IRREDUCIBLE RESTRICTION
AND ZEROS OF CHARACTERS

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Abstract. Let $G$ be a finite group, let $N$ be normal in $G$ and suppose that $\chi$ is an irreducible complex character of $G$. Then $\chi_N$ is not irreducible if and only if $\chi$ vanishes on some coset of $N$ in $G$.

1. Introduction

Let $N \triangleleft G$, where $G$ is an arbitrary finite group, and let $\chi \in \text{Irr}(G)$ be an irreducible complex character of $G$. In this note we give a characterization of when the restricted character $\chi_N$ is irreducible which at the same time extends Burnside’s theorem on zeros of characters.

Theorem A. Let $G$ be a finite group and let $N \triangleleft G$. Let $\chi \in \text{Irr}(G)$. Then $\chi_N$ is not irreducible if and only if $\chi$ vanishes on some coset $Nx$ of $N$ in $G$.

When $N = 1$ (or more generally if $N$ is abelian) Theorem A is Burnside’s theorem on zeros. We mention some easy consequences of Theorem A which again extend Burnside’s theorem.

Corollary B. Let $N \triangleleft G$ with $G/N$ a $\pi$-group and let $\chi \in \text{Irr}(G)$. If $\chi$ is nonzero on the $\pi$-elements of $G$, then $\chi_N$ is irreducible.

Corollary C. Let $N \triangleleft G$ and let $H \leq G$ be such that $G = HN$. Let $\chi \in \text{Irr}(G)$. If $\chi(h) \neq 0$ for $h \in H$, then $\chi_N$ is irreducible.

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2. Proofs

Our proof of Theorem A is another application of the theory of character triple isomorphisms. The reader is referred to [1] for its definition and main properties.

Proof of Theorem A. Suppose first that $\chi_N$ is irreducible. Let $x \in G$. By Lemma (8.14) of [1], we have that

$$\sum_{g \in Nx} |\chi(g)|^2 = |N|,$$

and therefore $\chi$ cannot be zero on the coset $Nx$. 

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Suppose now that $\chi_N$ is not irreducible. We want to find $x \in G$ such that $\chi(nx) = 0$ for all $n \in N$. Let $\theta \in \text{Irr}(N)$ be an irreducible constituent of $\chi_N$. Let $T$ be the stabilizer of $\theta$ in $G$ and by the Clifford correspondence (Theorem (6.11) of [1]), let $\psi \in \text{Irr}(T)$ be such that $\chi = \psi^G$. Assume that $T < G$. Then

$$\bigcup_{g \in G} T^g \subset G,$$

and we let $x \in G$ lie in no $G$-conjugate of $T$. Since $N$ is contained in every $G$-conjugate of $T$, it follows that for each element $n \in N$, the element $nx$ is contained in no $G$-conjugate of $T$. It follows that $\chi = \psi^G$ vanishes on the entire coset $Nx$, by the character induction formula.

We can now assume that $T = G$. Hence $\chi_N$ is a multiple of $\theta$. By Theorem (11.28) of [1], let $(G^*, M, \nu)$ be a character triple isomorphic to $(G, N, \theta)$ with $M \subseteq Z(G^*)$. Let $\chi^* \in \text{Irr}(G^*)$ correspond to $\chi$. Let us denote by $\nu$ the group isomorphism $G = M \to G$. By Gallagher’s theorem (Corollary (6.17) of [1]), notice that we may write $U/N = V/M$ is cyclic, we see that $\theta$ has an extension $\varphi \in \text{Irr}(U)$, by Corollary (11.22) of [1]. By Gallagher’s theorem (Corollary (6.17) of [1]), notice that we may write $\chi_U = \psi \varphi$ for some character $\psi$ of $U/N$. Let us denote by $\psi^*$ the character of $V/N$ satisfying

$$\psi^*((wN)^*) = \psi(wN)$$

for $w \in U$. By Definition (11.23.d) of [1], we have that

$$\chi^*_V = \psi^* \varphi^*,$$

where $\varphi^*$ is the extension of $\nu$ corresponding to $\varphi$ under the character triple isomorphism. Since $\nu$ is linear, we have that $\varphi^*$ is also linear.

Now, let the coset $Nx \subseteq U$ correspond to $My$ under the character triple isomorphism. Note that we can write $\psi(x) = \psi(Nx) = \psi^*(My) = \psi^*(y)$. Now

$$0 = \chi^*(y) = \psi^*(y) \varphi^*(y),$$

and the second factor is nonzero because $\varphi^*$ is linear. Thus $\psi^*(y) = 0$ and hence $\psi(x) = 0$. It follows that

$$\chi(x) = \psi(x) \varphi(x) = 0.$$

Since $x$ was an arbitrary element of the coset $Nx$, the result follows.

Proof of Corollary B. Suppose that $\chi_N$ is not irreducible. Then there exists a coset $Nx$ of $N$ in $G$ on which $\chi$ is zero. Now $Nx = Nx_\pi$ contains the $\pi$-element $x_\pi$, and this contradicts the hypothesis.

Proof of Corollary C. If $\chi_N$ is not irreducible, there exists a coset $Nx$ of $N$ in $G$ on which $\chi$ is zero. Now $Nx$ contains some $h \in H$, and this contradicts the hypothesis.
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