

DEPTH OF SYMMETRIC ALGEBRAS OF CERTAIN IDEALS

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ABSTRACT. We compute the depth of the symmetric algebra of certain ideals in terms of the depth of the ring modulo the ideal generated by the entries of a minimal presentation matrix.

The purpose of this paper is to give a formula for the depth of the symmetric algebra $S(I) = \bigoplus_{j \geq 0} S_j(I)$ for certain ideals. We also obtain an analogous result for the symmetric algebra $S(I/I^2)$ of the conormal module (as an R/I -module). Our method is a simple application of the recent results ([3], [5]) concerning the Cohen-Macaulay properties of the blow-up rings $R[It] = \bigoplus_{j \geq 0} I^j$ and $gr_I(R) = \bigoplus_{j \geq 0} I^j/I^{j+1}$. In particular, these blow-up rings are Cohen-Macaulay for the ideals under our consideration. It is well known that the symmetric algebra need not be Cohen-Macaulay, but it turns out that these symmetric algebras have fairly large depth in any case.

Let R be a noetherian local ring with infinite residue field k , and let I be an ideal. Recall that the *analytic spread* of I is $\ell(I) = \dim R[It] \otimes_R k$, or equivalently is the least number s of elements a_1, \dots, a_s in I such that $I^{k+1} = (a_1, \dots, a_s)I^k$ for some $k \geq 0$; the least integer k (over all such *reductions*) is the *reduction number* r of I . Note that $r = 0 \iff \ell = n \equiv \nu(I)$ (the minimal number of generators of I) $\iff I$ is generated by analytically independent elements. We say that I satisfies G_s if $\nu(I_p) \leq \dim R_p$ whenever $p \in V(I)$ and $\dim R_p < s$, and that I satisfies G_∞ if I satisfies G_s for every s . We will also need the Artin-Nagata property [7]: I satisfies AN_s (respectively AN_s^-) if $J = (a_1, \dots, a_s) : I$ is Cohen-Macaulay whenever $a_1, \dots, a_s \in I$ and $ht J \geq s \geq ht I$ (respectively, and $ht I + J > s$).

Theorem 1. *Let R be a local Cohen-Macaulay ring with infinite residue field, let I be an ideal with $ht I \geq 2$, analytic spread ℓ , reduction number r , minimally generated by $n = \ell + 1$ elements, let ϕ be a minimal presentation matrix of I , and assume that I satisfies G_ℓ and $AN_{\ell-2}^-$, and that $S_j(I) \cong I^j$ and $\text{depth } R/I^j \geq \dim R - \ell + r - j$ for $1 \leq j \leq r$. Then*

$$\text{depth } S(I) = \text{depth } R/I_1(\phi) + n,$$

$$\text{depth } S(I/I^2) = \min\{\text{depth } R/I_1(\phi) + n, \dim R\}.$$

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Proof. By the universal property of the symmetric algebra, there exists a canonical exact sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow S(I) \longrightarrow R[It] \longrightarrow 0.$$

However, by [3, Theorem 2.6 and Lemma 3.1], \mathcal{A} is cyclic, and is annihilated by $I_1(\phi)$. It then follows easily that $\text{Ann}(\mathcal{A}) = I_1(\phi)S(I)$. Hence $\mathcal{A} \cong S(I)/\text{Ann}(\mathcal{A}) \cong S(I)/I_1(\phi)S(I) \cong R/I_1(\phi) \otimes_R S(I)$. Now let $S = R[T_1, \dots, T_n]$. Then there is an epimorphism $S \longrightarrow S(I)$ whose kernel L is generated by all linear relations on I . Hence then there is an S -module isomorphism $\mathcal{A} \cong R/I_1(\phi) \otimes_R S/L \cong R/I_1(\phi) \otimes_R S$, and thus we obtain an exact sequence

$$0 \longrightarrow R/I_1(\phi) \otimes_R S \longrightarrow S(I) \longrightarrow R[It] \longrightarrow 0.$$

Tensoring with R/I , we obtain the corresponding exact sequence

$$0 \longrightarrow R/I_1(\phi) \otimes_R S \longrightarrow S(I/I^2) \longrightarrow gr_I(R) \longrightarrow 0.$$

Now [5, Theorem 3.1 and Corollary 3.4] shows that the rings $R[It]$ and $gr_I(R)$ are Cohen-Macaulay. Note that $\dim S(I) = \dim R + 1$, while $\dim S(I/I^2) = \dim R$ if and only if I satisfies G_∞ , and otherwise $\dim S(I/I^2) = \dim R + 1$ ([1, Proposition 6.5]). Thus the first formula is immediate, and the second one will follow once we show that $\text{depth } S(I/I^2) \leq \dim R$. Hence supposing that $\text{depth } S(I/I^2) > \dim R$, necessarily $S(I/I^2)$ must be Cohen-Macaulay of dimension $\dim R + 1$, and I does not satisfy G_∞ . In particular, $S(I/I^2)$ is (locally) equidimensional. However, one of its minimal components will be that of the associated graded ring, and $\dim gr_I(R) = \dim R$. This contradiction completes the proof. \square

As a corollary we find that, with our current assumptions, the two symmetric algebras can never be simultaneously Cohen-Macaulay.

Corollary 1. *With the assumptions of Theorem 1,*

*$S(I)$ is Cohen-Macaulay if and only if $I_1(\phi)$ is Cohen-Macaulay of height $n - 1$,
 $S(I/I^2)$ is Cohen-Macaulay if and only if $I_1(\phi)$ is Cohen-Macaulay of height n .*

Proof. If $I_1(\phi)$ is Cohen-Macaulay of height $n - 1$ (respectively n), then clearly $S(I)$ (respectively $S(I/I^2)$) is Cohen-Macaulay. To show the converse, note our assumptions imply that $ht I_1(\phi) \geq n - 1$ [5, Remark 2.7]. Hence if $S(I)$ is Cohen-Macaulay, then

$$\begin{aligned} \dim R + 1 = \dim S(I) &= \text{depth } S(I) = \text{depth } R/I_1(\phi) + n \leq \dim R/I_1(\phi) + n \\ &= \dim R - ht I_1(\phi) + n \leq \dim R + 1. \end{aligned}$$

Hence we must have that $\text{depth } R/I_1(\phi) = \dim R/I_1(\phi)$ and that $ht I_1(\phi) = n - 1$. For $S(I/I^2)$, note that as in the proof of Theorem 1, if $S(I/I^2)$ is Cohen-Macaulay, then I satisfies G_∞ ; hence $ht I_1(\phi) \geq n$. Thus in this case

$$\begin{aligned} \dim R = \dim S(I/I^2) &= \text{depth } S(I/I^2) = \min\{\text{depth } R/I_1(\phi) + n, \dim R\} \\ &\leq \min\{\dim R/I_1(\phi) + n, \dim R\} = \min\{\dim R - ht I_1(\phi) + n, \dim R\} \\ &\leq \dim R, \end{aligned}$$

and hence we conclude that $R/I_1(\phi)$ is Cohen-Macaulay and that $ht I_1(\phi) = n$. \square

Corollary 2. *With the assumptions of Theorem 1, assume further that I satisfies AN_ℓ . Then $S(I)$ is Cohen-Macaulay.*

Proof. By [3, Proposition 3.2], after elementary row operations, the entries of one row of ϕ generate $I_1(\phi)$. Thus $I_1(\phi) = (a_1, \dots, a_\ell) : a_{\ell+1} = (a_1, \dots, a_\ell) : I$, for some generating set $a_1, \dots, a_{\ell+1}$ of I . Since I satisfies AN_ℓ , it follows that $I_1(\phi)$ is a Cohen-Macaulay ideal and has height exactly ℓ [7, Proposition 1.7]. \square

One may view this latter corollary as a partial generalization of one of the main results of [1] (Theorem 6.10), which guarantees that the symmetric algebra is Cohen-Macaulay for ideals satisfying *sliding depth*.

In our situation we have a good description of the components of the symmetric algebra, as hinted at in the proof of the Theorem. Indeed, under those hypotheses, we have the decomposition $(0) = I_1(\phi)S(I) \cap \mathcal{A}$. This is clear since \mathcal{A} is cyclic, generated by a form not contained in the extension of the maximal ideal, and has annihilator $I_1(\phi)S(I)$.

Corollary 3. *With the assumptions of Theorem 1,*
 $S(I)$ *is unmixed if and only if* $I_1(\phi)$ *is unmixed of height* $n - 1$,
 $S(I/I^2)$ *is unmixed if and only if* $I_1(\phi)$ *is unmixed of height* n .

Corollary 4. *With the assumptions of Theorem 1, assume further that* I *satisfies* $AN_{\ell-1}^-$. *Then* $S(I)$ *is unmixed.*

Proof. As in the proof of Corollary 2, $I_1(\phi)$ is unmixed of height ℓ by [7, Proposition 1.7]. \square

We now are able to answer a question of Herzog, Simis and Vasconcelos [2, p.142]. They asked whether there is a Cohen-Macaulay (generic complete intersection) ideal for which $S(I/I^2)$ is Cohen-Macaulay and $S(I)$ is not. (Non-Cohen-Macaulay ideals with this property were known.) It is known [1, Corollary 6.9] that such an ideal must necessarily satisfy G_∞ . Recently, an example was found in [6, Example 6.15] but the ring R is not regular, which, as they mention, still leaves the question open in this case.

Corollary 1 now gives an answer to this question in principle. To show that such examples exist in great number, we will state the result in a less technical manner.

Corollary 5. *Let* R *be a local Gorenstein ring of dimension* d , *and let* I *be a Cohen-Macaulay ideal of grade* $g \geq 2$ *satisfying* G_∞ *and* $\text{depth } R/I^j \geq d - g - j + 1$ *for* $1 \leq j \leq d - g - 2$ *and suppose that* $\ell(I) < d$. *Then* $S(I/I^2)$ *is Cohen-Macaulay, but* $S(I)$ *is Cohen-Macaulay if and only if* $\nu(I) = \ell(I)$.

Proof. We may assume that the residue field is infinite. If $n = \nu(I) < d$, then by [7, Corollary 2.13] I is strongly Cohen-Macaulay; hence $S(I/I^2)$ (as well as $S(I)$) is Cohen-Macaulay by [2, Theorem 2.6]. This Theorem also shows that $S(I) \cong R[It]$; hence in particular $n = \ell$. Thus we may assume that $n = d$.

We now claim that $\ell = n - 1$. For otherwise, suppose that $\ell \leq n - 2$. Our assumption on the depth of the powers of I implies that I satisfies AN_{d-3} [7, Theorem 2.9], and hence in particular $AN_{\ell-1}$. Let $(a_1, \dots, a_\ell) \subset I$ be any reduction of I . Then $ht J = (a_1, \dots, a_\ell) : I \geq \ell + 1$ by [7, Proposition 1.11]. However, as I satisfies $AN_{\ell-1}$, J must have height exactly ℓ by [7, Proposition 1.7]. This contradiction proves the claim.

Thus we now have that $n = d = \ell + 1$ and that I satisfies $AN_{\ell-2}$. But now [7, Theorem 4.1] (or [5, Proposition 4.7]) gives that $r(I) = 1$. Thus I satisfies all the hypotheses of Theorem 1. Since $ht I_1(\phi) = d$, the result follows immediately from Corollary 1. \square

We conclude with several examples.

Example 1. Let (R, m) be a 6-dimensional regular local ring, and let X be a 2×4 matrix such that $I_1(X) = m$ and $ht\ I_2(X) \geq 3$. Then $I = I_2(X)$ is a perfect ideal of grade 3, a complete intersection on the punctured spectrum, $S(I/I^2)$ is Cohen-Macaulay, and $S(I)$ is not Cohen-Macaulay.

Proof. It is well known that I is a perfect ideal of grade 3, and it is readily seen to be a complete intersection on the punctured spectrum. As $n = \nu(I) = 6 = \dim R$, in particular I satisfies G_∞ . Since the well-known Plücker relation on the minors of X shows that $\ell < n$, the result follows from Corollary 5. \square

One may even achieve that I is an isolated singularity. For example, if x_1, \dots, x_6 is a regular system of parameters, one may take $I = I_2 \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & x_1 & x_2 \end{pmatrix}$.

The next example is obtained by linking a certain tensor product, as in [4].

Example 2. Let $A = k[[x_1, \dots, x_d]]$, and let J be an ideal such that A/J is Cohen-Macaulay, of dimension 2 and type 2, and a complete intersection in codimension one. Let $\underline{\alpha} = \alpha_1, \dots, \alpha_g$ be a regular sequence in J which generates J generically. Consider the isomorphism $\phi : k[[x_1, \dots, x_d]] \rightarrow k[[y_1, \dots, y_d]]$ given by $\phi(x_i) = y_i$, let J' and $\underline{\alpha}'$ be the image of J and $\underline{\alpha}$ under ϕ , and set $R = k[[x_1, \dots, x_d, y_1, \dots, y_d]]$ and $I = (\underline{\alpha}, \underline{\alpha}')R : (J, J')R$.

Then I is a perfect ideal of R , generically a complete intersection, $S(I/I^2)$ is Cohen-Macaulay, and $S(I)$ is not Cohen-Macaulay.

Proof. The result follows from Corollary 5, using [4, Remark 2.3 and Lemma 2.4].

The simplest example of this occurs when $d = 4$ and one takes $J = (x_1x_2, x_2x_3, x_3x_4)$. The resulting ideal I is perfect of grade 4 and deviation 4; it may be viewed as the edge ideal of a certain graph [4, Example 3.8].

Example 3. Let (S, m) be a 5-dimensional regular local ring containing a field, X an alternating 5×5 matrix with $I_1(X) = m$ and $ht\ I_4(X) \geq 3$, Y a generic 5×1 matrix, $R = S[[Y]]$, and $I = Pf_4(X) + I_1(XY)$.

Then I is a perfect Gorenstein ideal of grade 5, a complete intersection on the punctured spectrum, $S(I/I^2)$ is Cohen-Macaulay, and $S(I)$ is not Cohen-Macaulay.

Proof. In this case we apply Corollary 1, after specializing the generic case, using [7, Theorem 2.14, and Example 2.12].

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