MENGER CURVATURE
AND $C^1$ REGULARITY OF FRACTALS

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Abstract. We show that if $E$ is an $s$-regular set in $\mathbb{R}^2$ for which the triple integral
$$\int_E \int_E \int_E c(x, y, z)^s \, d\mathcal{H}^s x \, d\mathcal{H}^s y \, d\mathcal{H}^s z$$
of the Menger curvature $c$ is finite and if $0 < s \leq 1/2$, then $\mathcal{H}^s$ almost all of $E$ can be covered with countably
many $C^1$ curves. We give an example to show that this is false for $1/2 < s < 1$.

1. Introduction

The Menger curvature $c(x, y, z)$ of three points $x$, $y$ and $z$ in the plane $\mathbb{R}^2$ is defined as the reciprocal of the radius of the circle passing through these points. For a historical background, see [K]. In [Mc] Melnikov found a remarkable connection between the Menger curvature and the Cauchy kernel $1/z$, $z \in \mathbb{C}$. This led to a rapid development on singular integrals over 1-dimensional subsets of $\mathbb{R}^2$ and on removable sets of bounded analytic functions; see [MV], [MMV], [D], and for a survey [M2].

Another aspect of the Menger curvature is that its integrals can be used to measure smoothness properties of subsets of $\mathbb{R}^n$. Note that $c(x, y, z) = 0$ if and only if the points $x$, $y$ and $z$ lie on the same line. Let $\mathcal{H}^s$ be the $s$-dimensional Hausdorff measure. For $\mathcal{H}^s$ measurable sets $E \subset \mathbb{R}^n$ with $0 < \mathcal{H}^s(E) < \infty$ the proper quantity to use is
$$c^2s(E) = \int_E \int_E \int_E c(x, y, z)^2s \, d\mathcal{H}^s x \, d\mathcal{H}^s y \, d\mathcal{H}^s z.$$
Léger proved in [L] that if $\mathcal{H}^1(E) < \infty$ and $c^2(E) < \infty$, then there are rectifiable curves $\Gamma_1, \Gamma_2, \ldots$ such that
$$\mathcal{H}^1 \left( E \setminus \bigcup_{i=1}^{\infty} \Gamma_i \right) = 0.$$Sets with this property are called 1-rectifiable in [M1] and countably $(\mathcal{H}^1, 1)$ rectifiable in [F].

In this paper we study analogous questions for other values of $s$. It was shown in [Li, Theorem 1.4] that if $E \subset \mathbb{R}^n$ is $\mathcal{H}^s$ measurable and $0 < \mathcal{H}^s(E) < \infty$ for...
some $1 < s \leq n$, then $c^{2s}(E) = \infty$. Hence we restrict to $0 < s < 1$. We also study only subsets of $\mathbb{R}^2$, although with slight modifications the results would extend to $\mathbb{R}^n$. For reasons indicated in Example 2.5 we restrict to the so-called $s$-regular sets. This means that there is a constant $C$ such that
\begin{equation}
(1.1) \quad r^s/C \leq \mathcal{H}^s(E \cap B(x,r)) \leq Cr^s \quad \text{for } x \in E, \ 0 < r < d(E).
\end{equation}
Here $B(x,r)$ is the closed ball with centre $x$ and radius $r$, and $d(E)$ stands for the diameter of $E$.

When $0 < s < 1$ and $E$ is compact, (1.1) alone implies that $E$ is contained in one rectifiable curve; see the proof of Theorem 4.1 in [MM]. A rectifiable curve is the same as a Lipschitz image of the interval $[0,1]$. We study here whether Lipschitz images can be replaced by $C^1$ curves. By a $C^1$ curve we mean a curve with continuously varying tangent. It is the same as the image of an interval under a regular $C^1$ map, that is, a $C^1$ map with non-vanishing derivative. We shall prove in Corollary 2.2 that if $E$ satisfies (1.1), $c^{2s}(E) < \infty$ and $0 < s \leq 1/2$, then there are $C^1$ curves $\Gamma_1, \Gamma_2, \ldots$ such that
\begin{equation}
\mathcal{H}^s\left(E \setminus \bigcup_{i=1}^{\infty} \Gamma_i\right) = 0.
\end{equation}
We give in 2.4 an example showing that this is false if $1/2 < s < 1$. For $s = 1$ it is again true due to [L], even with the weaker condition $\mathcal{H}^1(E) < \infty$ instead of (1.1), because then covering $\mathcal{H}^1$ almost everything with Lipschitz images or $C^1$ curves are equivalent as a consequence of Rademacher’s theorem; see [F] 3.2.29.

There are also many other characterizations for 1-rectifiable sets. In [MM] analogous conditions for $s$-dimensional sets were investigated and this paper can be considered as a further contribution to that study.

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2. Covering with $C^1$ Curves

We begin with the following result on the existence of tangents. We say that a set $E \subset \mathbb{R}^2$ has a tangent $L$ at $x$ if $L$ is a line through $x$ such that for any $\alpha > 0$, $E \cap B(x,r) \subset C(x,\alpha)$ for all sufficiently small $r > 0$, where $C(x,\alpha)$ is the double-cone with centre $x$, axis $L$ and angle $\alpha$.

2.1. Theorem. Let $0 < s \leq 1/2$ and let $E \subset \mathbb{R}^2$ be $\mathcal{H}^s$ measurable and $s$-regular. If $x \in E$ and
\begin{equation}
(2.1) \quad c^{2s}(E, x) := \int_E \int_E c(x, y, z)^{2s} \, d\mathcal{H}^s y \, d\mathcal{H}^s z < \infty,
\end{equation}
then $E$ has a tangent at $x$.

Proof. Let $x \in E$. By the $s$-regularity of $E$ there are positive numbers $b$ and $d < d(E)$ such that for $i = 1, 2, \ldots$,
\begin{equation}
(2.2) \quad \mathcal{H}^s(A_i) \geq bd^{is},
\end{equation}
where
\begin{equation}
A_i = \{y \in E : d^{i+1} < |x - y| \leq d^i\}.
\end{equation}
Set

\[ \gamma_i = \int_{A_i} \int_{E} c(x, y, z)^{2s} \, d\mathcal{H}^s y \, d\mathcal{H}^s z. \]

Then

\[ \sum_i \gamma_i \leq \int_{E} \int_{E} c(x, y, z)^{2s} \, d\mathcal{H}^s y \, d\mathcal{H}^s z < \infty. \]

We shall show that for each \( i \) there is a line \( L_i \) through \( x \) such that

\[ \mathcal{H}^s (A_i \cap L_i(d^i)) \geq bd^{is}/16, \]

where

\[ \eta_i = (16b^{-1})^{1/s} \gamma_i^{1/(2s)}, \]

and

\[ B(\delta) = \{ x \in \mathbb{R}^2 : \text{dist}(x, \beta) \leq \delta \} \quad \text{for } B \subset \mathbb{R}^2, \delta > 0. \]

Suppose (2.4) fails for some \( i \). By (2.2) there is a closed quarter-plane \( Q \) (a sector with angle \( \pi/2 \)) with vertex at \( x \) such that \( \mathcal{H}^s (A_i \cap Q) \geq bd^{is}/4 \). Further, there is a line \( L \) through \( x \) such that

\[ \mathcal{H}^s (A_i \cap Q \cap H_j) \geq bd^{is}/8 \quad \text{for } j = 1, 2, \]

where \( H_1 \) and \( H_2 \) are the two closed half-planes whose boundary is \( L \). Since \( \mathcal{H}^s (A_i \cap L(\eta_i, d^i)) < bd^{is}/16 \), we have

\[ \mathcal{H}^s (A_i \cap Q \cap H_j \setminus L(\eta_i, d^i)) > bd^{is}/16 \quad \text{for } j = 1, 2. \]

Let \( x_j \in A_i \cap Q \cap H_j \setminus L(\eta_i, d^i) \) for \( j = 1, 2 \). We use the following formula, which is an exercise in elementary geometry:

\[ c(x, x_1, x_2) = \frac{2 \text{dist}(x_2, L_{y_1, x_1})}{|x - x_2| |x_1 - x_2|}, \]

where \( L_{y_1, x_2} \) denotes the line through two points \( y_1 \) and \( y_2 \). This gives

\[ c(x, x_1, x_2) \geq \frac{2\eta_i d^i}{d^{is} - d^i} = \frac{2\eta_i}{d^i}. \]

Thus by (2.6) and (2.5)

\[ \gamma_i > \left( \frac{\eta_i}{d^i} \right)^{2s} (bd^{is}/16)^2 = (b/16)^2 \eta_i^{2s} = \gamma_i, \]

which is a contradiction proving (2.4).

Next we show that if

\[ \zeta_i = \max \{ 12\eta_i/d, (16 \cdot 50^{2s}Cb^{-1}d^{-2s}\gamma_i)^{1/(3s)} \}, \]

and if \( \zeta_i < d \), then

\[ A_i \subset L_i(\zeta_i d^i). \]

Suppose this fails and let \( y_1 \in A_i \setminus L_i(\zeta_i d^i) \). Then \( \zeta_i < 1 \) and \( B(y_1, \frac{1}{2}\zeta_i d^i) \subset B(x, 2d^i) \setminus L_i(\frac{1}{2}\zeta_i d^i) \). Thus for all \( y \in B(y_1, \frac{1}{2}\zeta_i d^i) \) and \( z \in A_i \cap L_i(\frac{1}{2}\zeta_i d^{i+1}) \) we have by some elementary geometry \( d(x, L_{y, z}) \geq \frac{1}{\Pi \zeta_i d^{i+1}} \). Hence by (2.7),

\[ c(x, y, z) \geq \frac{\zeta_i d^{i+1}/12}{d^{i+1} \cdot 2d^i} = \frac{\zeta_i d}{24d^i}. \]
Since by (1.1) \( \mathcal{H}^s(E \cap B(y_i, \frac{1}{2}\zeta_i d^i)) \geq \frac{1}{4}\zeta_i d^i / C \), and by (2.4) \( \mathcal{H}^s(A_i \cap L_i(\frac{1}{2}\zeta_i d^i)) \geq \frac{1}{16} bd^8 \) (as \( L_i(\frac{1}{12}\zeta_i d^i) \supset L_i(\eta_i d^i) \) by (2.8)), we get

\[
\gamma_i \geq \left( \frac{\zeta_i d^i}{24 d^i} \right)^{2s} C^{-1} \left( \frac{1}{2}\zeta_i d^i \right)^s \frac{1}{16} bd^8
\]

which proves (2.9).

Let \( \alpha_i \in [0, \pi) \) be the angle between the lines \( L_i \) and \( L_{i+1} \). We claim that

(2.10) \[ \alpha_i \leq \max \left\{ 8d^{-1} \eta_i, 8d^{-1} \eta_{i+1}, 8(16/b)^{1/s} d^{-3} (\gamma_i + \gamma_{i+1})^{1/(2s)} \right\}. \]

Suppose this is false. Then if \( y \in A_i \cap L_i(\eta_i d^i) \) and \( z \in A_{i+1} \cap L_{i+1}(\eta_{i+1} d^{i+1}) \), we have by simple elementary geometry dist\( (z, L_{x,y}) \geq \frac{1}{4} \alpha_i d^{i+2} \). Hence (2.7) gives

\[
c(x,y,z) \geq \frac{\alpha_i d^{i+2}}{(2d^s)^2} = \frac{\alpha_i d^2}{8d^s}.\]

Integrating over \( A_i \cap L_i(\eta_i d^i) \) and \( A_{i+1} \cap L_{i+1}(\eta_{i+1} d^{i+1}) \) and using (2.4), we obtain

\[
\gamma_i + \gamma_{i+1} \geq \left( \frac{\alpha_i d^2}{8d^s} \right)^{2s} \left( \frac{1}{16} bd^{i+1} \right)^s = \frac{d^{6s} b^{2s} \alpha_i^{2s}}{16^2 \cdot 8^{2s}} > \gamma_i + \gamma_{i+1};
\]

a contradiction proving (2.10).

By the definition (2.5) of \( \eta_i \) and by (2.10) we have for some \( C_1 < \infty \) for all \( i \),

\[
\alpha_i \leq C_1 (\gamma_i + \gamma_{i+1})
\]

since \( 0 < s \leq 1/2 \). Using (2.3) we find that \( \sum \alpha_i < \infty \). This means that the lines \( L_i \) converge to a line \( L \) through \( x \). Applying (2.9) and the fact that \( \zeta_i \to 0 \), we see that \( L \) is a tangent to \( E \) at \( x \). This completes the proof.

2.2. Corollary. If \( 0 < s \leq 1/2, E \subset \mathbb{R}^2 \) is \( s \)-regular, \( \mathcal{H}^s \) measurable and \( c^{2s} (E) < \infty \), then there are \( C^1 \) curves \( \Gamma_1, \Gamma_2, \ldots \) such that

\[
\mathcal{H}^s \left( E \setminus \bigcup_{i=1}^{\infty} \Gamma_i \right) = 0.
\]

Proof. This follows from Theorem 2.1 and [MM Theorem 3.9(1)]. The proof is a relatively easy application of Whitney’s extension theorem.

2.3. Remark. Even if we would assume that the integral in (2.1) is uniformly bounded for \( x \in \mathbb{R}^2 \), \( E \) is not necessarily contained in a single \( C^1 \) curve, that is, the tangent need not vary continuously. For example, let \( C \) be a compact \( s \)-regular set lying on the unit circle \( S^1 \) and let \( D \) be an \( s \)-regular Cantor set on \( \{(x,y) : 0 \leq x \leq 1, y = 0\} \) with \( 0 \in D \). Choose a sequence \( I_j \) of complementary intervals of \( D \) with mid-points \( x_j \) and lengths \( l_j \) in such a way that \( x_j \to 0 \), and \( l_j / x_j \to 0 \) very quickly. Let

\[
E = D \cup \bigcup_{j=1}^{\infty} \left( \frac{1}{4} l_j C + x_j \right).
\]

Then \( E \) is \( s \)-regular. If \( l_j / x_j \to 0 \) sufficiently quickly, then \( c^{2s}(E, x) \) is uniformly bounded, but \( E \) is not contained in any \( C^1 \) curve.

We now give an example to show that Theorem 2.1 and Corollary 2.2 fail for \( 1/2 < s < 1 \).
2.4. Example. Let $1/2 < s < 1$. Then there is a compact $s$-regular set $E \subset \mathbb{R}^2$ such that $c^{2s}(E, x)$ is uniformly bounded for $x \in E$, but $\mathcal{H}^s(E \cap \Gamma) = 0$ for every $C^1$ curve $\Gamma$.

Proof. We shall construct $E$ with a von Koch-type construction similar to that used in [DS] §20. Define $r \in (0, 1)$ by

$$2r^s = 1.$$ 

Let $J_{0,1}$ be a closed oriented line-segment of length 1 in $\mathbb{R}^2$. Let $J_{1,1}$ and $J_{1,2}$ be the closed oriented line-segments of length $r$ in $\mathbb{R}^2$ such that the initial point of $J_{1,1}$ is the initial point of $J_{0,1}$, the initial point of $J_{1,2}$ is the mid-point of $J_{0,1}$, and the oriented angle from $J_{0,1}$ to both $J_{1,1}$ and $J_{1,2}$ is 1. Suppose we have constructed the closed oriented line-segments $J_{k,1}, \ldots, J_{k,2^k}$ of length $r^k$. We apply the above operation to each $J_{k,i}$ with the angle 1 replaced by $1/(k+1)$ to obtain the line-segments $J_{k+1,1}, \ldots, J_{k+1,2^{k+1}}$ of length $r^{k+1}$. It is clear that the unions $\bigcup_{i=1}^{2^k} J_{k,i}$ converge as $k \to \infty$ to a compact $s$-regular set $E$. For each $k$ and $j$ we denote by $E_{k,j}$ the subset of $E$ generated by $J_{k,j}$ (in the obvious way). Then for all $k$,

$$E = \bigcup_{j=1}^{2^k} E_{k,j}.$$ 

Since $\sum_k k^{-1} = \infty$, one sees easily that $E$ has tangent at none of its points. In fact, $E$ approaches all of its points along all directions in the sense that for any $x \in E$ and any line $L$ through $x$ there is a sequence $x_i \in E \setminus \{x\}$ such that $x_i \to x$ and $\text{dist}(x_i, L)/|x_i - x| \to 0$. This together with the $s$-regularity of $E$ implies that $\mathcal{H}^s(E \cap f([0,1])) = 0$ for any regular $C^1$ mapping $f : [0,1] \to \mathbb{R}^2$. This can be checked by using the regularity of $f$ to write $[0,1] = \bigcup_{k=1}^{\infty} A_k$ where each $A_k$ is a Borel set such that for some $c_k \in S^1$

$$\left|(f(x) - f(y))/|f(x) - f(y)| - c_k\right| < 1/2 \quad \text{for } x, y \in A_k.$$ 

Then $\mathcal{H}^s(E \cap f(A_k)) = 0$ for all $k$ by the above scatteredness property of $E$. It remains to show that $c^{2s}(E, x)$ is uniformly bounded.

Fix $x \in E$. For $y \in E$, $y \neq x$, let $k(y)$ be the largest $k$ such that $x, y \in E_{k-1,j}$ for some $j$. Here $E_{0,j} = E$. Let $y, z \in E \setminus \{x\}$ with $y \neq z$. Denote $k = k(y), l = k(z)$ and assume that $k \leq l$.

Suppose first that $k = l$. Then for some $j$, $y, z \in E_{k,j}$ whereas $x \notin E_{k,j}$. Let $m = m(y, z)$ be the largest $m$ such that $y, z \in E_{m-1,j}$ for some $j$. Then $m > k$. It follows from the construction that there is a positive number $b$, depending only on $r$, such that

$$|x - z| \geq b^{-1}r^k, \quad |y - z| \geq b^{-1}r^m,$$

$$\text{dist}(z, L_{x,y}) \leq b|y - z|.$$ 

If $m$ is not much bigger than $k$, we can improve the last estimate. Since for $m \leq 2k$ the angle between the lines $L_{x,y}$ and $L_{y,z}$ is at most a constant times

$$\sum_{j=k+1}^{m} \frac{1}{j} \approx \log \frac{m}{k} \approx \frac{m-k}{k},$$
we can choose \( b \) so that also
\[
\text{dist}(z, L_{x,y}) \leq b \frac{m-k}{k} r^m.
\]
Consequently by (2.7) we have both
\[
c(x, y, z) \leq 2b^3 r^{-k} \quad \text{and} \quad c(x, y, z) \leq 2b^3 \frac{m-k}{k} r^{-k}.
\]
If \( k < l \) we get in the same way interchanging \( x \) and \( y \) in the above argument
\[
c(x, y, z) \leq 2b^3 r^{-k} \quad \text{and} \quad c(x, y, z) \leq 2b^3 \frac{l-k}{k} r^{-k}.
\]
Set
\[
F_k = \{ y \in E : k(y) = k \} \quad \text{and} \quad F_m(y) = \{ z \in E : k(z) = k(y), k(y, z) = m \}.
\]
Then
\[
\mathcal{H}^s(F_k) \leq C_1 r^{sk} \quad \text{and} \quad \mathcal{H}^s(F_m(y)) \leq C_1 r^{sm}
\]
where \( C_1 \) depends only on \( r \). Therefore, changing \( m-k \) to \( n \),
\[
c^{2s}(E, x) = \int_E \int_E c(x, y, z)^{2s} d\mathcal{H}^s z d\mathcal{H}^s y
\]
\[
\leq 2 \sum_{k=1}^{\infty} \sum_{m=k+1}^{\infty} \int_{F_k} \int_{F_m(y)} c(x, y, z)^{2s} d\mathcal{H}^s z d\mathcal{H}^s y
\]
\[
+ 2 \sum_{k=1}^{\infty} \sum_{l=k+1}^{\infty} \int_{F_k} \int_{F_l} c(x, y, z)^{2s} d\mathcal{H}^s z d\mathcal{H}^s y
\]
\[
\leq 4 \cdot 2^{2s} b^{6s} C_1^2 \sum_{k=1}^{\infty} \sum_{m=k+1}^{\infty} \frac{m-k}{k}^{2s} r^{-2sk} r^{sk} r^{sm}
\]
\[
+ 4 \cdot 2^{2s} b^{6s} C_1^2 \sum_{k=1}^{\infty} \sum_{m=2k+1}^{\infty} \sum_{l=m}^{\infty} r^{-2sk} r^{sk} r^{sm}
\]
\[
= 4 \cdot 2^{2s} b^{6s} C_1^2 \sum_{n=1}^{\infty} \sum_{k=1}^{n} r^{-2sk} r^{2s} \sum_{k=n}^{\infty} r^{2s} < \infty
\]
since \( 2s > 1 \). Thus \( c^{2s}(E, x) \) is bounded.

We now show that Theorem 2.1 and Corollary 2.2 fail if we replace the regularity assumption (1.1) by \( \mathcal{H}^s(E) < \infty \).

**2.5. Example.** Given \( 0 < s < 1 \) there exists a compact set \( E \subset \mathbb{R}^2 \) such that
\( 0 < \mathcal{H}^s(E) < \infty \), \( c^{2s}(E, x) \) is uniformly bounded for \( x \in \mathbb{R}^2 \) and \( \mathcal{H}^s(E \cap \Gamma) = 0 \) for every \( C^1 \) curve \( \Gamma \).
Proof. Choose an integer $n_1 > 1$, let $J = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, y = 0\}$ and $J_i = \{(x, y) \in \mathbb{R}^2 : x = i/n_1, 0 \leq y \leq r_1\}, \quad i = 1, \ldots, n_1$, where $r_1$ is determined by

$$n_1 r_1^2 = 1.$$ 

Let $\mu_1$ be the normalized, i.e., $\mu_1 \left( \bigcup_i J_i \right) = 1$, length measure on $\bigcup_{i=1}^{n_1} J_i$. If we choose $n_1$ very large, $r_1$ will be very small and the distance $1/n_1 = r_1^2$ between any $J_i$ and $J_{i+1}$ will be much bigger than $r_1$. From this we see easily that choosing $n_1$ large enough, we have

$$c^{2s}(\mu_1, x) = \int \int c(x, y, z)^{2s} \, d\mu_1 y \, d\mu_1 z \leq C_0$$

for all $x \in \mathbb{R}^2$, where $C_0$ is an absolute constant.

Next we replace each vertical line segment $J_i$ by $n_2$ horizontal line segments $J_{i,j}$ of length $r_2$ such that $n_2 r_2^2 = r_1^2$ in the same way. Let $\mu_2$ be the normalized length measure on $\bigcup_{i,j} J_{i,j}$. Choosing $n_2$ sufficiently large we can keep $c^{2s}(\mu_2, x)$ as close to $c^{2s}(\mu_1, x)$, uniformly, as we want. We choose it so that $c^{2s}(\mu_2, x) \leq C_0 + \frac{1}{r}$ for $x \in \mathbb{R}^2$. The point here is that for some small $\delta > 0$ looking from any $x \in \mathbb{R}^2$ the part of $\mu_2$ outside $B(x, \delta)$ looks very much like $\mu_1$ and the contribution of $\mu_2$ in $B(x, \delta)$ to $c^{2s}(\mu_2, x)$ is very small.

Continuing this we get unions $E_k$ of line segments $J_{i_1, \ldots, i_k}$ of length $r_k$. Every second time these line segments are horizontal and every second time vertical. We also have uniformly distributed probability measures $\mu_k$ on $E_k$ satisfying $c^{2s}(\mu_k, x) \leq C_0 + \sum_{i=2}^{k} 2^{-i}$ for all $k$ and $x \in \mathbb{R}^2$. The sets $E_k$ converge to a compact set $E$ with $0 < \mathcal{H}^s(E) < \infty$ and the measures $\mu_k$ converge weakly to a probability measure $\mu$ supported on $E$ such that

$$c^{2s}(\mu, x) \leq 2 \quad \text{for } x \in \mathbb{R}^2.$$ 

It is easy to check that $\mu$ is comparable with $\mathcal{H}^s$ restricted to $E$. Thus $c^{2s}(E, x)$ is uniformly bounded.

Finally that $\mathcal{H}^s(E \cap \Gamma) = 0$ for every $C^1$ curve $\Gamma$ can be checked with the help of decompositions such as in the proof of Example 2.4.

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