IMAGINARY POWERS OF LAPLACE OPERATORS

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Abstract. We show that if $L$ is a second-order uniformly elliptic operator in divergence form on $\mathbb{R}^d$, then $C_1(1+|\alpha|)^{d/2} \leq \|L^{i\alpha}\|_{L^1 \to L^\infty} \leq C_2(1+|\alpha|)^{d/2}$. We also prove that the upper bounds remain true for any operator with the finite speed propagation property.

1. Introduction

Assume that $a_{ij} \in C^\infty(\mathbb{R}^d)$, $a_{ij} = a_{ji}$ for $1 \leq i, j \leq d$ and that $\kappa I \leq (a_{ij}) \leq \tau I$ for some positive constants $\kappa$ and $\tau$. We define a positive self-adjoint operator $L$ on $L^2(\mathbb{R}^d)$ by the formula

$$L = -\sum \partial_i a_{ij} \partial_j.$$ (1)

We refer readers to [8] for the precise definition and basic properties of $L$. In particular, $L$ admits a spectral resolution $E(t)$ and we can define the operator $L^{i\alpha}$ by the formula

$$L^{i\alpha} = \int_0^\infty t^{i\alpha} dE(t).$$

By spectral theory $\|L^{i\alpha}\|_{L^2 \to L^2} = 1$. It is well known that $L^{i\alpha}$ falls within the scope of classical Calderon-Zygmund theory (as described in [3] or [22]) and so it extends to a bounded operator on $L^p$, $1 < p < \infty$, and is also weak type $(1,1)$. The main aim of this paper is to obtain the sharp estimate for the weak type $(1,1)$ norm of $L^{i\alpha}$ in terms of $\alpha$.

The study of imaginary powers of operators is an important part of the theory of operators of type $\omega$ with $H^\infty$ functional calculus (see e.g., [3], [9] and [17]). What is perhaps more interesting and relevant from the point of view of this paper is that the weak type $(1,1)$ norm of imaginary powers of self-adjoint operators can play a central role in the theory of spectral multipliers. See [5] and [15]. Imaginary powers of Laplace operators on compact Lie groups were also investigated in [20]. Theorem 2 below applied to Laplace operators on compact Lie groups gives the sharp endpoint result of Theorem 3 in [20], pp. 58. See also Corollary 4 of [20], pp. 121.

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However, the starting point for this paper is the following observation from [2]. If we denote the weak type \((1,1)\) norm of an operator \(T\) on a measure space \((X, \mu)\) by \(\|T\|_{L^1 \to L^{1,\infty}} = \sup \lambda \mu(\{x \in X: |Tf(x)| > \lambda\})\) where the supremum is taken over \(\lambda > 0\) and functions \(f\) with \(L^1(X)\) norm less than one, then for the standard Laplace operator on \(\mathbb{R}^d\),
\begin{equation}
C_1(1 + |\alpha|)^{d/2} \leq \|(-\Delta)^{i\alpha}\|_{L^1 \to L^{1,\infty}} \leq C_2(1 + |\alpha|)^{d/2} \log(1 + |\alpha|).
\end{equation}
The classical Hörmander multiplier theorem (see [13]) states that a multiplier operator \(T_m\) on \(\mathbb{R}^d\) with multiplier \(m\) satisfies
\begin{equation}
\|T_m\|_{L^1 \to L^{1,\infty}} \leq C_s \sup_{t > 0} \|\eta(\cdot) m(t\cdot)\|_{H^s} \leq A
\end{equation}
for any \(s > d/2\) and any \(\eta \in C_0^\infty(\mathbb{R}^d)\) not identically zero. Here \(H_s\) is the Sobolev space of order \(s\) on \(\mathbb{R}^d\). Since the Sobolev norm in [3] behaves like \((1 + |\alpha|)^s\) for the multiplier \(m(x) = |x|^{i\alpha}\) of \((-\Delta)^{i\alpha}\) [2] shows that the exponent \(d/2\) in Hörmander’s theorem is sharp. Furthermore, if [3] is satisfied with \(A < \infty\), then the distribution \(K = m\) agrees with a locally integrable function away from the origin which satisfies
\begin{equation}
I(B) = \sup_{y \neq 0} \int_{|x| \geq B|y|} |K(x - y) - K(x)| \, dx \leq A
\end{equation}
for \(B \geq 2\) and Hörmander’s theorem actually shows that the weak type \((1,1)\) norm of \(T_m\) is bounded by \(I(B) + \|m\|_{L^\infty}^2 + B^d\). One can easily compute that for the convolution kernel \(K\) of \((-\Delta)^{i\alpha}\), the integral \(I(B)\) is bounded above and below by \((1 + |\alpha|)^{d/2} \log(1 + |\alpha|/B)\). Hence Hörmander’s theorem gives the upper bound in [2]. The lower bound is a simple consequence of the explicit formula for the kernel \(K\) of \((-\Delta)^{i\alpha}\). See for example, [21] pp. 51-52.

The main observation of this paper is to note that there is a slight improvement of the bound \(I(B) + \|m\|_{L^\infty}^2 + B^d\) to \(I(B) + (\|m\|_{L^\infty}^2 B^d)^{1/2}\). This can be achieved either by using C. Fefferman’s ideas in [11] of exploiting more information of \(L^p\) bounds or by varying the level of the Calderón-Zygmund decomposition and optimising. Hence we will be able to remove the \(\log\) term in [2]. We will show that this more precise estimate holds for a general class of operators.

**Theorem 1.** Suppose that \(L\) is defined by \([11]\). Then
\begin{equation}
C_1(1 + |\alpha|)^{d/2} \leq \|L^{i\alpha}\|_{L^1 \to L^{1,\infty}} \leq C_2(1 + |\alpha|)^{d/2}
\end{equation}
for all \(\alpha \in \mathbb{R}\).

**Proof of the lower bound.** We begin with some known estimates for the kernel \(p_t(x, y)\) of the heat operator \(\exp^{-tL}\) associated to \(L\). First, this kernel satisfies Gaussian bounds
\begin{equation}
C_1 \frac{1}{t^{d/2}} \exp\left(-b_1 \rho^2(x, y)/t\right) \leq p_t(x, y) \leq C_2 \frac{1}{t^{d/2}} \exp\left(-b_2 \rho^2(x, y)/t\right)
\end{equation}
(see [3]) for some positive constants \(C_1, C_2, b_1\) and \(b_2\) and where \(\rho(x, y)\) denotes the geodesic distance between \(x\) and \(y\) given by the Riemannian metric \((a_{ij})\). In this setting of uniform ellipticity, \(\kappa |x - y| \leq \rho(x, y) \leq \tau |x - y|\). Secondly, from the construction of a parametrix for the heat equation with respect to \(L\) (either via Hadamard’s construction, see §17.4 of [13], or using pseudodifferential operator
techniques, see chapter 7, §13 of [23], we have for each \( y \in \mathbb{R}^d \) a ball \( B(y, r) \) such that for \( x \in B(y, r) \) and \( 0 < t < 1 \),

\[
|p_t(x, y) - (\det a_{ij}(y))^{-1/2}(4\pi t)^{-d/2}e^{-\rho^2(x, y)/4t}| \leq Ct^{1/2}t^{-d/2}.
\]

Here we are using the fact that \( p_t \) is symmetric, \( p_t(x, y) = p_t(y, x) \). From (11) and (12), we have for \( x \in B(y, r) \) the bound

\[
|p_t(x, y) - (\det a_{ij}(y))^{-1/2}(4\pi t)^{-d/2}e^{-\rho^2(x, y)/4t}| \leq Ct^{1/2}t^{-d/2} \exp(-b'(x, y)^2/t)
\]

which translates into a bound for the kernel \( K_{L^{j\alpha}} \) of \( L^{j\alpha} \) since the functional calculus for \( L \) gives us the relationship

\[
L^{j\alpha} = \Gamma(-i\alpha)^{-1} \int_0^\infty t^{-i\alpha-1}e^{-tL}dt
\]

for \( \alpha \neq 0 \). Thus for \( x \in B(y, r) \),

\[
|K_{L^{j\alpha}}(x, y) - (\det a_{ij}(y))^{-1/2}4^\alpha\pi^{-d/2}\gamma(\alpha)\rho(x, y)^{-d-2i\alpha}| \leq C|\Gamma(-i\alpha)|^{-1}\rho(x, y)^{-d+1/2}
\]

where \( \gamma(\alpha) = \Gamma(i\alpha + d/2)/\Gamma(-i\alpha) \). Using (8) with \( y = 0 \) we obtain for \( \lambda \) large enough

\[
\mu(\{ |K_{L^{j\alpha}}(x, 0)| \geq \lambda \}) \\
\geq \mu(\{C_1|\gamma(\alpha)|\rho^{-d}(x, 0) \geq 2\lambda \}) - \mu(\{C_2|\Gamma(-i\alpha)|\rho^{-d+1/2}(x, 0) \geq \lambda \}) \\
= \mu(B(0, (2C_1|\gamma(\alpha)|/\lambda)^{1/d})) - \mu(B(0, (C_2|\Gamma(-i\alpha)|/\lambda)^{1/(d-1/2)})) \\
\geq C'|\gamma(\alpha)|/\lambda.
\]

Here \( \mu \) is Lebesgue measure and the sets above have the further restriction that \( x \in B(0, r) \). Since \( K_{L^{j\alpha}} \) is smooth away from the diagonal, we see that \( L^{j\alpha}\phi_0(x) \) tends to \( K_{L^{j\alpha}}(x, 0) \) as \( \delta \to 0 \) for any \( x \neq 0 \) and any approximation of the identity \( \{\phi_\delta\} \). Hence the above estimate shows that the weak type (1,1) norm of \( L^{j\alpha} \) is bounded below by \( |\gamma(\alpha)| = |\Gamma(i\alpha + d/2)/\Gamma(-i\alpha)| \sim (1 + |\alpha|)^{d/2} \) (see [10]).

The upper bound in Theorem [11] holds in a much more general setting which we describe now. Assume that \( (X, \mu, \rho) \) is a space with measure \( \mu \) and metric \( \rho \). If \( \|P\|_{L^2 \to L^\infty} < \infty \), then we can define the kernel \( K_P \) of the operator \( P \) by the formula

\[
\langle P(\psi), \phi \rangle = \int P(\psi)\overline{\phi}d\mu = \int K_P(x, y)\overline{\psi(x)}\phi(y)d\mu(x)d\mu(y).
\]

Note that \( \text{supp}_x \|K_P(x, \cdot)\|_{L^2} = \|P\|_{L^2 \to L^\infty} \). Next, we say that

\[
\text{supp} \ P \subset \{(x, y) \in X^2 : \rho(x, y) \leq r \}
\]

if \( \langle P(\psi), \phi \rangle = 0 \) for every \( \phi, \psi \in L^2 \) and every \( r_1 + r_2 + r < \rho(x', y') \) such that \( \psi(x) = 0 \) for \( \rho(x, x') > r_1 \) and \( \phi(x) = 0 \) for \( \rho(x, y') > r_2 \). This definition [11] makes sense even if \( \|P\|_{L^2 \to L^\infty} = \infty \). Now if \( L \) is a self-adjoint positive definite operator acting on \( L^2(\mu) \), then we say that it satisfies the finite speed propagation property of the corresponding wave equation if

\[
\text{supp} \ K_{C(t\sqrt{L})} \subset \{(x, y) \in X^2 : \rho(x, y) \leq t \},
\]

where \( C_t(\sqrt{L}) = \int \cos(t\sqrt{\lambda})dE(\lambda) \).
Theorem 2. Suppose that $L$ satisfies (10). Next assume that
\[ \|\exp(-tL)\|_{L_2 \to L_\infty}^2 \leq C_1 V_{d,D}(t^{1/2})^{-1} \leq C_2 V_{d,D}(t^{1/2})^{-1} \]
for all $t > 0$ and $x \in X$, where $B(x,t)$ is a ball with radius $t$ centred at $x$ and
\[ V_{d,D}(t) = \begin{cases} t^d & \text{for } t \leq 1, \\ t^D & \text{for } t > 1, \end{cases} \]
for $d, D \geq 0$. Then
\[ \|L^{\alpha}\|_{L^{1 \to L^{1,\infty}}} \leq C_2(1 + |\alpha|)^{\max(d,D)/2} \]
for all $\alpha \in \mathbb{R}$.

We remark that (10) and (11) are equivalent to having Gaussian upper bounds on the heat kernel and the associated volume growth on balls. See [18]. Furthermore, the upper bound in Theorem 1 follows from Theorem 2. Indeed, if $X = \mathbb{R}^d$, $\rho(x,y) = \tau|x-y|$, and $\mu$ is Lebesgue measure, then it is well known (see e.g. [8] and [19]) that (11) and (10) hold. We are going to prove Theorem 2 only in the case $d = D$. The argument for the other cases is similar.

2. Preliminaries

The following lemma is a very simple but useful consequence of (10).

Lemma 1. Assume that $L$ satisfies (10) and that $\hat{F}$ is a Fourier transform of an even bounded Borel function $F$ with $\mathrm{supp} \hat{F} \subset [-r,r]$. Then
\[ \mathrm{supp} K_{\hat{F}({\sqrt{L}})} \subset \{ (x,y) \in \mathbb{R}^2 : \rho(x,y) \leq r \}. \]

Proof. If $F$ is an even function, then by the Fourier inversion formula,
\[ F(\sqrt{L}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{F}(t)C_t(\sqrt{L}) \, dt. \]
But since $\mathrm{supp} \hat{F} \subset [-r,r]$,
\[ F(\sqrt{L}) = \frac{1}{2\pi} \int_{-r}^{r} \hat{F}(t)C_t(\sqrt{L}) \, dt \]
and Lemma 1 follows from (10).

Lemma 2. Let $\phi \in C_c^\infty(\mathbb{R})$ be even, $\phi \geq 0$, $\|\phi\|_{L_1} = 1$, $\mathrm{supp}(\phi) \subset [-1,1]$, and set $\phi_r(x) = 1/r \phi(x/r)$ for $r > 0$. Let $\Phi$ denote the Fourier transform of $\phi$ and $\Phi^r$ denote the Fourier transform of $\phi_r$. If (11) and (10) hold, then the kernel $K_{\phi^r(\sqrt{L})}$ of the self-adjoint operator $\Phi^r(\sqrt{L})$ satisfies
\[ \mathrm{supp} K_{\phi^r(\sqrt{L})} \subset \{ (x,y) \in X^2 : \rho(x,y) \leq r \} \]
and
\[ |K_{\phi^r(\sqrt{L})}(x,y)| \leq C r^{-d} \]
for all $r > 0$ and $x,y \in X$. 

Proof. (12) follows from Lemma 1. For any $m \in \mathbb{N}$ and $r > 0$, we have the relationship

$$(I + rL)^{-m} = \frac{1}{m!} \int_0^\infty e^{-rtL}e^{-t}t^{m-1}dt$$

and so when $m > d/4$, (11) implies

$$\|(I + rL)^{-m}\|_{L^2 \rightarrow L^\infty} \leq \frac{1}{m!} \int_0^\infty \|\exp(-rtL)\|_{L^2 \rightarrow L^\infty}e^{-t}t^{m-1}dt \leq C_1 r^{-d/4}$$

for all $r > 0$. Now $\|(I + r^2L)^{-m}\|_{L^1 \rightarrow L^2} = \|(I + r^2L)^{-m}\|_{L^2 \rightarrow L^\infty}$ and so

$$\|\Phi^r(\sqrt{L})\|_{L^1 \rightarrow L^\infty} \leq \|(I + r^2L)^{2m}\Phi^r(\sqrt{L})\|_{L^2 \rightarrow L^2} \|(I + r^2L)^{-m}\|_{L^2 \rightarrow L^\infty}^2.$$

The $L^2$ operator norm of the first term is equal to the $L^\infty$ norm of the function $(1 + r^2|t|)^{2m}\Phi(r\sqrt{|t|})$ which is uniformly bounded in $r > 0$ and so (13) follows by (14).

Next we recall the Calderón-Zygmund decomposition in the general setting of spaces of homogeneous type (see e.g. [3] or [22]).

**Lemma 3.** There exists $C$ such that, given $f \in L^1(X,\mu)$ and $\lambda > 0$, one can decompose $f$ as

$$f = g + b = g + \sum b_i$$

so that

1. $|g(x)| \leq C\lambda$, a.e. $x$ and $\|g\|_{L^1} \leq C\|f\|_{L^1}$.
2. There exists a sequence of balls $B_i = B(x_i,r_i)$ such that the support of each $b_i$ is contained in $B_i$ and

$$\int |b_i(x)|d\mu(x) \leq C\lambda\mu(B_i).$$

3. $\sum \mu(B_i) \leq C\frac{1}{\lambda} \int |f(x)|d\mu(x)$.
4. There exists $k \in \mathbb{N}$ such that each point of $X$ is contained in at most $k$ of the balls $B(x_i,2r_i)$.

We are now in a position to prove Theorem 2.

3. PROOF OF THEOREM 2

The proof follows closely the line of argument in [1] (which of course generalises to this setting). We are attempting to prove

$$\lambda \mu(\{x \in X : |L^{1\alpha}f(x)| \geq \lambda\}) \leq C(1 + |\alpha|\frac{d}{2}) \|f\|_{L^1}.$$

As usual we start by decomposing $f$ into $g + \sum b_i$ at the level of $\lambda$ according to Lemma 3. We will follow the idea of C. Fefferman [11] of using more information of the $L^2$ operator norm (in our case, $\|L^{1\alpha}\|_{L^2 \rightarrow L^2} = 1$) by smoothing out the bad functions $b_i$ at a scale smaller than the size of its support and considering this part of the good function where $L^2$ estimates can be used (see also [4]). In our case for each $b_i$, consider $\Phi^{s_i}(\sqrt{L})b_i$ where $s_i = \theta r_i$, $\theta = (1 + |\alpha|)^{-\frac{d}{2}}$, and let
Therefore the first term in (15) is bounded by \(1 + \lambda\). Hence \(f = G + B\) where \(B = \sum (I - \Phi^s_v)X\) and we write
\[
\lambda \mu(\{|L^\alpha f(x)| \geq \lambda\}) \leq \lambda \mu(\{|L^\alpha G(x)| \geq \lambda/2\}) + \lambda \mu(\{|L^\alpha B(x)| \geq \lambda/2\}).
\]
The first term is less than \(4/\lambda \|L^\alpha G\|_{L^2}^2 \leq 4/\lambda \|G\|_{L^2}^2\). However, according to Lemma 2,
\[
|\Phi^s_v(X)b_i(x)| \leq \int |K_{\Phi^s_v(X)}(x,y)b_i(y)| d\mu(y) \leq C (\theta r_i)^{-d}\|b_i\|_{L^1} \|B(x,2r_i)\|
\]
and therefore by Lemma 3 \(|G(x)| \leq C\theta^{-d} \lambda\) for a.e. \(x\). Using Lemma 2 again which shows that the \(L^p \to L^p\) operator norms of \(\Phi^s_v(X)\) are uniformly bounded in \(r > 0\), we also have that
\[
\|G\|_{L^1} \leq \|g\|_{L^1} + C \sum |\Phi^s_v(X)b_i| \leq \|g\|_{L^1} + C \sum \|b_i\|_{L^1} \leq C \|f\|_{L^1}.
\]
Therefore the first term in (15) is bounded by \((1 + |\alpha|)\|f\|_{L^1}\).

Since \(\mu\bigcup B(x_i,\theta^{-1}r_i)\) \(\leq C\theta^{-d} \sum \mu(B_i) \leq C(1 + |\alpha|)\|f\|_{L^1}/\lambda\), then to bound the second term in (15), it suffices to show
\[
\int_{x \not\in \bigcup B_i^*} |L^\alpha B(x)| d\mu(x) \leq C(1 + |\alpha|)^{1/2} \|f\|_{L^1},
\]
where \(B_i^* = B(x_i,\theta^{-1}r_i)\). Let \(H^\alpha(t) = |t|^{2\alpha}\) so that
\[
L^\alpha B(x) = \sum H^\alpha(1 - \Phi^s_v)(\sqrt{L})b_i(x)
\]
and therefore the left side of (16) is less than
\[
\sum_i \int_{x \not\in \bigcup B_i^*} \left| \int K_{H^\alpha(1 - \Phi^s_v)(\sqrt{L})}(x,y)b_i(y) d\mu(y) \right| d\mu(x)
\]
\[
\leq \sum_i \int |b_i(y)| \int_{x \not\in B_i^*} |K_{H^\alpha(1 - \Phi^s_v)(\sqrt{L})}(x,y)| d\mu(x) d\mu(y).
\]
Since \(F(L)^* = \overline{F(L)}\), we may interchange the roles of \(x\) and \(y\), and so (16) will follow from Lemma 3 once we establish
\[
\sup_{x,i} \int_{\rho(x,y) \geq \theta^{-1}r_i} |K_{H^\alpha(1 - \Phi^s_v)(\sqrt{L})}(x,y)| d\mu(y) \leq C(1 + |\alpha|)^{1/2}.
\]
We now fix \(x \in X\) and \(i\). Let \(\eta \in C^\infty_c(R)\) be an even function supported in \(\{t \in R : 1 \leq |t| \leq 4\}\) such that
\[
\sum_{n=\infty} \eta(2^{-n}t) = 1 \text{ for all } t \neq 0.
\]
We put \(H^\alpha_n(t) = \eta(2^{-n}t)H^\alpha(t)\) so that
\[
H^\alpha(1 - \Phi^s_v)(\sqrt{L}) = \sum H^\alpha_n(1 - \Phi^s_v)(\sqrt{L}).
\]
Thus
\begin{equation}
\int_{y \in B_i^*} |K_{H_n(1-\Phi^{s_1})}(\sqrt{T}) (x, y)| \, d\mu(y) \leq \sum_{n \neq B_i^*} \int_{y \in B_i^*} |K_{H_n(1-\Phi^{s_1})}(\sqrt{T}) (x, y)| \, d\mu(y)
\end{equation}
and we will estimate each term in the sum on the right side in terms of \(n\) and \(i\), uniformly in \(x \in X\).

Let \(k_0 = [d/2] + 1\) so that
\[
\int_{y \in B_i^*} (1 + 2^n\rho(x, y))^{-2k_0} \, d\mu(y) \leq C \int_{\theta^{-1} r_i} (1 + 2^n\rho(x, y))^{-2k_0} \, d\mu(y)
\]
and therefore by the Cauchy-Schwarz inequality,
\[
\int_{y \in B_i^*} |K_{H_n(1-\Phi^{s_1})}(\sqrt{T}) (x, y)| \, d\mu(y) \leq C 2^{-nk_0} (\theta^{-1} r_i)^{d-k_0}
\]
(19) \[
\int_{\rho(x, y) \geq \theta^{-1} r_i} |K_{H_n(1-\Phi^{s_1})}(\sqrt{T}) (x, y)|^2 (1 + 2^n\rho(x, y))^{2k_0} \, d\mu(y) \right)^{1/2}.
\]
We break up the integral on the right side of (19) where \(2^n\rho(x, y)\) is roughly constant and consider
\begin{equation}
\sum_{2^j \geq 2^n r_i \theta^{-1}} \int_{2^{j-1} - 2^n \rho(x, y) \leq 2^{j-1}} |K_{H_n(1-\Phi^{s_1})}(\sqrt{T}) (x, y)|^2 \, d\mu(y).
\end{equation}

Fix a nonnegative even \(\varphi \in C_c^\infty (\mathbb{R})\) such that \(\varphi = 1\) on \([-1/4, 1/4]\) and \(\varphi = 0\) on \(\mathbb{R} \setminus [-1/2, 1/2]\). Then the Fourier transforms of \(H_n(1-\Phi^{s_1})\) and \(H_n(1-\Phi^{s_1})\)
* \((\delta - \varphi_{2^{-j}})\) agree on \(\{ \xi : |\xi| \geq 2^{j-1} - 1\} \) and so by Lemma 1 the kernels of \(H_n(1-\Phi^{s_1})(\sqrt{T})\) and \(H_n(1-\Phi^{s_1})\) \* \((\delta - \varphi_{2^{-j}})(\sqrt{T})\) agree on the set \(\{ (x, y) \in X^2 : \rho(x, y) \geq 2^{j-1} - 1\}\). Here \(\delta\) denotes the Dirac mass at 0. For each \(j\), the integrals in (20) satisfy the bound
\[
\int_{2^{j-1} \leq \rho(x, y) \leq 2^{j-1}} |K_{H_n(1-\Phi^{s_1})}(\sqrt{T}) (x, y)|^2 \, d\mu(y) \leq \|K_{H_n(1-\Phi^{s_1})}(\sqrt{T})\|_{L^2 \to L^\infty}^2
\]
where we are defining \(F_{n,j}^\alpha(t) = H_n(1-\Phi^{s_1}) \* (\delta - \varphi_{2^{-j}})(t)\). So by (14), the right side of this inequality is bounded by \(\|(I + 2^{-2n}L)^m F_{n,j}^\alpha(\sqrt{T})\|_{L^2 \to L^2}^2 \) as long as \(m > d/4\). Everything then comes down to estimating the \(L^\infty\) norm of \((1 + 2^{-2n}t^2)^m F_{n,j}^\alpha(t)\). We make the following claim.

Claim. For each \(j\), \(n\) and \(m > d/4\),
\[
(1 + 2^{-2n}t^2)^m |F_{n,j}^\alpha(t)| \leq C_m |\alpha|^{k_0} 2^{-j k_0} \min (1, (2^n r_i \theta)^2) \min (1, |\alpha| 2^{-j})
\]
uniformly in \(t \in \mathbb{R}\).

The claim shows that
\[
\|K_{F_{n,j}^\alpha}(\sqrt{T})\|_{L^2 \to L^\infty} \leq C |\alpha|^{k_0} 2^{-j k_0} 2^{\frac{d}{2}} \min (1, (2^n r_i \theta)^2) \min (1, |\alpha| 2^{-j})
\]
and hence the sum in (20) is bounded by
\[
|\alpha|^{2k_o} \cdot 2^{n-d} \min(1, (2^n r_i \theta)^2) \sum_{2^j \geq 2^n r_i \theta^{-1}} \min(1, |\alpha|2^{-j}) \leq |\alpha|^{2k_o} \cdot 2^{nd} \min(1, (2^n r_i \theta)^2) \log(2 + \frac{|\alpha|}{2^n r_i \theta^{-1}}).
\]

Recall that \( \theta \) and \( \alpha \) are related so that \( \theta |\alpha| = |\alpha|/(1 + |\alpha|)^{\frac{1}{2}} \leq \theta^{-1} \). Plugging this into (19) gives
\[
\int_{y \notin B^*_i} |K_{H^{n}(1-\Phi^*)}(\sqrt{\xi}, y) \, dy | \leq \theta^{-d} (2^n r_i \theta)^{\frac{d}{2}} \min(1, (2^n r_i \theta)^2) \log(2 + \frac{1}{2^n r_i \theta})
\]
and this makes the sum in (28) bounded by \( \theta^{-d} = (1 + |\alpha|)^{\frac{1}{2}} \), proving (17) and hence Theorem 2.

**Proof of the Claim.** If \( G_n(t) = H^n(1-\Phi^*(t)) \), then \( F_{n,j}^{\alpha}(t) = 2^{(n-j)k_o} G_n^{(k_o)} \hat{\Psi}_{2n-j}(t) \) where \( \hat{\Psi} = \xi^{-k_o} (1 - \varphi(x)) \) (and so \( \hat{\Psi} \) is continuous, rapidly decreasing and has vanishing moments, \( \int t^\ell \hat{\Psi}(t) \, dt = 0, \ell = 0, 1, 2, \ldots \)). Hence
\[
F_{n,j}^{\alpha}(t) = 2^{(n-j)k_o} \int_{\mathbb{R}} \left[ G_n^{(k_o)}(t - s) - G_n^{(k_o)}(0) \right] \hat{\Psi}_{2n-j}(s) \, ds = 2^{(n-j)k_o} \int_{\mathbb{R}} \left[ G_n^{(k_o)}(t - 2^{n-j}s) - G_n^{(k_o)}(t) \right] \hat{\Psi}(s) \, ds.
\]
However \( G_n(t) = \eta(2^{-n}t) |t|^{2\alpha}(1 - \Phi(s,t)) \) and thereby each time we take a derivative, we gain a factor of \( 2^{-n} \). \( G_n^{(k_o)}(t) \) is thus a finite sum of terms of the form \( \alpha^p 2^{-nk_o} \eta(2^{-n}t) |t|^{2\alpha} \hat{\Psi}(s,t) \) where \( \hat{\Psi} \in C^\infty_c(\mathbb{R}) \), \( \text{supp}(\hat{\Psi}) \subset \text{supp}(\eta) \) and \( \Psi \) is a Schwartz function which is \( 0(t^2) \) as \( t \to 0 \) (note that \( \hat{\Psi}(0) = \int x^\xi \, dx = 0 \) since \( \phi \) is even). The worst power \( p \) is \( k_o \), which occurs when all derivatives land on the factor \( |t|^{2\alpha} \).

Without loss of generality, let us suppose that
\[
G_n^{(k_o)}(t) = \alpha^k_o 2^{-nk_o} \eta(2^{-n}t) |t|^{2\alpha} \hat{\Psi}(s,t).
\]
From the above integral representation of \( F_{n,j}^{\alpha}(t) \), we see that the main contribution to \((1 + 2^{-2n}t^2)^m |F_{n,j}^{\alpha}(t)| \) occurs when \( |t| \sim 2^n \) and in this case,
\[
|F_{n,j}^{\alpha}(t)| \leq C |\alpha|^{k_o} 2^{(n-j)k_o} 2^{-nk_o} \min(1, (s,t)^2) \leq C |\alpha|^{k_o} 2^{-j} \min(1, (2^n r_i \theta)^2).
\]
However we may write
\[
F_{n,j}^{\alpha}(t) = -2^{(n-j)k_o} 2^{n-j} \int_0^1 \int_{\mathbb{R}} G_n^{(k_o+1)}(t - s) \hat{\Psi}(s) \, ds \, ds \, d\sigma
\]
and therefore we also have
\[
|F_{n,j}^{\alpha}(t)| \leq C |\alpha|^{k_o+1} 2^{(n-j)k_o} 2^{-n(j+1)} \min(1, (s,t)^2) \leq C |\alpha|^{k_o-j} |\alpha|^{2^{-j}} \min(1, (2^n r_i \theta)^2),
\]
establishing the claim.
Remarks. Theorem 1 holds also for Laplace-Beltrami operators on compact manifolds of dimension $d$. The proof is essentially the same as the proof of Theorem 1.

The hypotheses of Theorem 2 are satisfied for Laplace operators on Lie groups of polynomial growth. However, if $L$ is a sub-Laplacian on the three dimensional Heisenberg group, then $d = 4$ but

$$C_1 (1 + |\alpha|)^{3/2} \leq \|L^{\alpha}\|_{L^1 \to L^{1,\infty}} \leq C_2 (1 + |\alpha|)^{3/2+\varepsilon}.$$ 

(See [16]; see also [12].) The same estimates hold for a sub-Laplacian on SU(2) for which $d = 4$ and $D = 0$ (see [7]). Thus there are situations where the upper bound is better than the one given by Theorem 2 and where the lower bound in Theorem 1 is false. For general groups of polynomial growth Theorem 2 gives the best known estimates; however as the above examples show, these bounds are not always best possible.

References


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