RESIDUALLY FINITE DIMENSIONAL AND AF-EMBEDDABLE $C^*$-ALGEBRAS

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Abstract. We show that every separable nuclear residually finite dimensional $C^*$-algebra satisfying the Universal Coefficient Theorem can be embedded into a unital separable simple AF-algebra.

The problem of determining when a $C^*$-algebra can be embedded as a $C^*$-subalgebra of an AF-algebra has been studied for a while (for example, in [Pi], [Vo], [Sp], [Br] and [D]—to name a few). J. Spielberg showed in [Sp], among other related things, that every separable CCR $C^*$-algebra can be embedded into an AF-algebra. In [D], it is shown that a separable nuclear residually finite dimensional $C^*$-algebra can be embedded into an AF-algebra if it is homotopically dominated by an AF-algebra. Consequently, from a result of Spielberg, Dadarlat shows (in [D]) that the AF-embeddability of a residually finite dimensional $C^*$-algebra $A$ (say in the “bootstrap” class of [RS]) depends only on the homotopy type of $A$.

The renewed interest in residually finite dimensional $C^*$-algebras arises from the result of Blackadar and Kirchberg ([BK1] and [BK2]) that simple separable nuclear quasidiagonal $C^*$-algebras are inductive limits of nuclear RFD $C^*$-algebras.

Since a $C^*$-subalgebra of a nuclear $C^*$-algebra must be exact ([K1]), non-exact $C^*$-algebras cannot be embedded into AF-algebras. So, we will consider nuclear $C^*$-algebras only. In this short note, we use recent results ([Ln3]) in classification of nuclear $C^*$-algebras to show that, in fact, every separable nuclear residually finite dimensional $C^*$-algebra can be embedded into a simple AF-algebra if it satisfies the Universal Coefficient Theorem (for example, if it is in the so-called “bootstrap” class of $C^*$-algebras).

Definition 1. Let $A$ be a $C^*$-algebra. An irreducible representation of $A$, $\pi$, is said to be finite dimensional, if $\pi(A)$ is finite dimensional. A $C^*$-algebra $A$ is said be residually finite dimensional (RFD) if there is a separating family of finite dimensional irreducible representations of $A$.

A separable RFD $C^*$-algebra has a separating sequence of finite dimensional irreducible representations.

Definition 2. Denote by $N_1$ the bootstrap class of $C^*$-algebras (see [RS]). Denote by $\mathcal{N}$ the class of $C^*$-algebras for which the Universal Coefficient Theorem (UCT)
holds. Every \( C^\ast \)-algebra in \( \mathcal{N}_1 \) satisfies the UCT (RS). In fact, \( A \in \mathcal{N} \) if \( A \) is \( kk \)-equivalent to a \( C^\ast \)-algebra in \( \mathcal{N}_1 \), \( \mathcal{N}_2 \) contains all separable type I \( C^\ast \)-algebras, and is closed under inductive limits, hereditary subalgebras, quotients, tensor products, cross products with \( \mathbb{Z} \), as well as extensions.

3. Let \( C \in \mathcal{N} \) be a unital separable residually finite dimensional \( C^\ast \)-algebra. Let 
\[ s(C) \quad \text{(with the weak-topology)} \]
be the set of tracial states defined by \( tr \circ \pi \), where \( \pi \) is an irreducible representation of finite rank and \( tr \) is the standard tracial state on \( \pi(C) \). There is a positive homomorphism \( \delta_C : K_0(C) \to C(s(A)) \) defined by 
\[ f \mapsto f(t) \quad (t \in s(A)). \]

Let \( A \) be a unital separable stably finite \( C^\ast \)-algebra and \( T(A) \) be its tracial state space. Let \( \text{Aff}(T(A)) \) be the continuous affine functions on \( T(A) \). Define 
\[ \rho_A : K_0(A) \to \text{Aff}(T(A)) \]
by 
\[ f \mapsto f(t) \quad (t \in T(A)). \]

**Definition 4 (2.1 in [La2]).** Let \( A \) be a unital simple \( C^\ast \)-algebra. We say that \( A \) is **tracially approximately finite dimensional** (TAF for brevity) if it satisfies the following: For any \( \varepsilon > 0 \) and any finite subset \( F \) of \( A \) which contains a non-zero element \( x_1 \) and a nonzero positive element \( a \in A_+ \) there exists a finite dimensional \( C^\ast \)-subalgebra \( B \subset A \) with \( p = 1_B \) such that 
\[
\begin{align*}
(1) & \quad \| px - xp \| < \varepsilon \quad \text{for all } x \in F; \\
(2) & \quad pxp \subseteq F \quad \text{for all } x \in F \quad \text{and } \| pxp \| \geq \| x_1 \| - \varepsilon; \\
(3) & \quad 1 - p \text{ is equivalent to a projection in } aAa.
\end{align*}
\]

Simple TAF \( C^\ast \)-algebras are quasidiagonal, have real rank zero, stable rank one, and weakly unperforated \( K_0 \).

5. Let \( C \in \mathcal{N} \) be a unital separable residually finite dimensional \( C^\ast \)-algebra. Fix a sequence of finite dimensional irreducible representations \( \{ \pi_k \} \) such that \( \{ \pi_k \}_{k=m}^\infty \) is separating for \( C \) for all \( m \). We also assume that \( \{ tr \circ \pi_k \}_{k=m}^\infty \) is dense in \( s(A) \). Suppose that \( \pi_k \) has rank \( r(k) \). In what follows, we may not distinguish \( \pi_k \) from \( \pi_k \otimes 1_K \) (on \( C \otimes M_K \)).

Let \( \bar{r}(2) = r(1) + 1, \bar{r}(n+1) = (n)! \bar{r}(n)r(n)(n+1) \), \( n = 1, 2, ..., \). Set \( R(i) = \bar{r}(i)r(i), i = 1, 2, ... \). Let \( C_1 = C, C_2 = C_1 \otimes M_{R(2)}, ..., C_{n+1} = C_n \otimes M_{R(n+1)}, ... \). Define 
\[ j_{n,n+1} : C_n \to C_{n+1} \]
by 
\[ f \mapsto \text{diag}(f, f, ..., f, \pi_n \otimes \text{id}_{\bar{r}(n)}(f), ..., \pi_n \otimes \text{id}_{\bar{r}(n)}(f)), \]
where \( f \in C_n \) repeats \( n! \) times and \( \phi_n \otimes \text{id}_{\bar{r}(n)}(f) \) repeats \( n(n)! \) times. Also, the image of \( \pi_n \otimes \text{id}_{\bar{r}(n)} \) is in \( (C : \text{id}_{C_n}) \otimes M_{R(n)} \). Define 
\[ A = \lim_n(C_n, j_{n,n+1}). \]
In what follows, the map \( C_n \to A \) will be denoted by \( j \).

**Lemma 6.** \( A \) is a unital separable nuclear simple TAF \( C^\ast \)-algebra with divisible \( K_0(A) \) and a unique tracial state and which satisfies the UCT.

**Proof.** Since \( A \) is a direct limit of \( C_n \) and each \( C_n \) is nuclear and satisfies the UCT, \( A \) is separable, nuclear and satisfies the UCT. It follows from 4.2, 4.3 and 4.4 in [La2] that \( A \) is a unital simple TAF \( C^\ast \)-algebra with a unique tracial state. We need only to show that \( K_0(A) \) is divisible. It suffices to show that, for any projection \( p \in M_K(A) \) (\( K \) is any positive integer) and any integer \( k > 0 \), there exists a projection \( q \in M_K(A) \) such that \( k[q] = [p] \) in \( K_0(A) \). Replacing \( p \) by an equivalent projection, without loss of generality, we may assume that \( p \in j(M_K(C_n)) \) for some large \( n \). There exists \( m \) such that \( k[q] = [p] \). Let \( p' \in M_K(C_n) \) such that \( j(p') = p \). From the construction in 5, it is clear that there exists \( q' \in M_K(C_{n+m}) \) such that \( k[q'] = [j_{n,n+m}(p')] \) in \( K_0(C_{n+m}) \). This implies that \( k[j(q')] = [p] \). \( \square \)
Let $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Fix $m$ and $n > 0$. Let $j_{m,m+n} = j_{m+n-1,m+n} \circ \cdots \circ j_{m,m+1}$. Note $C_{m+n} = C_m \otimes M_K(n)$, where $K(n) = \prod_{i=m}^{m+n-1} (R(i)! (i+1))$. Thus

\[
j_{m,m+n}(f) = \text{diag}(f, f, \ldots, f, \Pi_m(f), \Pi_{m+1}(f), \ldots, \Pi_{m+n-1}(f))
\]

where $f$ repeats $\prod_{i=1}^{m+n} (m+i-1)$ many times, $\Pi_m$ is

\[
m(m!) \prod_{j=1}^{n-1} (R(m+j)(m+j)!(m+j+1))
\]

copies of $\pi_m \otimes \text{id}_{R(m)}$, $\Pi_{m+1}$ is

\[
(m+1)(m+1)! \prod_{j=2}^{n-1} (R(m+j)(m+j)!(m+j+1))
\]

copies of $\pi_{m+1} \otimes \text{id}_{R(m)}$, $\ldots$, $\Pi_{m+n-1}$ is $(m+n-1)(m+n-1)!$ copies of $\pi_{m+n-1} \otimes \text{id}_{R(m)}$. Set $A = \lim_n (C_n, j_{n,n+1})$.

Define

\[
a_1^{(n)} = (m!) \prod_{j=1}^{n-1} (R(m+j)(m+j)!(m+j+1))/K(n) = m/(R(m)(m+1)),
\]

\[
a_2^{(n)} = (m+1)(m+1)! \prod_{j=2}^{n} (R(m+j)(m+j)!(m+j+1))/K(n)
\]

\[
= 1/R(m)m!(R(m+1)((m+1)+1)), \ldots,
\]

and

\[
a_k^{(n)} = (m+k)(m+k)!/[R(m+k)(m+k)!(m+k+1)]
\]

\[
\times \prod_{j=1}^{k-1} (R(m+j)(m+j)+1])
\]

\[
= 1/[R(m+k-1)(m+k-1)!R(m+k)(m+k+1)]
\]

\[
\times \prod_{j=1}^{k-2} R(m+j)(m+j)!(m+j+1)], \ldots
\]

Fix $j$, then $a_j^{(n)} = a_j^{(m+1)}$ for all $n > m$. Thus we set $a_j = a_j^{(n)}$. Note that $a_j^{(n)} \in \mathbb{Q}$. Let $p_1, p_2, \ldots, p_r \subset C_m \otimes M_K$ be nonzero projections and $tr$ be the normalized trace on $\pi_j \otimes \text{id}_{R(m)}(C_m)$ ($j = m, m+1, \ldots, m+n$). Set $x_{ij} = tr(\pi_{m+j} \otimes \text{id}_K(p_i))$, $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, n$. So $x_{ij} \in \mathbb{Q}$. Let $P_{m,m+n} : C_m \rightarrow C_{m+n}$ be defined by

\[
f \mapsto (0, \Pi_m(f), \Pi_{m+1}(f), \ldots, \Pi_{m+n}(f)).
\]

We compute that

\[
b_j^{(n)} = tr(P_{m,m+n} \otimes \text{id}_K(p_i)) = \sum_{i=1}^{n} x_{ij} a_j.
\]
It is important to note that \( x_{ij}, a_j^{(n)}, b_j^{(n)} \in \mathbb{Q} \). Let \( \tau \) be the unique tracial state of \( A \). Set \( \tau(j(p_j)) = z_j \). Note that
\[
\left( \prod_{j=m}^{m+n} (m+j-1)/K(n) \to 0 \quad (n \to \infty) \right).
\]
Therefore, it is easy to see that \( b_j^{(n)} \to z_j \) as \( n \to \infty \).

**Lemma 8.** Let \( \tau \) be the unique tracial state on \( A \). Then \( \tau|_{j(C_m)} \) is in the weak closure of the convex hull of \( s(C_m) \). Consequently, \( \ker \tau|_{j(C_m)} \supset \ker \delta_{C_m} \).

**Proof.** First, if \( t \in s(C_{m+n}) \), it is easy to compute that \( t|_{j(C_m)} \) is in the convex hull of \( s(C_m) \). Take a sequence of \( \{t_n\} \subset s(C_{m+n}) \). Extend \( t_n \) to a state on \( A \) and let \( s \) be a weak limit of \( t_n \) (on \( A \)). Then \( s \) is a tracial state on \( A \). Therefore \( s = \tau \). This implies that \( \tau|_{j(C_m)} \) is in the weak closure of the convex hull of \( s(C_m) \).

**Lemma 9.** For any finitely generated subgroup \( G \subset K_0(A) \), there is a finitely generated free subgroup \( G_0 \subset K_0(C_m) \) for some \( m > 0 \) which has \( \mathbb{Z} \)-rank \( r \) and is generated by \( r \) positive \( \mathbb{Z} \)-linearly independent elements \( f_1, \ldots, f_r \in G_0 \) such that \( \rho_A \circ j_s(G_0) = \rho_A(G) \) and \( \rho_A \circ j_s \) is injective on \( G_0 \).

**Proof.** Note that \( \rho_A(K_0(A)) \) is a divisible subgroup of \( \mathbb{R} \). Let \( g_1, g_2, \ldots, g_r \) be \( \mathbb{Q} \)-linearly independent elements in \( \rho_A(K_0(A))_+ \) such that \( \bigoplus_{i=1}^r \mathbb{Z}g_i = \rho_A(G) \). Let \( \iota : \bigoplus_{i=1}^r \mathbb{Z}g_i \to K_0(A) \) be an injection such that \( \rho_A \circ \iota = \text{id} \bigoplus_{i=1}^r \mathbb{Z}g_i \). Note that \( \iota(g_i) \) is positive \( (i = 1, 2, \ldots, r) \). Therefore there are \( f_1, f_2, \ldots, f_r \in K_0(C_m)_+ \) for some large \( m \) such that \( j_s(f_i) = \iota(g_i) \), \( i = 1, 2, \ldots, r \). Let \( G_0 \) be the subgroup generated by \( f_1, \ldots, f_r \). We see that \( G_0 \) has rank \( r \) and \( j_s(f_1, \ldots, f_r) \in K_0(C_m)_+ \) are \( \mathbb{Z} \)-linearly independent.

**Lemma 10.** Let \( F \) be a divisible ordered subgroup of \( \mathbb{R} \). Let \( \{x_{ij}\}_{0 \leq i \leq r, 0 \leq j < \infty} \) be an \( r \times \infty \) matrix having rank \( r \) and with each \( x_{ij} \in \mathbb{Q}_+ \), and let \( \{a_j^{(n)}\} \) be sequences of positive rational numbers such that \( a_j^{(n)} \to a_j \) as \( n \to \infty \). For each \( n \),
\[
(x_{ij})_{r \times n} v_n = y_n,
\]
where \( v_n = (a_j^{(n)})_{n \times 1} \) is an \( n \times 1 \) column vector and \( y_n = (b_i^{(n)}) \in \mathbb{Q}^r \) is an \( r \times 1 \) column vector.

Suppose that \( y_n \to z \) for some \( z = (z_j)_{r \times 1} \in F^r \) with \( z_j \geq 0 \) (in \( \mathbb{R}^r \) norm). Then for some sufficiently large \( n \), there is \( u = (c_j)_{n \times 1} \in F_n^r \) \( (c_j > 0) \) such that
\[
(x_{ij})_{r \times n} u = z.
\]

**Proof.** To save notation, without loss of generality, we may assume that \( (x_{ij})_{r \times r} \) has rank \( r \). Set \( A_n = (x_{ij})_{r \times n} \) \( (n \geq r) \). Then there exists an invertible matrix \( B \in M_r(\mathbb{Q}) \) (which does not depend on \( n \)) such that \( BA_n = C_n \), where \( C = (c_{ij})_{r \times n} \), \( c_{ii} = 1 \) for \( i = 1, 2, \ldots, r \), and \( c_{ij} = 0 \) if \( i \neq j \). Let \( I_r \) be the \( r \times r \) identity matrix. We may write
\[
C_n = (I_r, D'_n),
\]
where \( D'_n \) is a \( r \times (n - r) \) matrix. Thus we have
\[
C_n v_n = B y_n \quad \text{and} \quad I_r v'_n = B y_n - D_n v_n,
\]
where \( v_n' = (a_1^{(n)}, a_2^{(n)}, ..., a_r^{(n)}) \) (as a column) and \( D_n = (0, D_n') \) is a \( r \times n \) matrix. Note that for any \( n \times 1 \) column vector \( v \) with the form \( (t_1, t_2, ..., t_r, a_r^{(n)}, a_{r+1}^{(n)}, ..., a_n^{(n)}) \), \( D_nv = D_nv_n \). Since \( a_j^{(n)} \to a_j > 0 \), there is an \( N_1 > 0 \) such that
\[
a_j^{(n)} \geq a_j/2 > 0
\]
for all \( n \geq N_1 \) and \( j = 1, 2, ..., r \). Let \( 0 < \varepsilon < \min \{ a_j/4 : j = 1, 2, ..., r \} \). There is \( N_2 > 0 \) such that
\[
\|By_n - Bz\|_{\infty} < \varepsilon
\]
if \( n \geq N_2 \). Set \( N = \max \{ N_1, N_2 \} \). Let \( u' = (c_1, c_2, ..., c_r) \) (column vector) satisfy the equation
\[
I_r u' = Bz - D_nv_n.
\]
Since \( I_r v_n' = v_n' \) and \( I_r u' = u' \), we have
\[
\|u' - v'\|_{\infty} < \varepsilon
\]
if \( n \geq N \). Therefore \( c_j > 0 \) for \( j = 1, 2, ..., r \). Set \( u = (c_1, ..., c_r, a_r^{(n)}, a_{r+1}^{(n)}, ..., a_n^{(n)}) \). Then
\[
I_r u' = Bz - D_nu
\]
(\( n \geq N \)). Since \( B \in M_r(\mathbb{Q}) \), \( D_nu = D_nu_n \in \mathbb{Q}^r \), \( z \in F^r \) and \( F \) is divisible, we conclude that \( u' \in F^r \). Since \( D_n = C_n - I_r \), we have
\[
C_nu = Bz.
\]
Finally, since \( B \) is invertible, we have
\[
A_nu = z.
\]
\( \square \)

11. Let \( B \) be a unital separable simple AF-algebra with \( (K_0(B), K_0(B)_+, [1_B]) = (\rho_A(K_0(A)), \rho_A(K_0(A))_+, 1) \). Let \( \alpha_0 : K_0(A) \to K_0(B) \) be the positive homomorphism defined by \( \rho_A \). By the Universal Coefficient Theorem, there is \( \alpha \in KL(A, B)_+ \) such that \( \alpha|_{K_0(A)} = \alpha_0 \). As in [DL], we identify \( KL(A, B) \) with \( Hom_A(K(A), K(B)) \). An element \( \alpha \in KL(A, B)_+ \) is an element such that \( \alpha(K_0(A) \setminus \{ 0 \}) \subset (K_0(A) \setminus \{ 0 \}) \).

Since \( K_1(B) = 0 \) and \( \text{Tor}(K_0(B)) = 0 \), from the following commutative diagram,
(the map from $K_i(\cdot)$ to $K_i(\cdot)$ is the multiplication by $k$), we have
\[ \alpha|_{K_i(A)} = 0 ~ \text{and} ~ \alpha|_{K_i(A,Z/k\mathbb{Z})} = 0. \]

We claim that
\[ \alpha|_{K_0(A,Z/k\mathbb{Z})} = 0 \]
for all $k > 0$.

Since $\rho_A(K_0(A)) = K_0(B)$ is divisible, $K_0(B)/kK_0(B) = 0$. Then (since $K_1(B) = 0$)
\[ K_0(B,Z/k\mathbb{Z}) = 0. \]

Therefore $\alpha|_{K_0(A,Z/k\mathbb{Z})} = 0$.

In [Ln3], we show the following.

**Theorem 12.** Let $A$ and $B$ be unital separable simple nuclear TAF $C^*$-algebras satisfying the UCT. Let $\alpha \in KL(A,B)_+$. Suppose that for every finite subset $G \subset K_1(A,Z/k\mathbb{Z})$ $(i = 0, 1, k = 0, 1, ...)$, there exists a sequence of contractive completely positive linear maps $L_n : A \to B$ such that
\[ \|L_n(ab) - L_n(a)L_n(b)\| \to 0 \quad (n \to \infty) \]
and
\[ [L_n]|_G = \alpha|_G. \]

Then there is a homomorphism $h : A \to B$ such that $[h] = \alpha$. (see 1.8 in Ln3 for the definition of $[L_n]|_G$).

13. We need only a version of Theorem 12 in which $B$ is an AF-algebra. In the next theorem, we let $B$ be a unital separable simple AF-algebra with
\[ (K_0(B), K_0(B)_+, [1_B]) = (\rho_A(A), \rho_A(A)_+, 1). \]

Let $\alpha \in KL(A,B)_+$ be as in 11. From 11 we only need to consider $G \subset K_0(A)$.

**Theorem 14.** Every unital separable residually finite dimensional $C^*$-algebra in $\mathcal{N}$ can be embedded into a unital simple AF-algebra.

**Proof.** From the construction in 5 $C$ maps into $A$. Since each $j_{n,n+1}$ is injective, we see that $C$ is embedded into $A$. Let $B$ be as in 11. It suffices to show that $A$ can be embedded into $B$. Let $\alpha$ be as in 13. Let $G$ be any finitely generated subgroup of $K_0(A)$. By 11 Theorem 12 and 13 to show that $A$ can be embedded into $B$, it suffices to show that there is a sequence of contractive completely positive linear maps $L_n : A \to B$ such that
\[ \|L_n(ab) - L_n(a)L_n(b)\| \to 0 \]
for all $a, b \in A$ and
\[ [L_n]|_G = \alpha|_G. \]

Let $m$, $G_0$ and $f_1, ..., f_r$ be as in Lemma 9. There are $z_1, z_2, ..., z_r \in \rho_B(B)_+ \setminus \{0\} = \rho_A(A)_+ \setminus \{0\}$ such that
\[ \rho_A \circ j_i(f_i) = z_i \quad (i = 1, 2, ..., r). \]

Note that $(j_{m,m+t})|_{G_0}$ is injective. So we may assume that there are projections $p_1, p_2, ..., p_r \in C_m \otimes M_K$ such that $[p_i] = f_i$ in $K_0(C_m)$, $i = 1, 2, ..., r$. We now use the notation developed in 7. Since $f_1, ..., f_r$ are $\mathbb{Z}$-linearly independent, by Lemma
Let $D = \bigoplus_{i=m}^{m+n} \pi_i \otimes \text{id}(C_m)$. Note $K_0(D) = \mathbb{Z}^n$. Define a positive map $\lambda : K_0(D) \to \rho_B(B) = \rho_A(A)$ by

$$(l_1, l_2, ..., l_n) = \sum_{j=1}^{n} \left( \frac{l_j}{R(m + j - 1)} \right) c_j$$

$(u = (c_1, c_2, ..., c_n) \in (\rho_A(A)^n)^+)$. It is well known that there exists a homomorphism $h : D \to B$ such that

$$[h]_{K_0(D)} = \lambda.$$

Let $L'_n = h \circ \bigoplus_{j=m}^{m+n} \pi_i : C_m \to B$. Note for each $i$,

$$(\bigoplus_{j=m}^{m+n} \pi_j)_*(f_i) = (R(m)x_{i1}, R(m + 1)x_{i2}, ..., R(m + n)x_{in})$$

and

$$\rho_B \circ [L'_n](f_i) = \sum_{j=1}^{n} x_{ij}c_j = z_i.$$

Thus

$$[L'_n]_{G_0} = \alpha \circ (j)_*|G_0.$$
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REFERENCES


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