal{D}_q^k.$$

The operator L_q is a linear combination of the operators $T^{2N}, T^{2N-1}, \dots, T, T^0 = I$, while the operator \mathcal{L}_q is a linear combination of the operators $T^N, T^{N-1}, \dots, T^{1-N}, T^{-N}$.

For $q \rightarrow 1$ both operators L_q and \mathcal{L}_q become $2N$ -order differential operators with polynomial coefficients:

$$(1.9) \quad \lim_{q \rightarrow 1} L_q = \lim_{q \rightarrow 1} \mathcal{L}_q = \sum_{k=0}^{2N} a_k^{(0)}(x) D^k,$$

where $a_k^{(0)}(x) = \lim_{q \rightarrow 1} a_k(x)$.

The operator \mathcal{L}_q will prove more practical in searching for orthogonal polynomials $P_n(x)$ satisfying eigenvalue equations of the kind

$$(1.10) \quad \mathcal{L}_q P_n(x) = \lambda_n P_n(x).$$

It is known that for $N = 1$, the little q -Jacobi polynomials satisfy an equation of the form (1.10) with $N = 1$ [9]. We wish to determine other systems of orthogonal polynomials satisfying equation (1.10) with $N > 1$.

To this end, we shall extend to q -difference operators the method proposed in [12].

The main idea of the method is the following. Consider the action of the operator \mathcal{L}_q upon the monomials x^n . From (1.6) and (1.7) we get

$$(1.11) \quad \mathcal{L}_q x^n = \sum_{s=0}^{2N} A_n^{(s)} x^{n-s},$$

where

$$(1.12) \quad A_n^{(s)} = q^{N(s-n)} [n][n-1] \dots [n-s+1] \pi_s(q^n),$$

and

$$(1.13) \quad \pi_s(q^n) = \alpha_{s0} + \sum_{i=1}^{2N-s} \alpha_{s+i,i} [n-s][n-s-1] \dots [n-s-i+1]$$

are polynomials in $z = q^n$ of degrees not exceeding $2N-s$. It is clear that, moreover,

$$(1.14) \quad A_n^{(s)} = 0, \quad s > 2N.$$

The coefficients $A_n^{(s)}$ completely characterize the operator \mathcal{L}_q . We will call $A_n^{(s)}$ the representation coefficients of the operator \mathcal{L}_q .

Proposition 1.1. *Assume that there are coefficients $A_n^{(s)}$ expressible as in (1.12), where $\pi_s(q^n)$ are arbitrary polynomials in q^n of degrees not exceeding $2N - s$. Assume also that $A_n^{(s)} = 0$, $s > 2N$ (i.e. $\pi_s(q^n) = 0$ for $s > 2N$). Then there exists a unique operator \mathcal{L}_q of the form (1.8) such that $A_n^{(s)}$ are its representation coefficients.*

Proof. Any polynomial $\pi_s(q^n)$ of degree not exceeding $2N - s$ can be presented in form (1.13) with some coefficients α_{ik} . These coefficients are determined using Newton's interpolation formula

$$(1.15) \quad \alpha_{s+i,i} = \frac{(q-1)^i q^{si+i(i-1)/2}}{[i]!} \mathcal{D}_q^i \pi_s(x) \Big|_{x=q^s}, \quad i = 0, 1, \dots, 2N - s,$$

where $[i]! = [1][2] \dots [i]$ is the q -factorial. Clearly, the coefficients α_{ik} are determined uniquely by (1.15) from the given polynomials $\pi_s(q^n)$. Hence the operator \mathcal{L}_q is defined uniquely.

2. BASIC RELATIONS FOR POLYNOMIALS SATISFYING EIGENVALUE EQUATIONS

In this section we consider the basic relations between the representation coefficients $A_n^{(s)}$ and the expansion coefficients of polynomials $P_n(x)$ satisfying eigenvalue equations. Assume that

$$(2.1) \quad P_n(x) = \sum_{s=0}^n B_n^{(s)} x^{n-s},$$

where $B_n^{(s)}$ are expansion coefficients. In what follows we will assume that the polynomials $P_n(x)$ are monic, i.e. that

$$(2.2) \quad B_n^{(0)} = 1.$$

Substituting (2.1) into the eigenvalue equation (1.10) we arrive at the following set of algebraic relations:

$$(2.3) \quad \sum_{i=0}^s B_n^{(s-i)} A_{n-s+i}^{(i)} = \lambda_n B_n^{(s)}, \quad s = 0, 1, 2, \dots, n.$$

These will be central in our analysis.

For $s = 0$, (2.3) gives

$$(2.4) \quad \lambda_n = A_n^{(0)}.$$

Thus from (1.12) we find that the eigenvalues λ_n have the expression

$$(2.5) \quad \lambda_n = q^{-Nn} \pi_0(q^n),$$

where $\pi_0(q^n)$ is a polynomial in q^n of degree not exceeding $2N$.

Similarly, for $s = 1$, (2.3) yields

$$(2.6) \quad A_n^{(1)} = \Omega_n B_n^{(1)},$$

where $\Omega_n = \lambda_n - \lambda_{n-1}$. From this relation we find that

$$(2.7) \quad B_n^{(1)} = [n] \frac{\pi_1(q^n)}{q^{-N} \pi_0(q^n) - \pi_0(q^{n-1})}.$$

Relation (2.3) can be rewritten in the form

$$(2.8) \quad (\lambda_n - \lambda_{n-s}) B_n^{(s)} = B_n^{(s-1)} A_{n-s+1}^{(1)} + B_n^{(s-2)} A_{n-s+2}^{(2)} + \dots + A_n^{(s)}.$$

From this relation we can conclude, by induction, that the coefficients $B_n^{(s)}$ are rational functions in q^n , namely that

$$(2.9) \quad B_n^{(s)} = [n][n-1] \dots [n-s+1] \frac{Q_{1,s}(q^n)}{Q_{2,s}(q^n)},$$

where $Q_{1,s}(q^n)$ is a polynomial of degree not exceeding $2Ns - s$, whereas the degree of the polynomial

$$(2.10) \quad Q_{2,s}(q^n) = \prod_{i=1}^s (q^{-iN} \pi_0(q^n) - \pi_0(q^{n-i}))$$

does not exceed $2Ns$.

The problem considered up to this point of finding the expansion coefficients $B_n^{(s)}$ of the polynomials $P_n(x)$ when the representation coefficients $A_n^{(s)}$ are given always has a unique solution.

Proposition 2.1. *Assume that the representation coefficients $A_n^{(s)}$ of the operator \mathcal{L}_q satisfy the requirement*

$$(2.11) \quad A_n^{(0)} \neq A_m^{(0)}, \quad n \neq m.$$

Then there exists a unique set of monic polynomials $P_n(x)$, $n = 0, 1, \dots$, satisfying equation (1.10).

Proof. We find λ_m from (2.4); in view of condition (2.11), a unique $B_n^{(1)}$ is found from (2.6). Assuming that all $B_n^{(1)}, B_n^{(2)}, \dots, B_n^{(s-1)}$ have thus been found recursively, $B_n^{(s)}$ is then determined in an unambiguous way owing to (2.11).

The polynomials $P_n(x)$ are not orthogonal in general. The requirement that they form an orthogonal set implies strong additional restrictions upon the coefficients $B_n^{(s)}$ and $A_n^{(s)}$. Indeed, orthogonal polynomials satisfy a three-term recurrence relation of the form [1]

$$(2.12) \quad P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = xP_n(x),$$

with b_n and u_n referred to as the recurrence parameters. From (2.12) and (2.1) we get the set of relations

$$(2.13) \quad B_{n+1}^{(s+1)} - B_n^{(s+1)} + u_n B_{n-1}^{(s-1)} + b_n B_n^{(s)} = 0, \quad s = 0, 1, \dots, n,$$

where it is assumed that $B_n^{(-1)} = B_n^{(n+1)} = 0$. Putting $s = 0$ and $s = 1$ in (2.13), we find

$$(2.14) \quad \begin{aligned} b_n &= B_n^{(1)} - B_{n+1}^{(1)}, \\ u_n &= B_n^{(2)} - B_{n+1}^{(2)} - b_n B_n^{(1)}. \end{aligned}$$

Taking into account that the coefficients $B_n^{(s)}$ are rational functions in q^n we arrive at the following proposition.

Proposition 2.2. *If the orthogonal polynomials $P_n(x)$ are eigenfunctions of the operator (1.8), their recurrence coefficients b_n, u_n are rational functions of the argument q^n .*

The problem of reconstructing the representation coefficients $A_n^{(s)}$ when the expansion coefficients $B_n^{(s)}$ are given is more difficult. The coefficients $B_n^{(s)}$ must of course be rational functions in q^n , since otherwise the problem has no solutions. Assume therefore that

$$(2.15) \quad B_n^{(s)} = [n][n-1] \dots [n-s+1] \frac{G_{1,s}(q^n)}{G_{2,s}(q^n)},$$

where the degree of the polynomial $G_{1,s}(q^n)$ does not exceed $2Ms - s$ (for some positive integer $M \leq N$) whereas the degree polynomial $G_{2,s}(q^n)$ does not exceed $2Ms$. We assume that the polynomials $G_{1,s}(x)$ and $G_{2,s}(x)$ have no common divisors. Note that in this case the polynomial $G_{2,s}(q^n)$ need not coincide with expression (2.10) because in the expression (2.9) polynomials $Q_{1,s}(x)$ and $Q_{2,s}(x)$ may have coinciding zeroes.

Consider relation (2.6) written in the form

$$(2.16) \quad \frac{A_n^{(1)}}{\Omega_n} = [n] \frac{G_{1,1}(q^n)}{G_{2,1}(q^n)}.$$

Since both $q^{nN} A_n^{(1)}$ and $q^{nN} \Omega_n$ should be polynomials in q^n of degrees not exceeding $2N$, we have

$$(2.17) \quad \begin{aligned} A_n^{(1)} &= q^{N(1-n)} \rho_1(q^n) [n] G_{1,1}(q^n), \\ \Omega_n &= q^{N(1-n)} \rho_1(q^n) G_{2,1}(q^n), \end{aligned}$$

where $\rho_1(q^n)$ is some polynomial of degree not exceeding $2N - 2M$.

The relation (2.3), for $s = 2$, can then be rewritten in the form

$$(2.18) \quad A_n^{(2)} = (\lambda_n - \lambda_{n-2}) B_n^{(2)} - A_{n-1}^{(1)} B_n^{(1)} = (\Omega_n + \Omega_{n-1}) B_n^{(2)} - A_{n-1}^{(1)} B_n^{(1)}.$$

The representation coefficient $A_n^{(2)}$ is thus determined uniquely if $A_n^{(1)}$ and Ω_n are known.

Assume that the coefficients $A_n^{(2)}, A_n^{(3)}, \dots, A_n^{(k-1)}$ have been determined iterating this process. With $s = k$, (2.3) can now be rewritten in the form

$$(2.19) \quad A_n^{(k)} = (\lambda_n - \lambda_{n-k}) B_n^{(k)} - B_n^{(k-1)} A_{n-k+1}^{(1)} - B_n^{(k-2)} A_{n-k+2}^{(2)} - \dots - B_n^{(1)} A_{n-1}^{(k-1)}.$$

Taking into account the fact that

$$\lambda_n - \lambda_{n-k} = \Omega_n + \Omega_{n-1} + \dots + \Omega_{n-k+1},$$

we see that $A_n^{(k)}$ is completely determined from the coefficients $\Omega_n, A_n^{(1)}, \dots, A_n^{(k-1)}$. For the $A_n^{(s)}$ thus obtained to actually be representation coefficients of an operator \mathcal{L}_q , they necessarily need to satisfy, in addition, the conditions of Proposition 1.1. When this is so, the corresponding polynomials are eigenfunctions of the operator \mathcal{L}_q .

3. LITTLE q -JACOBI POLYNOMIALS

The monic little q -Jacobi polynomials [9] are defined as

$$(3.1) \quad P_n(x; a, b) = (-1)^n \frac{q^{n(n-1)/2} (aq; q)_n}{(abq^{n+1}; q)_n} {}_2\phi_1 \left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix} \middle| qx \right),$$

where $(a; q)_n = (1-a)(1-aq) \dots (1-aq^{n-1})$ is the q -shifted factorial and ${}_2\phi_1$ denotes the q -hypergeometric function (see, e.g., [2]).

The orthogonality relation is

$$(3.2) \quad \sum_{k=0}^{\infty} w_k P_n(q^k; a, b) P_m(q^k; a, b) = h_n \delta_{nm},$$

where h_n are appropriate normalization constants, and the normalized weight function is

$$(3.3) \quad w_k = \frac{(aq; q)_{\infty}}{(abq^2; q)_{\infty}} \frac{(bq; q)_k (aq)^k}{(q; q)_k}.$$

It is assumed that $0 < aq < 1$, $b < q^{-1}$. The expansion coefficients of the little q -Jacobi polynomials are

$$(3.4) \quad B_n^{(s)} = b^{-s} \frac{(q^{-n}, a^{-1}q^{-n}; q)_s}{(q, a^{-1}b^{-1}q^{-2n}; q)_s}.$$

It is easily verified that equations (2.8) have the following solutions:

$$(3.5) \quad \begin{aligned} A_n^{(0)} &= \lambda_n = [n](q^{1-n} - abq^2), \\ A_n^{(1)} &= [n](aq - q^{1-n}), \\ A_n^{(s)} &= 0, \quad s \geq 2. \end{aligned}$$

Hence, the little q -Jacobi polynomials satisfy a second-order q -difference equation, as is well known [9].

We also need the value of the function of second kind, $Q_n(z)$, at $z = 0$ (the point $z = 0$ is an accumulation point of the orthogonality measure):

$$(3.6) \quad Q_n(0; a, b) = - \sum_{k=0}^{\infty} \frac{P_n(q^k; a, b) w_k}{q^k}.$$

This sum can be evaluated using the q -binomial theorem and the q -Saalschütz formula (see, e.g., [2]):

$$(3.7) \quad Q_n(0; a, b) = (-1)^{n+1} a^n q^{n(n-1)/2} \frac{1 - abq}{1 - a} \frac{(q; q)_n (bq; q)_n}{(abq; q)_n (abq^{n+1}; q)_n}.$$

Taking into account that

$$(3.8) \quad P_n(0; a, b) = (-1)^n q^{n(n-1)/2} \frac{(aq; q)_n}{(abq^{n+1}; q)_n},$$

we note that if $a = q^j$, $j = 1, 2, 3, \dots$, then

$$(3.9) \quad \begin{aligned} \Phi_n &= Q_n(0; q^j, b) + \beta P_n(0; q^j, b) \\ &= (-1)^n q^{n(n-1)/2} \frac{(q^{j+1}; q)_n}{(bq^{n+j+1}; q)_n} \left(\beta - q^{nj} \frac{(1 - bq^{j+1}) (bq; q)_j (q; q)_j}{(1 - q^j) (q^{n+1}; q)_j (bq^{n+1}; q)_j} \right). \end{aligned}$$

4. TRANSFORMED q -JACOBI POLYNOMIALS

Let $P_n(x)$ be arbitrary orthogonal polynomials with measure localized on the interval $[a, b]$. The corresponding weight function $w(x)$ is assumed to be normalized to 1, i.e.

$$\int_a^b w(x) dx = 1.$$

Introduce the functions of second kind,

$$(4.1) \quad Q_n(z) = \int_a^b \frac{P_n(x) w(x)}{z - x} dx.$$

Let c be a point beyond the orthogonality interval $[a, b]$ such that $Q_n(c)$ exists.

Consider the polynomials

$$(4.2) \quad \tilde{P}_n(x) = \mathcal{G}(c)\{P_n(x)\} = P_n(x) - \frac{\Phi_n}{\Phi_{n-1}} P_{n-1}(x), \quad n = 1, 2, \dots, \quad \tilde{P}_0(x) = 1,$$

where

$$(4.3) \quad \Phi_n = Q_n(c) + \beta P_n(c).$$

The notation $\mathcal{G}(c)\{P_n(x)\}$ stands for the Geronimus transformation [3], [4] of the polynomials $P_n(x)$ at the point $x = c$ (for details see, e.g., [11]). The weight function $\tilde{w}(x)$ of the polynomials $\mathcal{G}(c)\{P_n(x)\}$ is

$$(4.4) \quad \tilde{w}(x) = \kappa \left(\frac{w(x)}{x - c} - \beta \delta(x - c) \right),$$

where κ is an appropriate normalization constant. The Geronimus transformation thus inserts a concentrated mass at the point $x = c$. The value of this mass depends on the parameter β .

Return to the case of the little q -Jacobi polynomials with $a = q^j$, $j = 1, 2, 3, \dots$. Perform the Geronimus transformation (4.2) with Φ_n given by (3.9). (In this case $c = 0$.)

The weight function $\tilde{w}(x)$ for the polynomials $\mathcal{G}(c)\{P_n(x)\}$ is

$$(4.5) \quad \tilde{w}(x) = \kappa \left(\sum_{k=0}^{\infty} \tilde{w}_k \delta(x - q^k) - \beta \delta(x) \right),$$

where

$$(4.6) \quad \tilde{w}_k = \frac{(q^{j+1}; q)_{\infty}}{(bq^{j+2}; q)_{\infty}} \frac{(bq; q)_k q^{jk}}{(q; q)_k}.$$

The weight function (4.5) can be rewritten in the form

$$(4.7) \quad \tilde{w}(x) = \kappa_1 (w(x; j - 1) + M \delta(x)),$$

where

$$M = -\beta \frac{1 - q^j}{1 - bq^{j+1}}, \quad \kappa_1 = \kappa \frac{1 - bq^{j+1}}{1 - q^j},$$

and $w(x; j - 1)$ is the weight function corresponding to the little q -Jacobi polynomials with the parameter $a = q^j$ replaced with $a = q^{j-1}$, i.e.

$$(4.8) \quad w(x; j - 1) = \sum_{k=0}^{\infty} w_k(j - 1) \delta(x - q^k),$$

and

$$(4.9) \quad w_k(j - 1) = \frac{(q^j; q)_{\infty}}{(bq^{j+1}; q)_{\infty}} \frac{(bq; q)_k q^{jk}}{(q; q)_k}.$$

Thus the weight function $\tilde{w}(x; j)$ for the polynomials $\mathcal{G}(0)\{P_n(x)\}$ is obtained from the weight function $w(x; j - 1)$ through the addition of an arbitrary mass M at the point $x = 0$.

In the expansion

$$(4.10) \quad \mathcal{G}(0)\{P_n(x; q^j, b)\} = \sum_{s=0}^n B_n^{(s)} x^{n-s},$$

the coefficients $B_n^{(s)}$ are found from (3.1) and (3.9):

$$(4.11) \quad B_n^{(s)} = b^{-s} \frac{(q^{-n}; q)_s (q^{-n-j}; q)_s}{(q; q)_s (b^{-1}q^{-j-2n}; q)_s} \times \left(1 - q^{n+j-s} \frac{(1 - bq^n)(1 - q^s)}{(1 - q^{n+j})(1 - bq^{j+2n-s})} \frac{Y_j(n)}{Y_j(n-1)} \right),$$

with

$$(4.12) \quad Y_j(n) = \beta q^{-jn} (q^{n+1}; q)_j (bq^{n+1}; q)_j - (q; q)_{j-1} (bq; q)_{j+1}.$$

5. CONSTRUCTION OF THE COEFFICIENTS $A_n^{(s)}$

In this section we construct the coefficients $A_n^{(s)}$ for the q -difference operator \mathcal{L}_q that has the polynomials $\tilde{P}_n(x)$ as eigenfunctions.

We start from the relation

$$(5.1) \quad A_n^{(1)} = \Omega_n B_n^{(1)},$$

where

$$(5.2) \quad \Omega_n = \lambda_n - \lambda_{n-1} = A_n^{(0)} - A_{n-1}^{(0)}.$$

Choose Ω_n proportional to the denominator of $B_n^{(1)}$:

$$(5.3) \quad \Omega_n = bq^{n+j-1} (q-1) (1 - b^{-1}q^{1-j-2n}) \times \left(\beta q^{-j(n-1)} (q^n; q)_j (bq^n; q)_j - (bq; q)_{j+1} (q; q)_{j-1} \right).$$

Then from (5.1) we get for $A_n^{(1)}$ the expression

$$(5.4) \quad A_n^{(1)} = (1 - q^{-n}) \left(\beta q^{j(1-n)} (q^{n+1}; q)_j (bq^n; q)_j (1 - q^{n-1}) - (q; q)_{j-1} (bq; q)_{j+1} (1 - q^{n+j-1}) \right).$$

Using (5.2) it is not difficult to show that

$$(5.5) \quad A_n^{(0)} = \lambda_n = \frac{\beta (q-1) q^{-n(j+1)-1} (q^n; q)_{j+1} (bq^n; q)_{j+1}}{1 - q^{-j-1}} - (q^{-n} - 1) (1 - bq^{n+j}) (bq; q)_{j+1} (q; q)_{j-1}.$$

(Note that $A_0^{(0)} = 0$.)

Now using the explicit expressions for $B_n^{(1,2)}$ and $A_n^{(0,1)}$, we find

$$(5.6) \quad A_n^{(2)} = \beta (q-1) q^{(2-n)(j+1)-1} \frac{(1 - q^{-j}) (q^{n-2}; q)_{j+3} (bq^n; q)_{j-1}}{(q; q)_2}.$$

Repeating this procedure for $s = 3, 4, \dots$ one can guess the expression

$$(5.7) \quad A_n^{(s)} = \beta(q-1)q^{(s-n)(j+1)-1} \frac{(q^{-j}; q)_{s-1}(q^{n-s}; q)_{s+j+1}(bq^n; q)_{j-s+1}}{(q; q)_s} + \xi_n \delta_{s,1} + \eta_n \delta_{s,0},$$

where

$$\begin{aligned} \xi_n &= (q^{-n} - 1)(q; q)_{j-1}(bq; q)_{j+1}(1 - q^{j+n-1}), \\ \eta_n &= (1 - q^{-n})(1 - bq^{n+j})(bq; q)_{j+1}(q; q)_{j-1}, \end{aligned}$$

and $s = 0, 1, 2, \dots$

Proposition 5.1. *The coefficients $A_n^{(s)}$ given by (5.7) satisfy the basic relations*

$$(5.8) \quad \sum_{i=0}^s B_n^{(s-i)} A_{n-s+i}^{(i)} = A_n^{(0)} B_n^{(s)}.$$

Proof. Using the explicit expressions for $B_n^{(s)}$ and $A_n(s)$ we can rewrite the lhs of (5.8) in the form

$$(5.9) \quad \sum_{i=0}^s B_n^{(s-i)} A_{n-s+i}^{(i)} = \eta_{n-s} B_n^{(s)} + \xi_{n-s+1} B_n^{(s-1)} + \kappa_n (S_1 + \nu_n S_2),$$

where

$$\begin{aligned} \kappa_n &= \beta(q-1)b^{-s} q^{(j+1)(s-n)-1} \frac{(q^{-n}; q)_s (q^{-j-n}; q)_s (q^{n-s}; q)_{j+1} (bq^{n-s}; q)_{j+1}}{(q; q)_s (b^{-1}q^{-j-2n}; q)_s (1 - q^{-j-1})}, \\ \nu_n &= \frac{q^{n+j}(1 - bq^n)(1 - q^{-s})Y_j(n)}{(1 - q^{n+j})(1 - bq^{j+2n-s})Y_j(n-1)}, \end{aligned}$$

and S_1, S_2 are the sums

$$\begin{aligned} S_1 &= \sum_{i=0}^s \frac{q^i (q^{-s}; q)_i (bq^{j+2n+1-s}; q)_i (q^{-j-1}; q)_i}{(q; q)_i (q^{n+1-s}; q)_i (bq^{n-s}; q)_i}, \\ S_2 &= \sum_{i=0}^s \frac{q^i (q^{1-s}; q)_i (bq^{j+2n-s}; q)_i (q^{-j-1}; q)_i}{(q; q)_i (q^{n+1-s}; q)_i (bq^{n-s}; q)_i}. \end{aligned}$$

These sums can be evaluated using the q -analog of the Saalschütz formula [2]:

$$\begin{aligned} S_1 &= q^{(j+1)s} \frac{(b^{-1}q^{-j-n}, q^{-1-j-n}; q)_s}{(b^{-1}q^{1-n}, q^{-n}; q)_s}, \\ S_2 &= q^{(j+1)(s-1)} \frac{(b^{-1}q^{1-j-n}, q^{-j-n}; q)_{s-1}}{(b^{-1}q^{2-n}, q^{1-n}; q)_{s-1}}. \end{aligned}$$

Relation (5.8) now becomes

$$(5.10) \quad (\eta_{n-s} - \lambda_n)B_n^{(s)} + \xi_{n-s+1}B_n^{(s-1)} + \kappa_n (S_1 + \nu_n S_2) = 0$$

and is seen to be identically satisfied. This proves the proposition.

From expression (5.7) it follows that

$$(5.11) \quad A_n^{(s)} = 0, \quad \text{if } s \geq j + 2.$$

Moreover, for $s < j + 2$ the coefficients $A_n^{(s)}$ have the form $A_n^{(s)} = q^{-(j+1)n} Q_{2j+2}(q^n; s)$, where $Q_{2j+2}(q^n; s)$ is a polynomial in q^n of degree $2j + 2$.

Hence we have

Proposition 5.2. *The polynomials $\mathcal{G}(0)\{P_n(x; q^j, b)\}$ are the eigenfunctions of a q -difference operator \mathcal{L}_q of order $2N = 2j + 2$.*

We know that the polynomials $\mathcal{G}(0)\{P_n(x; q^j, b)\}$ coincide with the polynomials $P_n(x; q^{j-1}, b; M)$ obtained from the little q -Jacobi polynomials by adding to the orthogonality measure a mass M at $x = 0$. We thus have equivalently the following

Proposition 5.3. *The polynomials $P_n(x; q^j, b; M)$ obtained from the little q -Jacobi polynomials by inserting a discrete mass at $x = 0$ in the orthogonality measure are the eigenfunctions of a q -difference operator of order $2N = 2j + 4$.*

This proposition is a q -analogue of the corresponding proposition for the ordinary Jacobi polynomials [7], [11].

Note that the first explicit example of the generalized little q -Jacobi polynomials satisfying a fourth-order q -differential equation was found in [5].

Remark. As the referee pointed out, when $a \neq q^j$, $j = 0, 1, 2, \dots$, then the coefficients $A_n^{(s)}$ (given by the expression (5.7)) do not vanish for all s . In this case one can expect that the corresponding polynomials are eigenfunctions of a q -difference operator of infinite order. When $q = 1$, corresponding differential operators of infinite order were found e.g. in [6], [7].

6. THE CASE OF LITTLE q -LAGUERRE POLYNOMIALS

The monic little q -Laguerre polynomials [9]

$$(6.1) \quad P_n(x; a) = (-1)^n q^{n(n-1)/2} (aq; q)_n {}_2\phi_1 \left(\begin{matrix} q^{-n}, 0 \\ aq \end{matrix} \middle| qx \right)$$

are obtained from the little q -Jacobi polynomials by setting $b = 0$. Hence, these polynomials also satisfy a q -difference equation.

Consider the polynomials $\mathcal{G}(0)\{P_n(x; q^j)\}$ obtained from the little q -Laguerre polynomials by the Geronimus transformation at $x = 0$. All formulas for these polynomials are obtained from those for little q -Jacobi polynomials by putting $b = 0$.

In particular, their coefficients $A_n^{(s)}$ are easily obtained from (5.7).

We thus have

Proposition 6.1. *The polynomials $\mathcal{G}(0)\{P_n(x; q^j)\}$ are the eigenfunctions of a q -difference operator of order $2N = 2j + 2$.*

In this case, the polynomials $\mathcal{G}(0)\{P_n(x; q^j)\}$ coincide with polynomials $P_n(x; q^{j-1}; M)$ obtained from the little q -Laguerre polynomials by adding to the orthogonality measure a mass M at $x = 0$. Hence

Proposition 6.2. *The polynomials $P_n(x; q^j; M)$ are the eigenfunctions of a q -difference operator of order $2N = 2j + 4$.*

When $q \rightarrow 1$ we get Koornwinder's generalized Laguerre polynomials $L_n^{(j; M)}(x)$ [10] whose measure differs from that of the ordinary Laguerre polynomials $L_n^{(j)}(x)$ by inserting a concentrated mass M at the endpoint $x = 0$ of the orthogonality interval $(0, \infty)$. These polynomials are known to satisfy a differential equation of order $2j + 4$ [6], [8].

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